THE THEORY OF THE ELECTROMAGNETIC FIELD
SECOND EDITION
DAVID M. COOK
Department of Physics
Lawrence University
711 E Boldt Way SPC24
Appleton, Wisconsin 54911

- IDL          MATLAB          OCTAVE
  MAXIMA        MAPLE          • MATHEMATICA
  PYTHON

First Edition 1975
1975 Copyright © by Prentice-Hall, Inc., Englewood Cliffs, NJ
1991 Copyright © reassigned to David M. Cook
2003 Printing of 1981 republished by Dover

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License (creativecommons.org/licenses/by-nc-sa/4.0/). Any use not permitted by this license requires authorization in writing from David M. Cook.

Library of Congress Call Number: QC665.E4C66

Corresponding information for the Second Edition has not yet been assigned.
Preface to Second Edition

I taught Lawrence’s sophomore course in electromagnetic theory covering the material in Chapters 0–7 and 9–12 in the spring term almost every year from 1966 through 1993. The first edition of this book, written during the first decade of that period, was published by Prentice-Hall in 1975, was reprinted in 1981, and remained in print for about a decade and a half. That edition was reprinted by Dover in 2003 and remained in print for another decade or so. Not too long after the original publication in 1975, as I was learning \LaTeX, I began the generation of the second edition, which I—or one of my colleagues at Lawrence—then used for offering the course from about 1977 until 1993. In addition, I taught Lawrence’s junior/senior course in advanced electromagnetic theory covering the material in Chapters 8 and 13–15 in the spring term every other year from the late 1960s through 1996.

Because it still accurately describes both my motivation for writing the book in the first place and the overall perspective adopted, the preface to the first edition of this book is reproduced starting on page v. The bulk of the text has been polished a bit from that in the first edition but not substantially revised. A number of enhancements, however, have been incorporated in the second edition. In particular,

- The use of computational tools, already a feature of the first edition, has been significantly expanded. Original sample coding has been adjusted to avoid outmoded commands and processes. In addition, students are directed more frequently to such tools as MAXIMA, Maple, Mathematica, IDL, MATLAB, OCTAVE, and/or PYTHON as resources for addressing assigned problems, though little description of these tools is incorporated in this book. Students are assumed to have acquired knowledge of one or more of these tools in other contexts.\(^1\),\(^2\)

- Even though they are now somewhat dated and are perhaps less readily available than

\(^1\) At Lawrence, introduction to Maple or Mathematica and to IDL or PYTHON is incorporated in a course called *Computational Mechanics* (“Comp Mech” in the students’ jargon), which sophomores are required to take in the term before they take the course in electromagnetic theory. Indeed, the book *Computation and Problem Solving in Undergraduate Physics*, CPSUP, which I have been developing since the late 1980s, provides the text for the computational part of the mechanics course. Reflecting a long-standing objective within the Department of Physics at Lawrence (see “Computation in Undergraduate Physics: The Lawrence Approach”, Am. J. Phys. *76*, 321-326 (April-May, 2008)), the inclusion of this course in the undergraduate curriculum at Lawrence aims not only to encourage instructors to direct students to computational tools but also—and perhaps more importantly—to encourage students to make use of computer tools on their own initiative, even if not explicitly assigned to do so by their instructors.

\(^2\) Various versions of CPSUP constructed to include several different subsets of the tools mentioned above are now available for download from the AAPT digital library ComPADRE by going to the URL [psrc.aapt.org/psrc/curricula/cpsup](psrc.aapt.org/psrc/curricula/cpsup). Various versions of this E and M text are now available for download from ComPADRE at the URL [psrc.aapt.org/curricula/Theory-of-EMField](psrc.aapt.org/curricula/Theory-of-EMField).
they were in 1975, all of the original references have been preserved. In addition, a few more recent sources have been added.

- The index has been substantially expanded and, in particular, includes pointers to the computational inclusions.

- Footnotes providing full name, birth date and place, and death date and place of each mentioned scientist have been added. Typically the footnote for each individual is placed at the first mention of the name in the text. To facilitate finding that information at later mentions, the page on which the footnote occurs has been included in the index.

As with CPSUP, this book can be customized to include only those computational tools available at the user’s site. Typically, only one of IDL, MATLAB, OCTAVE, and PYTHON and only one of MAXIMA, Maple, and Mathematica will be included. Text that relates to an omitted tool will be omitted altogether from the version reflecting that choice. Thus, one printing will include only IDL and Maple while another may include only OCTAVE and MAXIMA. To preserve easy communication among users of different versions, when an omitted tool is discussed in a section by itself, that section will be omitted, and the corresponding section number will also be omitted. Similarly, while I have tried to phrase exercises generically, I have not been able fully to achieve that objective. Thus, there may be gaps in section numbers and/or exercise numbers in a version that does not include everything. Page numbers will always run continuously from the beginning to the end of each version while equation numbers, figure numbers, table numbers, and footnote numbers will run continuously chapter by chapter. Consequently, specific equations, figures, tables, and footnotes will have different numbers in different versions but section numbers and chapter numbers will be the same in all versions.

David M. Cook
Appleton, WI
13 April 2023
Preface to First Edition

This book evolved concurrently with a two-term intermediate-level course that I have offered for upper-division undergraduates nearly every year since 1965. Its writing was motivated largely by a decision to treat the traditional topics of electricity and magnetism in a non-traditional order. The more significant departures from the traditional sequencing of topics include:

1. Immediate parallel development of the phenomenologies both of charge and of current, using observed experimental properties to guide the selection of operational definitions for these fundamental entities.

2. Early discussion of the differences among the common systems of units, emphasizing the way these differences are reflected in the definitions adopted for charge and current and simultaneously setting the stage for the later selection of the mksa system for most of the book.

3. Introduction of the surface integral in the context of current calculations, where the flux of the current density represents something actually flowing across the surface and hence is more readily understood than the more abstract flux of, say, the electric field.

4. Simultaneous introduction of both the electric and magnetic induction fields, thereby permitting a unified and compact treatment of particle trajectories and of forces and torques on general charge and current distributions.

5. Postponement of the serious use of differential equations until after Maxwell’s equations have been fully developed, thereby making it possible for students to take electricity and magnetism concurrently with introductory differential equations.

6. Full development of Maxwell’s equations in vacuum before introducing the complications brought about by the presence of matter, regarding the modifications introduced by matter as an important special topic rather than as a part of the basic theory and (at the same time) permitting students to become fully familiar with the fields in vacuum and with vector field theory in general before thrusting them into the subtle and sometimes confusing distinctions among the four fields used in the macroscopic theory of the fields in matter.

The flow chart following this preface displays the sequencing of topics in this book.

This flow chart can also be used as an aid to selecting that portion of the text to be covered if temporal constraints force an abbreviation of the course. The essential theory of
the electromagnetic field in vacuum is covered in Chapters 1–6, which should be included
in any course. (Sections 3.2, 4.12, and 5.9 can be omitted if desired.) The remaining
chapters deal with a variety of applications, consequences, and extensions of the basic
theory and, with the exception of Chapters 9–13, are largely independent of one another.
The interrelationships among Chapters 7–15 are illustrated in the flow chart. If, for example,
Section 12.8 is to be included, then the first four sections of Chapter 8 must be included; if
Chapter 13 or 14 is to be covered, then portions of Chapter 7 must be included; Section 7.4
can be omitted unless Section 13.6 is to be studied; and so on. The flow chart does not
show that most of Chapter 8 (and particularly Section 8.7) can be studied any time after
completion of Section 4.6.

Some of the features of this book are not apparent in the flow chart. In the first
place, fewer than the average number of examples have been worked out explicitly in the
text, partly to make the compactness of the basic theory the more apparent and partly
to save some of these standard examples (which are almost invariably easier than most
end-of-chapter problems) for assignment to students. When a standard example is set
as a problem, the desired final result is either incorporated in the problem statement or
included in the table of answers at the end of the book, and these problems are indexed
along with the textual material. The text therefore does not suffer as a reference work
because it fails to treat some of the standard examples explicitly. Secondly, the details
of symmetry arguments, particularly in Sections 4.4 and 5.6, have been given more than
average space, hopefully to prevent the student from developing either a gnawing mistrust of
these powerful arguments or a willingness to use symmetry to support all sorts of outlandish
claims. Thirdly, some of the problems draw specific attention to discussions in the periodical
literature or in other texts, so that the student is encouraged to avoid complete reliance
on any single source. Fourthly, throughout the text purely mathematical developments
have been identified as such (and in some cases have been physically separated from the
main text) so that the fundamental physics of electricity and magnetism stands out more
prominently.

Finally, the growing availability of digital computers for instructional purposes and
the growing sophistication of intermediate level students in the use of these computers
have prompted the inclusion of several problems whose solutions can best be carried out
on a computer. Some of these problems simply use the computer to calculate a table of
values or plot a graph of an unfamiliar function obtained as the analytic solution to the
problem. Others describe an algorithm and lead the student through a numerical solution
of a suggested problem, for example, the tracing of field lines for some distribution of point
charges. In two cases (trajectory problems and Laplace’s equation) a brief discussion of
numerical approaches is incorporated in the text itself.

I conclude this Preface with an attempt to enumerate and acknowledge the several
debts I owe to other individuals for their contributions—both direct and indirect—to this
book. First, I wish to record my gratitude to several anonymous reviewers, to J. Bruce
Brackenridge, to John R. Brandenberger, and to James S. Evans for numerous criticisms
and suggestions that have led to improvements in the final version of this book; to John
R. Merrill for many conversations about the uses of computers in physics instruction; and
to dozens of students for their patience with endless drafts and for innumerable valuable
suggestions. In addition I wish to acknowledge my debt to the numerous authors whose
works on electricity and magnetism I have studied over the years. I am particularly aware
of being influenced by the texts by E. M. Purcell; R. P. Feynman, R. B. Leighton, and M. Sands; and J. D. Jackson; and I have acknowledged these and other works that I can identify as having contributed in a general way to my own thinking by citing them more explicitly at appropriate points throughout this book. Further, I am grateful to Miss Jean St. Pierre for her expert and efficient typing of the manuscript. Last in order but first in importance, I owe to my wife and children a debt of a very different sort. They have patiently endured while the task of writing this book not only occupied all of my time but also went on and on and on. I recognize and am grateful for the love and understanding expressed by their patience.

David M. Cook
Appleton, WI
January 1975
Introduction

The major subdisciplines within classical physics are mechanics, electricity and magnetism, and thermodynamics. Although we cannot ignore completely the interactions among these subdisciplines, this book is concerned primarily with the classical theory of electricity and magnetism. Somewhat more accurately, we treat the classical theory of the electric and magnetic fields to be defined in Chapter 3. The physical phenomena that motivated the development of this theory were noticed and puzzled over in antiquity: Rubbed amber attracts bits of paper, and pieces of naturally occurring lodestone experience mutual forces of interaction. The development of the contemporary formal theory, however, did not begin until the late 1700's. The experimental work of Charles-Augustin de Coulomb\(^3\) (1785; Chapter 2), André Marie Ampère\(^4\) and Hans Christian Ørsted\(^5\) (1800–1820; Chapter 2), and Michael Faraday\(^6\) (1830–1860; Chapter 6) played a prominent role. Less than 100 years after Coulomb’s work, the theory was brought to its present form by James Clerk Maxwell\(^7\) (early 1860’s; Chapter 6). Faraday and Maxwell discovered interrelationships between electric and magnetic phenomena, but a reexpression of Maxwell’s theory in the terminology of Albert Einstein’s\(^8\) special theory of relativity (1905) revealed even deeper connections (Chapter 15) and we now view these initially distinct phenomena as a single phenomenon, which we call \textit{electromagnetism}. Again more accurately, we now view the electric and magnetic fields as different aspects of a more inclusive electromagnetic field. Although the interrelationships supplied by the theory of relativity can be exploited to develop the classical theory of the electromagnetic field from a very small number of experimental observations, we nevertheless adopt a more traditional approach that requires a larger (but still small) number of experimental observations but does not require knowledge of transformations between frames of reference in relative motion. In Chapter 15, we shall show that the results of this development are consistent with the principles of relativity.

This book can be divided into two main sections. In the first section, we begin by enumerating the observed properties of charge and current and by using these properties to deduce suitable units for the quantitative measurement of charge and current (Chapter 1), continue by introducing the concepts of charge and current densities needed for describing

\(^{3}\)b. 14 June 1736 in Angoulême, France; d. 23 August 1806 in Paris, France.
\(^{4}\)b. 20 January 1775 in Lyon, France; d. 10 June 1836 in Marseille, France. We shall hereafter follow English convention and omit the accents.
\(^{5}\)b. 14 August 1777 in Rudkøbing, Denmark; d. 9 March 1851 in Copenhagen, Denmark. The Danish letter Ø is usually rendered ‘Oe’ in English, and we will use that convention hereafter.
\(^{6}\)b. 22 September 1791 in Newington Butts, England; d. 25 August 1867 in Hampton Court, Middlesex, England.
\(^{7}\)b. 13 June 1831 in Edinburgh, Scotland; d. 5 November 1879 in Cambridge, England.
\(^{8}\)b. 14 March 1879 in Ulm, Kingdom of Württemberg, German Empire; d. 18 April 1955 in Princeton, New Jersey, United States.
the “state” of arbitrary charge distributions (Chapter 2) and by defining the electric and magnetic fields (Chapter 3), and conclude by examining the properties of these fields, first when they are time-independent (Chapters 4 and 5) and then when they are time-dependent (Chapter 6). Maxwell’s equations, which are at once the goal of the first section and the basis of the second section, are first stated in Section 6.3. From that point on, we shall be concerned with the consequences of Maxwell’s equations and with their generalization to include the response of matter to externally applied fields. In the second main section of this book (Chapters 7–15), we examine potential theory, plane waves, properties of matter, radiation from accelerated particles, and the relativistic formulation of Maxwell’s equations. The main course of the development in Chapters 1–6 and the interrelationships among the later chapters in the book are both shown in the flow chart following the preface. In this chart, items displayed entirely in capital letters are points of experimental contact, basic laws are placed in boxes with extra heavy borders, and each dotted line connects two boxes, the first of which provides a suggestion for (but not a deduction of) the second. The readers may find it useful to refer to this chart occasionally as they work their way through this book.

For many readers, this book probably represents a first (or perhaps a second) encounter with mathematical physics. It will therefore be necessary now and again to discuss mathematical techniques per se. To forestall the common (but wholly erroneous) conclusion that theoretical physics is mathematics, a special effort has been made to identify mathematical developments as such and to stress the role of experimental properties in guiding the development of the theory. In some cases (Chapter 0, Sections 2.3 and 2.5, and the Appendices) mathematical topics have been physically separated from the main text so that, once the mathematics has been mastered, the reader can approach the main text without being distracted from the physics by these mathematical digressions. The mathematical component of this text is important and the reader must strive for fluency in mathematical manipulation. On the other hand, the mathematics cannot be allowed to overshadow the physics, which—after all—is what the mathematical model is constructed to represent.

Several additional comments and suggestions to guide the reader through this book and assist in a careful study of the electromagnetic field follow:

1. Throughout this book, equations are referred to by a two-part identification, the first part denoting the chapter or appendix in which the equation occurs. Equation (12.28) is the 28th equation in Chapter 12, Eq. (C.8) is the 8th equation in Appendix C, etc. Problems are referred to by a similar two-part number prefixed with the letter P.

2. Your attention is drawn to the identities in Appendix C, to the data in Appendix E, and to the final page of the index, on which you will find a quick directory to important formulas contained in the text proper.

3. Even if you are already familiar with the preliminaries treated in Chapter 0, skim the chapter; you will find not only a review of background material but also the setting out of a notation whose meaning will subsequently be assumed without further definition.

4. Fill in the omitted steps in each argument on your own initiative; only a few problems requesting the completion of these arguments are specifically included.

5. Read all of the problems even if only a few are actually solved in detail; in total,
these problems indicate something of the broad spectrum of circumstances to which the basic theory can be applied.

6. Do not spend time laboriously evaluating standard integrals; exploit the capabilities of computer-aided algebra programs like MAXIMA, MAPLE, and Mathematica or, if such programs are unavailable, use integral tables. Three of the many available tables are

- *CRC Standard Mathematical Tables and Formulae* (CRC Press, Boca Raton, FL). Available in several editions, the most recent of which (32nd) is dated 2011.

Unfortunately, as the years pass and computer-aided algebra programs become more common and more capable, integral tables are harder and harder to find.

7. Refer regularly to other books on the same topics. Many may be found by browsing through the shelves in a library in the proper area (537s and 538s in the Dewey decimal system; QC501–766 or so in the Library of Congress system) or by searching on-line at amazon.com or bn.com for “electromagnetic theory” or “electromagnetism”.

- *Some* books at a level more elementary than this book have been authored by R. P. Feynman, R. B. Leighton, and M. Sands; D. Halliday and R. Resnick; A. F. Kip; and E. M. Purcell.
- Books at a level comparable to that of this book include those by Paul Lorrain and Dale R. Corson; David J. Griffiths; J. R. Reitz and F. J. Milford; W. M. Schwarz; and W. T. Scott.
- Finally, books at levels ranging from somewhat to very much more advanced than this book include those by R. Becker and F. Sauter; J. D. Jackson; E. C. Jordan and K. G. Balmain; L. D. Landau and L. M. Lifshitz; J. B. Marion; W. K. H. Panofsky and M. Phillips; W. R. Smythe; and J. A. Stratton.

Over the years, there have been many editions of some of these books, some with added authors.

8. The reader wanting assistance in computational approaches to problems in electromagnetism might browse through the shelves in a library in the proper area (518s, 537s, and 538s in the Dewey decimal system; QA75–77 and QC501–766 in the Library of Congress system) or search on-line at amazon.com or bn.com for “applied numerical methods” or “numerical methods in physics”. Books on these topics include those by B. Carnahan, H. A. Luther, and J. O. Wilkes; S. D. Conte; K. Atkinson; R. H. Landau; N. J. Giordano; F. S. Acton, H. Gould and J. Tobochnik; and Wolfgang Christian.
# Table of Contents

Preface to Second Edition iii
Preface to First Edition v
Introduction ix
Table of Contents xiii

## 0 Mathematical and Physical Preliminaries

0.1 Scalars, Vectors, and Vector Algebra ........................................ 1
0.2 The Representation of Fields .................................................. 8
0.3 Static Force Fields ..................................................................... 13
  0.3.1 Equations of Motion ............................................................ 13
  0.3.2 Work and the Line Integral ................................................... 13
  0.3.3 The Work-Kinetic Energy Theorem ......................................... 15
  0.3.4 Conservative Force Fields and Potential Energy ...................... 15
  0.3.5 Torque ............................................................................... 19
  0.3.6 Power ................................................................................. 20
0.4 Coordinate-Free Definitions for Gradient and Curl ......................... 22

## 1 Quantifying Charge and Current

1.1 The Phenomenon of Electric Charge .......................................... 29
1.2 The Interaction of Point Charges ............................................... 31
1.3 The Phenomenon of Current ..................................................... 33
1.4 The Interaction of Parallel Currents ............................................ 34
1.5 Current as Charge in Motion ..................................................... 36
1.6 Units in Electricity and Magnetism ............................................. 37
  1.6.1 CGS-ESU ........................................................................... 37
  1.6.2 CGS-EMU ......................................................................... 38
  1.6.3 Rationalized MKS Units ..................................................... 38
  1.6.4 Other Systems of Units ....................................................... 39
  1.6.5 Summary ........................................................................... 39
  1.6.6 Conversion of Units ............................................................ 39
1.7 Recent Changes in Definitions of Units ........................................ 41

## 2 Specifying Arbitrary Distributions

2.1 Charge Density .......................................................................... 49
2.2 Current Density .......................................................................... 51
2.3 Digression I: Stokes’ and Divergence Theorems ................................. 57
  2.3.1 Stokes’ Theorem .................................................................. 57
  2.3.2 The Divergence Theorem ..................................................... 58
  2.3.3 A Useful Integral: Solid Angle .............................................. 61
2.4 The Equation of Continuity ....................................................... 63
2.5 Digression II: Operators Involving $\nabla$ .................................... 65
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Definition of The Electromagnetic Field</td>
<td>69</td>
</tr>
<tr>
<td>3.1</td>
<td>Definition of the Electromagnetic Field</td>
<td>69</td>
</tr>
<tr>
<td>3.2</td>
<td>Trajectories of Particles in Prescribed Fields</td>
<td>72</td>
</tr>
<tr>
<td>3.3</td>
<td>Forces and Torques on General Distributions</td>
<td>78</td>
</tr>
<tr>
<td>4</td>
<td>The Electric Field of Static Charges</td>
<td>85</td>
</tr>
<tr>
<td>4.1</td>
<td>Finding the Electrostatic Field from its Sources</td>
<td>85</td>
</tr>
<tr>
<td>4.1.1</td>
<td>The Electric Monopole (Point Charge)</td>
<td>87</td>
</tr>
<tr>
<td>4.1.2</td>
<td>The Electric Dipole</td>
<td>89</td>
</tr>
<tr>
<td>4.1.3</td>
<td>Electric Multipoles</td>
<td>90</td>
</tr>
<tr>
<td>4.1.4</td>
<td>Other Distributions of Point Charges</td>
<td>91</td>
</tr>
<tr>
<td>4.1.5</td>
<td>An Infinite Sheet</td>
<td>91</td>
</tr>
<tr>
<td>4.2</td>
<td>Mapping Electric Fields</td>
<td>93</td>
</tr>
<tr>
<td>4.3</td>
<td>Numerical Evaluation of Fields</td>
<td>95</td>
</tr>
<tr>
<td>4.4</td>
<td>Gauss's Law</td>
<td>100</td>
</tr>
<tr>
<td>4.5</td>
<td>The Restricted Faraday Law</td>
<td>106</td>
</tr>
<tr>
<td>4.6</td>
<td>The Electrostatic Potential</td>
<td>107</td>
</tr>
<tr>
<td>4.7</td>
<td>Numerical Evaluation of Potentials</td>
<td>110</td>
</tr>
<tr>
<td>4.8</td>
<td>Mapping Electrostatic Potentials</td>
<td>112</td>
</tr>
<tr>
<td>4.9</td>
<td>Conductors as Equipotential Regions</td>
<td>114</td>
</tr>
<tr>
<td>4.10</td>
<td>Capacitance</td>
<td>114</td>
</tr>
<tr>
<td>4.11</td>
<td>Potential Energy in the Electrostatic Field</td>
<td>117</td>
</tr>
<tr>
<td>4.12</td>
<td>The Multipole Expansion of the Electrostatic Potential</td>
<td>120</td>
</tr>
<tr>
<td>5</td>
<td>The Magnetic Induction Field of Steady Currents</td>
<td>127</td>
</tr>
<tr>
<td>5.1</td>
<td>The Law of Biot-Savart</td>
<td>128</td>
</tr>
<tr>
<td>5.2</td>
<td>Applications of the Law of Biot-Savart</td>
<td>131</td>
</tr>
<tr>
<td>5.2.1</td>
<td>The Infinitely Long Straight Wire</td>
<td>131</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Circular Loops</td>
<td>132</td>
</tr>
<tr>
<td>5.3</td>
<td>Mapping Magnetic Fields</td>
<td>136</td>
</tr>
<tr>
<td>5.4</td>
<td>Numerical Evaluation of Fields</td>
<td>136</td>
</tr>
<tr>
<td>5.5</td>
<td>The Magnetic Flux Law</td>
<td>141</td>
</tr>
<tr>
<td>5.6</td>
<td>Ampere's Circuital Law</td>
<td>142</td>
</tr>
<tr>
<td>5.7</td>
<td>The Magnetic Vector Potential</td>
<td>147</td>
</tr>
<tr>
<td>5.8</td>
<td>Energy in the Static Magnetic Induction Field</td>
<td>151</td>
</tr>
<tr>
<td>5.9</td>
<td>The Multipole Expansion of $\mathbf{A}$</td>
<td>152</td>
</tr>
<tr>
<td>6</td>
<td>Maxwell's Equations in Vacuum</td>
<td>157</td>
</tr>
<tr>
<td>6.1</td>
<td>Electromagnetic Induction: Faraday's Law</td>
<td>158</td>
</tr>
<tr>
<td>6.1.1</td>
<td>Electromotive Force</td>
<td>158</td>
</tr>
<tr>
<td>6.1.2</td>
<td>Motional EMF</td>
<td>160</td>
</tr>
<tr>
<td>6.1.3</td>
<td>Faraday's Law of Electromagnetic Induction</td>
<td>163</td>
</tr>
<tr>
<td>6.1.4</td>
<td>Energy in the Static Magnetic Induction Field</td>
<td>166</td>
</tr>
<tr>
<td>6.2</td>
<td>Displacement Current</td>
<td>170</td>
</tr>
<tr>
<td>6.3</td>
<td>Maxwell's Equations</td>
<td>173</td>
</tr>
<tr>
<td>6.4</td>
<td>Energy in the Electromagnetic Field</td>
<td>176</td>
</tr>
<tr>
<td>6.5</td>
<td>Momentum in the Electromagnetic Field</td>
<td>177</td>
</tr>
<tr>
<td>6.6</td>
<td>Maxwell's Equations for the Potentials</td>
<td>178</td>
</tr>
<tr>
<td>6.7</td>
<td>Decoupling the Equations for the Fields</td>
<td>181</td>
</tr>
<tr>
<td></td>
<td>Interlude: A Change of View</td>
<td>185</td>
</tr>
<tr>
<td>7</td>
<td>Plane Electromagnetic Waves in Vacuum</td>
<td>187</td>
</tr>
<tr>
<td>7.1</td>
<td>Plane Electromagnetic Waves</td>
<td>188</td>
</tr>
<tr>
<td>7.2</td>
<td>Energy and Momentum in Plane Waves</td>
<td>192</td>
</tr>
<tr>
<td>7.3</td>
<td>Superposition of Waves of the Same Frequency</td>
<td>195</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.4 Superposition of Waves of Different Frequencies</td>
<td>203</td>
</tr>
<tr>
<td>7.5 Plane Waves in Three Dimensions</td>
<td>205</td>
</tr>
<tr>
<td>8 Potential Theory</td>
<td>211</td>
</tr>
<tr>
<td>8.1 Boundary Conditions</td>
<td>211</td>
</tr>
<tr>
<td>8.2 Superposition and Uniqueness</td>
<td>212</td>
</tr>
<tr>
<td>8.3 One-Dimensional Problems</td>
<td>215</td>
</tr>
<tr>
<td>8.4 Two-Dimensional Problems by Separation of Variables</td>
<td>215</td>
</tr>
<tr>
<td>8.4.1 Laplace’s Equation in 2D Cartesian Coordinates x, y</td>
<td>215</td>
</tr>
<tr>
<td>8.4.2 Laplace’s Equation in 2D Cylindrical Coordinates r, φ</td>
<td>217</td>
</tr>
<tr>
<td>8.4.3 Laplace’s Equation in 2D Cylindrical Coordinates r, z</td>
<td>218</td>
</tr>
<tr>
<td>8.4.4 Laplace’s Equation in 2D Spherical Coordinates r, θ</td>
<td>218</td>
</tr>
<tr>
<td>8.5 Two-Dimensional Problems Using Complex Variables</td>
<td>224</td>
</tr>
<tr>
<td>8.6 The Method of Images</td>
<td>227</td>
</tr>
<tr>
<td>8.7 Numerical Solution of Laplace’s Equation</td>
<td>231</td>
</tr>
<tr>
<td>8.8 The Method of Analogy</td>
<td>238</td>
</tr>
<tr>
<td>8.9 Poisson’s Equation</td>
<td>238</td>
</tr>
<tr>
<td>9 Properties of Matter I: Conduction</td>
<td>243</td>
</tr>
<tr>
<td>9.1 Conductivity and Ohm’s Law</td>
<td>244</td>
</tr>
<tr>
<td>9.2 Carrier Mobility and Collision Times</td>
<td>247</td>
</tr>
<tr>
<td>10 Properties of Matter II: Dielectric Polarization</td>
<td>253</td>
</tr>
<tr>
<td>10.1 The Microscopic Description: Electric Polarizability</td>
<td>255</td>
</tr>
<tr>
<td>10.2 The Macroscopic Description: Dielectric Polarization</td>
<td>259</td>
</tr>
<tr>
<td>10.3 Potential and Field at an Exterior Point</td>
<td>260</td>
</tr>
<tr>
<td>10.4 Field at an Interior Point</td>
<td>263</td>
</tr>
<tr>
<td>10.5 Basic Equations When Dielectrics Are Present</td>
<td>269</td>
</tr>
<tr>
<td>10.6 The Clausius-Mossotti Relation</td>
<td>274</td>
</tr>
<tr>
<td>11 Properties of Matter III: Magnetization</td>
<td>279</td>
</tr>
<tr>
<td>11.1 The Microscopic Description: Magnetic Polarizability</td>
<td>279</td>
</tr>
<tr>
<td>11.2 The Macroscopic Description: Magnetization</td>
<td>285</td>
</tr>
<tr>
<td>11.3 Field at an Exterior Point</td>
<td>286</td>
</tr>
<tr>
<td>11.4 An Alternative Approach to the Exterior Field</td>
<td>292</td>
</tr>
<tr>
<td>11.5 Magnetic Induction Field an an Interior Point</td>
<td>294</td>
</tr>
<tr>
<td>11.6 Basic Equations when Magnetic Matter is Present</td>
<td>297</td>
</tr>
<tr>
<td>11.7 Connecting the Microscopic with the Macroscopic</td>
<td>301</td>
</tr>
<tr>
<td>11.8 Ferromagnetism</td>
<td>303</td>
</tr>
<tr>
<td>12 Time-Dependent Fields When Matter Is Present</td>
<td>311</td>
</tr>
<tr>
<td>12.1 Maxwell’s Equations in Matter</td>
<td>311</td>
</tr>
<tr>
<td>12.2 The Equation of Continuity</td>
<td>315</td>
</tr>
<tr>
<td>12.3 The Energy Theorem</td>
<td>315</td>
</tr>
<tr>
<td>12.4 The Momentum Theorem</td>
<td>317</td>
</tr>
<tr>
<td>12.5 On Which Fields are Basic</td>
<td>318</td>
</tr>
<tr>
<td>12.6 The Potentials</td>
<td>318</td>
</tr>
<tr>
<td>12.7 Boundary Conditions at Discontinuities in the Medium</td>
<td>320</td>
</tr>
<tr>
<td>12.8 Static Potentials</td>
<td>326</td>
</tr>
<tr>
<td>13 Plane Electromagnetic Waves</td>
<td>331</td>
</tr>
<tr>
<td>13.1 Maxwell’s Equations for Monochromatic Fields</td>
<td>331</td>
</tr>
<tr>
<td>13.2 Boundary Conditions on Monochromatic Fields</td>
<td>336</td>
</tr>
<tr>
<td>13.3 Plane Monochromatic Waves in Linear Media</td>
<td>339</td>
</tr>
<tr>
<td>13.3.1 Expressions for Plane Waves</td>
<td>339</td>
</tr>
<tr>
<td>13.3.2 Interpretation of the Propagation Vector</td>
<td>341</td>
</tr>
</tbody>
</table>
### TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.3.3 Waves in Non-Conducting Media</td>
<td>344</td>
</tr>
<tr>
<td>13.4 Transmission and Reflection at Plane Interfaces</td>
<td>346</td>
</tr>
<tr>
<td>13.4.1 The Grand Strategy</td>
<td>346</td>
</tr>
<tr>
<td>13.4.2 Reflection/Transmission at Normal Incidence</td>
<td>348</td>
</tr>
<tr>
<td>13.4.3 Reflection/Transmission at Oblique Incidence</td>
<td>350</td>
</tr>
<tr>
<td>13.5 Wave Guides and Cavity Resonators</td>
<td>359</td>
</tr>
<tr>
<td>13.5.1 A Simple Transverse Electric Mode</td>
<td>359</td>
</tr>
<tr>
<td>13.5.2 Other Modes</td>
<td>363</td>
</tr>
<tr>
<td>13.5.3 Cavity Resonators</td>
<td>363</td>
</tr>
<tr>
<td>13.6 Dispersion</td>
<td>367</td>
</tr>
<tr>
<td>14 Radiation from Prescribed Sources in Vacuum</td>
<td>377</td>
</tr>
<tr>
<td>14.1 The Inhomogeneous Wave Equation: Retardation</td>
<td>378</td>
</tr>
<tr>
<td>14.2 Radiation from the Oscillating Electric Dipole</td>
<td>381</td>
</tr>
<tr>
<td>14.3 The Liénard-Wiechert Potentials</td>
<td>385</td>
</tr>
<tr>
<td>14.4 The Fields of a Moving Point Charge</td>
<td>389</td>
</tr>
<tr>
<td>14.5 Radiation from Accelerated Point Charges</td>
<td>393</td>
</tr>
<tr>
<td>14.6 The Radiation Reaction</td>
<td>398</td>
</tr>
<tr>
<td>15 Relativistic Form of Maxwell’s Equations</td>
<td>403</td>
</tr>
<tr>
<td>15.1 A Review of Special Relativity</td>
<td>406</td>
</tr>
<tr>
<td>15.2 Maxwell’s Equations in Covariant Form</td>
<td>414</td>
</tr>
<tr>
<td>15.3 Transformation of the Electromagnetic Field</td>
<td>419</td>
</tr>
<tr>
<td>15.4 The Stress-Energy-Momentum Tensor</td>
<td>420</td>
</tr>
<tr>
<td>15.5 A New Viewpoint: The Law of Biot-Savart Revisited</td>
<td>423</td>
</tr>
<tr>
<td>A Linear Equations, Determinants, and Matrices</td>
<td>431</td>
</tr>
<tr>
<td>A.1 Simultaneous Linear Equations and Determinants</td>
<td>431</td>
</tr>
<tr>
<td>A.2 Matrix Algebra</td>
<td>435</td>
</tr>
<tr>
<td>B Binomial and Taylor Expansions</td>
<td>439</td>
</tr>
<tr>
<td>C Vector Identities and Relationships</td>
<td>443</td>
</tr>
<tr>
<td>D Complex Numbers and Fourier Analysis</td>
<td>445</td>
</tr>
<tr>
<td>D.1 The Algebra of Complex Numbers</td>
<td>445</td>
</tr>
<tr>
<td>D.2 Fourier Series</td>
<td>449</td>
</tr>
<tr>
<td>D.3 Fourier Transforms</td>
<td>451</td>
</tr>
<tr>
<td>E Reference Tables</td>
<td>453</td>
</tr>
<tr>
<td>E.1 Selected Physical Constants</td>
<td>453</td>
</tr>
<tr>
<td>E.2 Abbreviations of Unit Names</td>
<td>454</td>
</tr>
<tr>
<td>E.3 Dot Products between Unit Vectors</td>
<td>454</td>
</tr>
<tr>
<td>E.4 Cross Products between Unit Vectors</td>
<td>454</td>
</tr>
<tr>
<td>F Listings of LU Procedures</td>
<td>455</td>
</tr>
<tr>
<td>F.1 IDL functions/procedures</td>
<td>455</td>
</tr>
<tr>
<td>F.1.1 IDL Procedure <em>ugen_grid</em></td>
<td>455</td>
</tr>
<tr>
<td>F.1.2 IDL Procedure <em>ludiffeq_23</em></td>
<td>456</td>
</tr>
<tr>
<td>F.1.3 IDL Procedure <em>ludiffeq_45</em></td>
<td>461</td>
</tr>
<tr>
<td>F.1.4 IDL Procedures <em>electfield, calcefield, arrowhead, getpointE</em></td>
<td>466</td>
</tr>
<tr>
<td>F.1.5 IDL Procedures <em>magnetfield, calcbfield, arrowhead, getpointB</em></td>
<td>471</td>
</tr>
<tr>
<td>F.1.6 IDL Procedures <em>magnetloop, calcbloop, arrowhead, getpointB, looprad, loopax</em></td>
<td>475</td>
</tr>
<tr>
<td>F.1.7 IDL Procedure <em>lisjus</em></td>
<td>480</td>
</tr>
<tr>
<td>F.1.8 IDL Procedure <em>antena</em></td>
<td>481</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

**G** Instructions for LU Procedures

G.1 Programs for IDL ................................................. 483
  G.1.1 Instructions for `electfield` for IDL .................. 483
  G.1.2 Instructions for `magnetfield` for IDL ............... 486
  G.1.3 Instructions for `magnetoop` for IDL ................. 488
  G.1.4 Instructions for `lisjus` for IDL ..................... 490
  G.1.5 Instructions for `antena` for IDL .................... 492

**H** Answers to Selected Problems ............................... 493

H.0 Chapter 0 .................................................. 493
H.1 Chapter 1 .................................................. 493
H.2 Chapter 2 .................................................. 494
H.3 Chapter 3 .................................................. 494
H.4 Chapter 4 .................................................. 495
H.5 Chapter 5 .................................................. 497
H.6 Chapter 6 .................................................. 497
H.7 Chapter 7 .................................................. 498
H.8 Chapter 8 .................................................. 499
H.9 Chapter 9 .................................................. 501
H.10 Chapter 10 ............................................... 501
H.11 Chapter 11 ............................................... 502
H.12 Chapter 12 ............................................... 503
H.13 Chapter 13 ............................................... 504
H.14 Chapter 14 ............................................... 505
H.15 Chapter 15 ............................................... 507
H.16 Appendix A ............................................... 507
H.17 Appendix B ............................................... 508
H.18 Appendix D ............................................... 508

Index ........................................................................ 509

Brief Index to Frequently Used Information .................... 530
Chapter 0

Mathematical and Physical Preliminaries

0.1 Scalars, Vectors, and Vector Algebra

Nearly every important physical quantity can be classified as a scalar, as a vector, or as a still more complicated entity called a tensor. For most of this book, we shall identify as a scalar any quantity (e.g., temperature, mass, density) that is completely determined by a magnitude alone, we shall identify as a vector any quantity (e.g., force, displacement, velocity) that is completely determined only if both a magnitude and a direction are specified, and we shall not identify tensors. We shall adopt these common definitions, however, only after recognizing that they fail to mention the following important property: True scalars, vectors, and tensors represent quantities (e.g., displacements) having a physical significance (e.g., direction and magnitude in space) that must remain unchanged even if a particular coordinate system used to represent these quantities by numbers is changed. That is, the physical content of a scalar, vector, or tensor must be invariant to such coordinate transformations as translations, rotations, and reflections. Consequently, the numbers used to express scalars, vectors, and tensors must exhibit very specific and determinable behaviors under coordinate transformations. In this book, however, we shall explore this behavior only incompletely, particularly in P0.34 and in Chapter 15.

We shall adopt the usual notational conventions. Scalars will be denoted by lightface letters, vectors by boldface letters, and the magnitude of a vector $\mathbf{A}$ by $|\mathbf{A}|$ or $\mathbf{A}$. In figures the vector $\mathbf{A}$ will be represented by an arrow whose length is proportional to the magnitude of $\mathbf{A}$ and whose direction coincides with that of $\mathbf{A}$.

The algebraic properties of scalars and vectors are assigned so that scalars and vectors combine mathematically in the same way that quantities represented by scalars and vectors combine physically. Without attempting any further motivation for our choice, we shall take scalars to be manipulated by the rules of ordinary algebra and we shall adopt the following properties for the algebraic manipulation of vectors:

1. Equality. Two vectors $\mathbf{A}$ and $\mathbf{B}$ are defined to be equal, $\mathbf{A} = \mathbf{B}$, if and only if they have the same magnitude and direction. This definition does not require the two vectors to originate at the same point.
2. Addition. A vector \( C \) is defined to be the sum of two other vectors \( A \) and \( B \), \( C = A + B \), if and only if \( A, B, \) and \( C \) are related geometrically as the sides of a triangle (Fig. 0.2). If the vectors represent displacements, the sum \( C \) is evidently the single displacement representing the net effect of displacement \( A \) followed by displacement \( B \).

3. Vector addition is commutative, \( A + B = B + A \). That is, the order in which the vectors are added does not affect the sum.

4. Vector addition is associative, \( A + (B + C) = (A + B) + C \). That is, the manner in which grouping is done to evaluate the sum of three (or more) vectors does not affect the final result.

5. Multiplication by a scalar. A vector \( B \) is defined to be equal to the vector \( A \) times the scalar \( s \), \( B = sA \), if \( |B| = |s||A| \) and the direction of \( B \) is that of \( A \) (\( s > 0 \)) or opposite to that of \( A \) (\( s < 0 \)). In particular, if \( s = -1 \), \( B \) has the same magnitude as \( A \) but the opposite direction. Finally, if \( s = 1/|A| \), then \( |B| = 1 \) and \( B \) is called a unit vector; it has the direction of \( A \) and is commonly denoted by \( \hat{A} \). Thus,

\[
\hat{A} = \frac{A}{|A|} = \frac{A}{A} \quad \text{or} \quad A = A\hat{A} \tag{0.1}
\]

6. Multiplication of a vector by a scalar is distributive, both with respect to addition of scalars, \( (s + t)A = sA + tA \), and with respect to addition of vectors, \( s(A + B) = sA + sB \).

7. Subtraction. A vector \( C \) is defined to be the difference between two vectors \( A \) and \( B \), \( C = A - B \), if \( C = A + (-B) \). Note that, if the vectors \( A \) and \( B \) are placed tail to tail, the vector \( A - B \) is a vector from the head of \( B \) (the second vector) to the head of \( A \) (the first vector).

8. Dot multiplication of vectors. A scalar \( s \) is defined to be the dot product of two vectors \( A \) and \( B \), \( s = A \cdot B \), if \( s = |A||B|\cos \theta \), where \( \theta \) is the angle between \( A \) and \( B \) (Fig. 0.1).\(^1\) The dot product is commutative, \( A \cdot B = B \cdot A \); is distributive

\(^1\) The angle \( \theta \) is measured in the plane defined by \( A \) and \( B \). Because \( \cos \theta \) is an even function of \( \theta \) and because \( \cos(2\pi - \theta) = \cos \theta \), however, \( A \cdot B \) has the same value whether \( \theta \) is measured counterclockwise or clockwise from \( A \) to \( B \) or counterclockwise or clockwise from \( B \) to \( A \).
with respect to vector addition, \( \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \); and is zero if \( \mathbf{A} \) and \( \mathbf{B} \) are nonzero and perpendicular. Geometrically, \( \mathbf{A} \cdot \mathbf{B} \) can be interpreted as \( |\mathbf{A}| \) times the projection of \( \mathbf{B} \) on \( \mathbf{A} \) or as \( |\mathbf{B}| \) times the projection of \( \mathbf{A} \) on \( \mathbf{B} \). Finally, the dot product of a vector with itself is the square of the magnitude of the vector \( \mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2 \).

9. Cross multiplication of vectors. A vector \( \mathbf{V} \) is defined to be the cross product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \), \( \mathbf{V} = \mathbf{A} \times \mathbf{B} \), if \( \mathbf{V} = |\mathbf{A}||\mathbf{B}| \sin \theta \, \hat{n} \), where \( \theta \) is the angle measured in the plane of \( \mathbf{A} \) and \( \mathbf{B} \) from \( \mathbf{A} \)—the first-named vector—to \( \mathbf{B} \)—the second-named vector—and \( \hat{n} \) is a unit vector in the direction determined by the thumb of the right hand when the fingers are extended along \( \mathbf{A} \) and the palm faces \( \mathbf{B} \) through the angle \( \theta \) (the right-hand rule) (Fig. 0.3).\(^2\) The cross product is anti-commutative, \( \mathbf{A} \times \mathbf{B} = - (\mathbf{B} \times \mathbf{A}) \); is distributive with respect to vector addition, \( \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \); and is zero if \( \mathbf{A} \) and \( \mathbf{B} \) are nonzero and parallel. Geometrically, \( |\mathbf{A} \times \mathbf{B}| \) is the area of the parallelogram whose adjacent edges are \( \mathbf{A} \) and \( \mathbf{B} \) placed tail to tail.

10. Multiple vector products can be evaluated by successive application of the definitions in (8) and (9). In particular, the triple scalar product \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \) and the two (unequal) triple vector products \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \) and \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \), occur frequently, and the identities

\[
\begin{align*}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\
(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}
\end{align*}
\]

(0.2)

(0.3)

are extremely useful.\(^3\)

\(^2\)Because \( \sin(2\pi - \theta) = -\sin \theta \), it makes no difference whether one measures \( \theta \) clockwise or counterclockwise from \( \mathbf{A} \) to \( \mathbf{B} \).

\(^3\)The author finds the following mnemonic helpful: Write down the product of the second vector with the dot product of the first and third vectors; then subtract the quantity obtained from this first term by exchanging the role of the two vectors occurring in parentheses in the original triple product. This mnemonic sounds cumbersome, but it has the particular advantage of working equally well for both forms of the triple vector product.
Figure 0.4: Resolution of an arbitrary vector $\mathbf{A}$ into components.

PROBLEMS

P0.1. Prove geometrically that (a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, (b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$, and (c) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$.

P0.2. Interpret $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ geometrically.

P0.3. Combine Eq. (0.2) with the anticommutativity of the cross product to derive Eq. (0.3). Hint: Rename the vectors.

P0.4. Let $\mathbf{A}$ be a known vector and $\mathbf{X}$ an unknown vector. (a) Describe geometrically the freedom remaining in $\mathbf{X}$ if $\mathbf{A} \cdot \mathbf{X} = s$ is fixed. (b) Describe geometrically the freedom remaining in $\mathbf{X}$ if $\mathbf{A} \times \mathbf{X} = c$ is fixed. (c) Let both $s$ and $c$ be given. Show that $\mathbf{X} = (s \mathbf{A} + c \times \mathbf{A})/A^2$.

Analytically, vectors are often conveniently represented by their components in some coordinate system. The component $A_n$ of a vector $\mathbf{A}$ in the direction specified by a unit vector $\hat{n}$ is defined by

$$A_n = \mathbf{A} \cdot \hat{n} = A \cos \theta \quad (0.4)$$

where $\theta$ is the angle between $\mathbf{A}$ and $\hat{n}$; equivalently, $A_n$ is the projection of $\mathbf{A}$ in the direction $\hat{n}$. If at some point in space we introduce a right-handed triad of mutually orthogonal unit vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, then the vector $\mathbf{A}$ has the three components $A_i = \mathbf{A} \cdot \hat{e}_i$, $i = 1, 2, 3$, and can itself be written as the sum of three vectors

$$\mathbf{A} = \sum_{i=1}^{3} A_i \hat{e}_i = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \quad (0.5)$$

as illustrated geometrically in Fig. 0.4. All of the definitions and theorems summarized in the previous paragraph have corresponding expressions in terms of components. Equality

---

4Right-handed triads are those in which $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$ and are almost universally preferred to left-handed triads, in which $\hat{e}_3 = -\hat{e}_1 \times \hat{e}_2$. 
of two vectors, for example, implies separate equality of the corresponding components,

\[ \mathbf{A} = \mathbf{B} \implies A_1 = B_1, \quad A_2 = B_2, \quad A_3 = B_3 \]  

(0.6)

A single vector equation is therefore equivalent to three separate scalar equations. Likewise,

\[ \mathbf{C} = \mathbf{A} \pm \mathbf{B} \implies C_1 = A_1 \pm B_1, \quad C_2 = A_2 \pm B_2, \quad C_3 = A_3 \pm B_3 \]  

(0.7)

\[ \mathbf{B} = s\mathbf{A} \implies B_1 = sA_1, \quad B_2 = sA_2, \quad B_3 = sA_3 \]  

(0.8)

and the commutative, associative, and distributive properties of vector algebra therefore reduce to a threefold application of the same properties of scalar algebra. Finally, in terms of components, one can show that

\[ \mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 \]  

(0.9)

\[ |\mathbf{A}|^2 = (A_1)^2 + (A_2)^2 + (A_3)^2 = \mathbf{A} \cdot \mathbf{A} \]  

(0.10)

\[ \mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\hat{\mathbf{e}}_1 + (A_3B_1 - A_1B_3)\hat{\mathbf{e}}_2 + (A_1B_2 - A_2B_1)\hat{\mathbf{e}}_3 \]  

(0.11)

\[ = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \]  

(0.12)

In Eq. (0.12), the vertical bars denote a determinant. (See P0.6 and Section A.1.)

one must, of course, be familiar enough with the dot and cross products to be able to evaluate simple ones easily by hand, symbolic manipulating computer programs can also evaluate these products. In Mathematica, the function \texttt{Cross} will produce the dot product of two three-component lists while the function \texttt{Dot} or, equivalently, the operator \texttt{.}' will produce the dot product of two three-component lists. For example,

```
In[1]:= R = {R1, R2, R3}; ! Create list R.
In[2]:= Q = {Q1, Q2, Q3}; ! Create list Q.
In[3]:= Cross[ R, Q ] ! Returns R × Q.
In[4]:= R.Q ! Returns R • Q.
In[5]:= Dot[R,Q] ! Returns R • Q.
```

will evaluate \texttt{R × Q} and \texttt{R • Q}.

---

**PROBLEMS**

**P0.5.** A vector \( \mathbf{A} \) is often usefully separated into a longitudinal part, \( A_{||} \), parallel to a given unit vector \( \mathbf{\hat{n}} \) and a transverse part, \( A_\perp \), perpendicular to \( \mathbf{\hat{n}} \), where \( \mathbf{A} = A_{||} + A_\perp \). (a) Show that \( A_{||} = (\mathbf{A} \cdot \mathbf{\hat{n}})\mathbf{\hat{n}} \) and \( A_\perp = \mathbf{\hat{n}} \times (\mathbf{A} \times \mathbf{\hat{n}}) \). (b) Show that \( \mathbf{\hat{n}} \times (\mathbf{A} \times \mathbf{\hat{n}}) = 0 \) implies that \( \mathbf{A} \times \mathbf{\hat{n}} = 0 \).

---

5The symbol \( \implies \) should be read “implies” or “implies that”.

P0.6. (a) Construct a table giving the nine possible dot products between two unit vectors chosen from \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \). (b) Construct a similar table giving the nine possible cross products between two unit vectors chosen from \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \). (c) Using these products and the algebraic properties of vectors, manipulate the expression

\[
(A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3) \otimes (B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3)
\]

where \( \otimes \) represents either a dot or a cross, and derive Eqs. (0.9) and (0.11).

P0.7. (a) Show by hand that

\[
A \cdot (B \times C) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}
\]

and prove that the dot and the cross can be interchanged, i.e., that \( A \cdot (B \times C) = (A \times B) \cdot C \). (b) Prove, again by hand, that

\[
(A \times B) \cdot (C \times D) = \begin{vmatrix} A \cdot C & A \cdot D \\ B \cdot C & B \cdot D \end{vmatrix}
\]

and then set \( A = C \) and \( B = D \) to show that \( \sin^2 \theta + \cos^2 \theta = 1 \), where \( \theta \) is the (arbitrary) angle between \( A \) and \( B \).

P0.8. Let \( A = 3\hat{e}_1 + 2\hat{e}_2 - 6\hat{e}_3 \) and \( B = \hat{e}_1 + \hat{e}_3 \). Find \( \hat{A}, \ A \cdot B, \ A \times B, \) and the angle between \( A \) and \( B \). First, do this problem by hand but then evaluate each quantity using an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica as well.

P0.9. Verify Eq. (0.2) by expressing all vectors in terms of their components and showing that both sides of the equation reduce to the same vector. \textit{Hints:} (1) Choose the basic unit vectors wisely. For example, choose \( \hat{e}_3 \) parallel to \( C \) and choose \( \hat{e}_2 \) in the plane of \( B \) and \( C \). (2) Accept the theorem that, by virtue of their transformation properties, vector identities that are valid in one coordinate system are necessarily valid in any coordinate system obtained by arbitrary translation and/or rotation of the first system.

P0.10. Using the resources of an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica, create four three-component vectors \( A, B, C, \) and \( D \), show that

a. \( A \times B = -B \times A \), or, equivalently, that \( A \times B + B \times A = 0 \).

b. \( A \times (B \times C) = (A \cdot C)B - (A \cdot B)C \)

c. \( (A \times B) \times C = (A \cdot C)B - (B \cdot C)A \)

d. \( A \cdot (B \times C) = (A \times B) \cdot C \)

e. \( (A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C) \)

P0.11. Let two points have position vectors \( r_1 \) and \( r_2 \), respectively. Obtain an expression for the vector pointing \textit{from} point 1 \textit{to} point 2.

When specific components of a vector are needed, they will almost always be expressed in one of three commonly occurring coordinate systems. In \textit{Cartesian coordinates}, a point in space is located by the three numbers \((x, y, z)\) indicated geometrically in Fig. 0.5(a) and having the range \(-\infty < x, y, z < \infty \); the mutually orthogonal unit vectors \( \hat{i}, \hat{j}, \) and \( \hat{k} \) at any point have, respectively, the direction of increasing \( x, y, \) and \( z \) at that point; and an arbitrary vector \( A \) has the components \( A_x = A \cdot \hat{i}, A_y = A \cdot \hat{j}, \) and \( A_z = A \cdot \hat{k} \), in terms of which

\[
A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad \text{ (Cartesian)}
\]
In *cylindrical coordinates*, a point in space is located by the three numbers \((r, \phi, z)\) indicated geometrically in Fig. 0.5(b) and having the range \(0 \leq r < \infty\), \(0 \leq \phi < 2\pi\), \(-\infty < z < \infty\); the mutually orthogonal unit vectors \(\hat{r}\), \(\hat{\phi}\), and \(\hat{k}\) at any point have, respectively, the direction of increasing \(r\), \(\phi\), and \(z\) at that point; and an arbitrary vector \(\mathbf{A}\) has the components \(A_r = \mathbf{A} \cdot \hat{r}\), \(A_\phi = \mathbf{A} \cdot \hat{\phi}\), and \(A_z = \mathbf{A} \cdot \hat{k}\), in terms of which

\[
\mathbf{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{k} \quad \text{(cylindrical)} \tag{0.14}
\]

\(A_r\), \(A_\phi\), and \(A_z\) are referred to, respectively, as the (cylindrical) radial, azimuthal, and axial components of \(\mathbf{A}\). Finally, in *spherical coordinates*, a point in space is located by the three numbers \((r, \theta, \phi)\) indicated geometrically in Fig. 0.5(c) and having the range \(0 \leq r < \infty\), \(0 \leq \theta < \pi\), \(0 \leq \phi < 2\pi\); the mutually orthogonal unit vectors \(\hat{r}\), \(\hat{\theta}\), and \(\hat{\phi}\) have, respectively, the direction of increasing \(r\), \(\theta\), and \(\phi\) at that point; and an arbitrary vector \(\mathbf{A}\) has the components \(A_r = \mathbf{A} \cdot \hat{r}\), \(A_\theta = \mathbf{A} \cdot \hat{\theta}\), and \(A_\phi = \mathbf{A} \cdot \hat{\phi}\), in terms of which

\[
\mathbf{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi} \quad \text{(spherical)} \tag{0.15}
\]

\(A_r\), \(A_\theta\), and \(A_\phi\) are referred to, respectively, as the (spherical) radial, polar, and azimuthal components of \(\mathbf{A}\), and the \(z\)-axis is often called the polar axis. Several relationships among these three sets of coordinates and unit vectors are summarized in Table 0.1. The following observations are pertinent to the use of vectors expressed in these coordinate systems:

1. Only in the Cartesian system are the unit vectors constant and independent of the point in space to which they apply. In the cylindrical and spherical systems, the unit vectors change direction as the point to which they apply moves about in space.

2. The rules—Eqs. (0.6)–(0.12)—for manipulating vectors expressed in terms of their components followed from the orthogonality and right-handedness of the basic unit

---

6 Though \(r\) or \(\rho\) are commonly used for the cylindrical radial coordinate, I choose to use the symbol \(r\)—script \(r\)—to avoid confusion with \(r\) as used for the spherical radial coordinate and \(\rho\) as used for charge density to be introduced in Chapter 1.

7 Be warned that mathematicians often reverse the roles of \(\theta\) and \(\phi\) as conventionally used by physicists.
vectors. Since the triads \((\hat{i}, \hat{j}, \hat{k}), (\hat{\xi}, \hat{\phi}, \hat{k}), \) and \((\hat{r}, \hat{\theta}, \hat{\phi})\) all exhibit these properties, Eqs. (0.6)–(0.12) apply in particular to the specific expressions in Eqs. (0.13)–(0.15).

(3) The above discussion presupposed no specific choice for the coordinate origin or for the orientation of the \(x, y,\) and \(z\) axes; these features of the coordinate system will in general be selected for convenience in application to the problem at hand.

(4) A number of different notations, particularly for the cylindrical coordinates, can be found in the literature, and suitable care must be exercised in comparing the writings of two or more authors when these coordinates are used.

---

**PROBLEM**

P0.12. Familiarize yourself with the entries in Table 0.1 and verify those in Table 0.1(a). Note: The expressions for \(\partial \hat{r} / \partial \phi, \partial \hat{r} / \partial \theta,\) and \(\partial \hat{\xi} / \partial \phi\) given in Table 0.1 can be conveniently used to obtain \(\hat{\phi}\) and \(\hat{\theta}\) by differentiating \(\hat{r}\) or \(\hat{\xi}\), which themselves can be written in terms of \(\hat{i}, \hat{j},\) and \(\hat{k}\) by inspection.

---

**0.2 The Representation of Fields**

Only a very few physical quantities are absolute constants. Most frequently, the value of a physical quantity (e.g., temperature, force) depends on where in space and when in time the quantity is determined. To denote the totality of values of a physical quantity at all points in some region of space-time, we introduce the word field, examples of which are the (scalar) temperature field in a room, the (vector) force field produced by a massive object, and the (scalar) probability field of a quantum mechanical particle.

For theoretical manipulations, fields are most usefully represented as (scalar or vector) functions of position and time. We introduce a coordinate system—probably one of those described in Section 0.1—and a clock, setting the origin and the orientation of the axes of each for maximum convenience to the problem at hand. A point in space-time is then located by its three spatial coordinates, say \((r, \theta, \phi)\), and its temporal coordinate \(t\) in the (arbitrarily selected) coordinate system, and the dependence of some field \(S\) on space-time is indicated by writing the space-time coordinates as arguments of a function, e.g., \(S(x, y, z, t)\). This dependence may alternatively be indicated by recognizing that the single point \(P\) variously identified by \((x, y, z), (r, \phi, z),\) or \((r, \theta, \phi)\) could equally well be labeled by its position vector \(r\), which is the vector from the origin to the point \(P\). The dependence of \(S\) on space-time might then be indicated by \(S(r, t)\). This more compact notation permits postponing introduction of \(r\) in one of the equivalent forms

\[
   r = x \hat{i} + y \hat{j} + z \hat{k} \quad \text{(Cartesian)}
\]
(a) Relationships among Cartesian and cylindrical coordinates:

\begin{align*}
x &= \tau \cos \phi \\
y &= \tau \sin \phi \\
\hat{\tau} &= \cos \phi \hat{i} + \sin \phi \hat{j} \\
\hat{\phi} &= -\sin \phi \hat{i} + \cos \phi \hat{j}
\end{align*}

Expressions relating the Cartesian and cylindrical components of a vector \( \mathbf{A} \) may be obtained by taking the dot products of \( \mathbf{A} \) with the above relationships among unit vectors, e.g., \( A_\tau = \mathbf{A} \cdot \hat{\tau} = A_x \cos \phi + A_y \sin \phi \).

\begin{align*}
\frac{\partial \hat{\tau}}{\partial \tau} &= 0 \\
\frac{\partial \hat{\phi}}{\partial \phi} &= 0 \\
\frac{\partial \hat{\phi}}{\partial \tau} &= 0 \\
\frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{\tau}
\end{align*}

(b) Relationships among Cartesian and spherical coordinates:

\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \phi
\end{align*}

Expressions relating the Cartesian and spherical components of a vector \( \mathbf{A} \) may be obtained by taking the dot products of \( \mathbf{A} \) with the above relationships among unit vectors, e.g., \( A_r = \mathbf{A} \cdot \hat{r} = A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \phi \).

\begin{align*}
\frac{\partial \hat{r}}{\partial r} &= 0 \\
\frac{\partial \hat{\theta}}{\partial \theta} &= \hat{\phi} \\
\frac{\partial \hat{\phi}}{\partial \phi} &= \sin \theta \hat{\phi}
\end{align*}
or in some other form until explicit selection of a coordinate system is required. It is, of course, not necessary for a field to depend on all four space-time coordinates, the time-independent or static field, denoted by \( S(\mathbf{r}) \), being particularly common.

Among the more important manipulations with scalar and vector fields are differentiation and integration. Differentiation of a scalar field is accomplished by the usual means. In Cartesian coordinates, the simple derivative of a vector field is that vector field whose components are the derivatives of the components of the field itself, e.g.,

\[
Q(\mathbf{r}, t) = Q_x(x, y, z, t) \hat{i} + Q_y(x, y, z, t) \hat{j} + Q_z(x, y, z, t) \hat{k}
\]  

(0.19)

then

\[
\frac{\partial Q}{\partial y} = \frac{\partial Q_x}{\partial y} \hat{i} + \frac{\partial Q_y}{\partial y} \hat{j} + \frac{\partial Q_z}{\partial y} \hat{k}
\]  

(0.20)

If the vector field is expressed in some other coordinate system, say,

\[
Q(\mathbf{r}, t) = Q_r(r, \phi, z, t) \hat{r} + Q_\phi(r, \phi, z, t) \hat{\phi} + Q_z(r, \phi, z, t) \hat{k}
\]  

(0.21)

then differentiation is more complicated, for the unit vectors may not be constant. Each term must be differentiated as a product, using the information in Table 0.1 to evaluate the derivatives of the unit vectors.

Simple integration of a scalar field is accomplished by familiar procedures; in particular, we shall encounter the volume integral, variously denoted by

\[
\int S(x, y, z) \, dx \, dy \, dz ; \quad \int S(\mathbf{r}) \, dv ; \quad \int S(\mathbf{r}) \, d^3 \mathbf{r}
\]

where the volume element has the expressions

\[
dv = dx \, dy \, dz \quad \text{(Cartesian)} \quad (0.22)
\]

\[
dv = r \, dr \, d\phi \, dz \quad \text{(cylindrical)} \quad (0.23)
\]

\[
dv = r^2 \sin \theta \, dr \, d\theta \, d\phi \quad \text{(spherical)} \quad (0.24)
\]

in the three common coordinate systems. In Cartesian coordinates, simple integration of a vector field produces another vector field whose components are the integrals of the components of the field being integrated. Caution must again be exercised in integrating vector fields expressed in other coordinate systems. More involved derivatives, particularly the gradient, the curl, and the divergence, and more involved integrals, particularly line integrals and surface integrals, will be introduced as our development proceeds.

**PROBLEMS**

**P0.13.** A scalar function of position is given in terms of the position vector \( \mathbf{r} \) by \( S(\mathbf{r}) = \mathbf{r} \cdot \mathbf{r} \). Write this function out more explicitly in Cartesian, cylindrical, and spherical coordinates.

**P0.14.** A scalar function \( S \) and a vector function \( \mathbf{Q} \) are given in terms of two position vectors \( \mathbf{r} \) and \( \mathbf{r}' \) by

\[
S(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad \mathbf{Q}(\mathbf{r}, \mathbf{r'}) = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}
\]
Letting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$, write $S$ and $Q$ explicitly in terms of $x$, $y$, $z$, $x'$, $y'$, and $z'$ and then show that:

$$Q = -\frac{\partial S}{\partial x}\mathbf{i} - \frac{\partial S}{\partial y}\mathbf{j} - \frac{\partial S}{\partial z}\mathbf{k}$$

Do this problem by hand first but then try it with an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica.

Although analytic representations of fields are essential to efficient theoretical manipulations, graphical representations often facilitate mental visualization of the fields. A graphical representation for a scalar field need convey only the magnitude of the field, and a diagram showing labeled level contours—lines or surfaces at all points of which the scalar field has the labeled value—suffices; the contour lines representing a particular (two-dimensional) scalar field are shown in Fig. 0.6. A graphical representation for a vector field, on the other hand, must convey both the magnitude and the direction of the field. One way to convey these features is to attach a vector to each of several representative points in space, drawing the vector at each point so that its direction coincides with the direction of the field and its length is proportional to the magnitude of the field; Fig. 0.7(a) shows a sample vector field represented in this way. Alternatively, a vector field can be mapped by drawing a set of field lines—lines that are everywhere tangent to the field vector. Let these field lines be spaced so that the number of lines crossing a unit area placed perpendicular to the field at some point is proportional to the magnitude of the field at that point. The resulting picture then conveys the direction of the field lines and the magnitude of the field by the spacing of the field lines. Adherence to this convention on spacing may, of course, require that field lines start and stop at some points in space. The field shown in Fig. 0.7(a) may also be represented by the field lines shown in Fig. 0.7(b). We offer two warnings: (1) Graphical representations in two dimensions are not entirely satisfactory for conveying the details of fields in three dimensions, and (2) graphical representations show the fields only at representative points in space; the fields themselves have values at all points in space.

PROBLEMS

P0.15. Explain why the level contours representing a scalar field cannot intersect.

P0.16. Describe the (three-dimensional) level contours of a scalar field given as a function of the position vector $\mathbf{r}$ by $S(\mathbf{r}) = 1/|\mathbf{r} - \mathbf{r}_0|$, where $\mathbf{r}_0$ is a fixed vector.

P0.17. A scalar field is given as a function of position $\mathbf{r}$ by $S(\mathbf{r}) = (\mathbf{k} \cdot \mathbf{r})/r^3$. (a) Describe the three-dimensional contour diagram. Consider both $S > 0$ and $S < 0$. Hint: Think in spherical coordinates. (b) Use IDL’s `lugen_grid` and `contour` commands, MATLAB’s `meshgrid` and `contour` commands, OCTAVE’s `meshgrid` and `contour` commands, or PYTHON’s `meshgrid` command in the `numpy` module and `contour` command in the `matplotlib.pyplot` module will produce these diagrams for scalar fields in two dimensions. (The file `lugen_grid.pro` may be installed in your computational environment but is also listed in Appendix F; that listing includes instructions for its use. If needed, the source file can be copied to your default directory from `$HEADEM/idl`, where the translation of `$HEADEM` for your site is provided in your Local Guide.)

P0.18. IDL’s `vel` and `velovect` commands, MATLAB’s `quiver` command, OCTAVE’s `quiver` command, and PYTHON’s `quiver` command in the `matplotlib.pyplot` module can be used to produce diagrams of this sort for vector fields in two dimensions.
Figure 0.6: Representation of a scalar field by contour lines. Physically, the field illustrated represents the altitude of points on the surface of a hypothetical earth and each contour line is labeled with the altitude (in feet) of points along that line. Wherever the contour lines are close together, the terrain is steep; wherever they are widely separated, the terrain slopes only gradually. In this figure, a mountain occurs near the upper right corner, a valley runs from center left to center top, and this same valley opens into a reasonably flat area in the lower right corner.

Figure 0.7: A vector field in (a) the vector representation and (b) the field line representation. In (a) the vectors indicate both the magnitude and the direction of the field; in (b) the lines indicate the direction of the field and their separation indicates its magnitude.

PYTHON’s *meshgrid* command in the *numpy* module and *contour* command in the *matplotlib.pyplot* module to obtain accurate drawings of half a dozen of the level contours of this field in the *yz*-plane. Probably here Cartesian coordinates will be better, and you will certainly have to invent some ruse for dealing with the divergence at the origin. *Hint:* The IDL command ?contour, the MATLAB commands *help meshdom* and *help contour*, the OCTAVE commands *help meshdom* and *help contour*, and, if *matplotlib.pyplot* has been imported as *plt* and *numpy* has been imported as *np*, the PYTHON commands *help(plt.contour)* and *help(np.meshgrid)* will provide information about the use of these commands.

**P0.18.** Show that the field lines representing the vector field \( \mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{r^3} \) are straight lines radiating away from the origin and then show that, in three dimensions, these field lines need be started only at the origin in order that \( |\mathbf{F}| \) at every point \( P \) be proportional to the number of lines crossing a unit area oriented perpendicular to \( \mathbf{F} \) at \( P \).

**P0.19.** Use IDL’s *vel* or *velovect* commands, MATLAB’s *quiver* command, OCTAVE’s *quiver* command, or PYTHON’s *quiver* command in the *matplotlib.pyplot* module to obtain
0.3 Static Force Fields

In this section we shall summarize a number of concepts associated with static force fields. Throughout this section \( \mathbf{F}(\mathbf{r}) \) denotes a force—it may be the total force or only a portion of the total force—experienced by a particle located at point \( \mathbf{r} \).

0.3.1 Equations of Motion

The most important role played by a force field lies in its control of the dynamics of the motion of a particle of mass \( m \) moving under its influence. Newton’s second law,\(^{11}\)

\[
\mathbf{F} = m \mathbf{a} \implies m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r})
\]

provides the starting point for predicting that behavior. With mass measured in kilograms (kg) and acceleration measured in meters/second\(^2\) (m/s\(^2\)), force is measured in kg·m/s\(^2\), a combination called the newton (N).

0.3.2 Work and the Line Integral

We ask next about the work done on a particle by the force field \( \mathbf{F}(\mathbf{r}) \) as the particle is moved from a point \( \mathbf{r}_a \) to a point \( \mathbf{r}_b \) along some specified path \( \Gamma \).\(^{12,13}\) Since in general the force will not have the same value at all points on the path, we seek an answer to this question by dividing the path into small approximately straight segments along any one of which the force may be regarded as approximately constant. If the center of the \( i \)-th segment is located at \( \mathbf{r}_i \), then the force experienced by the particle at all points on this segment is approximately \( \mathbf{F}(\mathbf{r}_i) \). Now, by definition the work \( \Delta W_i \) done by a constant force moved through a displacement along a straight line is the product of the displacement and the component of the force in the direction of the displacement,

\[
\Delta W_i \approx |\mathbf{F}(\mathbf{r}_i)| |\Delta \mathbf{r}_i| \cos \theta_i = \mathbf{F}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i
\]

where \( \Delta \mathbf{r}_i \) is the net displacement represented by the \( i \)-th segment and \( \theta_i \) is the angle between \( \mathbf{F}(\mathbf{r}_i) \) and \( \Delta \mathbf{r}_i \) (Fig. 0.8). Summing contributions from all segments gives

---

\(^{11}\) English physicist and mathematician Isaac Newton, b. 25 December 1642 in Woolsthorpe, Lincolnshire, England; d. 20 March 1726/7 in Kensington, Middlesex, England.

\(^{12}\) To bring about this motion, there will in general be forces acting on the particle in addition to those exerted by the specified force field.

\(^{13}\) Take particular note of the implicit sign convention in the statement that we are evaluating the work done on a particle by a force.
Figure 0.8: The path of a particle moving in a force field.

\[ W \approx \sum_i \Delta W_i = \sum_i \mathbf{F}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i \]  

(0.27)

as an estimate of the work done by \( \mathbf{F}(\mathbf{r}) \) as the particle moves over the entire path. We now refine the division of the path, making all segments smaller (and simultaneously increasing their number). As this refinement continues, the approximation becomes better, and in the limit, as \( |\Delta \mathbf{r}_i| \to 0 \) for all \( i \) and the number of segments approaches infinity, the expression becomes exact, i.e.,

\[ W = \lim_{|\Delta \mathbf{r}_i| \to 0} \sum_i \mathbf{F}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i = \int_{\Gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \]  

(0.28)

Equation (0.28) both defines the line integral and identifies the line integral of a force along some path with the work done by the force on a particle that moves along the path. In general, if a more explicit (analytic) evaluation of a line integral is needed, the integral must be written out in more detail, probably using one of the expressions

\[ d\mathbf{r} = dx \hat{i} + dy \hat{j} + dz \hat{k} \]  

(Cartesian)  

(0.29)

\[ d\mathbf{r} = dx \hat{\xi} + r d\phi \hat{\phi} + dz \hat{k} \]  

(cylindrical)  

(0.30)

\[ d\mathbf{r} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \]  

(spherical)  

(0.31)

for the infinitesimal displacement \( d\mathbf{r} \). In Cartesian coordinates, for example, we have that

\[ W = \int_{\Gamma} [F_x(x, y, z) \, dx + F_y(x, y, z) \, dy + F_z(x, y, z) \, dz] \]  

(0.32)

but in general we can proceed no further until the path has been specified, perhaps by stipulating relationships that determine both \( y \) and \( z \) along the path as functions of \( x \). Once such relationships are available, we may reduce the line integral to an integral on a single variable and evaluate it by familiar techniques.

In consequence of its definition as a force times a distance, work is measured in the mks system in newton-m (N·m), a unit called the joule (J).\(^{14,15}\)

\(^{14}\)James Prescott Joule, see footnote 11 in Chapter 4, page 108.

\(^{15}\)Note that, in a cgs system, where the unit of force is the dyne (dyn = gm·cm/s\(^2\)), the dyn·cm as the unit of work is called the erg.
0.3.3 The Work-Kinetic Energy Theorem

If the force \( \mathbf{F}(\mathbf{r}) \) in Eq. (0.28) is the resultant of all forces experienced by a particle of mass \( m \) at point \( \mathbf{r} \), then by Newton’s second law

\[
\mathbf{F}(\mathbf{r}) = m \frac{d^2 \mathbf{r}}{dt^2} = m \frac{d\mathbf{v}}{dt}
\]

and we find that

\[
W = \int_{\Gamma} m \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} = m \int \mathbf{v} \cdot d\mathbf{v} = \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2
\]

where \( v_a \) (\( v_b \)) is the velocity of the particle at the start (end) of the path. This result, called the work-kinetic energy theorem, equates the work done by the resultant of all forces acting on a particle to the change in kinetic energy of the particle.

As with work, kinetic energy is measured in joules in the mks system and ergs in a cgs system of units.

0.3.4 Conservative Force Fields and Potential Energy

Although the line integral of the most general force field cannot be evaluated until after a path has been specified, many commonly occurring force fields permit an evaluation of \( \int \mathbf{F} \cdot d\mathbf{r} \) if only the end points of the path are known. For these fields, the particular route by which the particle moves from start to finish does not affect the total work done on it by the force field. The line integral of these special fields is said to be path-independent and the fields themselves are said to be conservative. Symbolically, we can indicate the dependence of the line integral only on the initial and final positions \( \mathbf{r}_a \) and \( \mathbf{r}_b \) by writing

\[
\int_{\Gamma_c} \mathbf{F}_c \cdot d\mathbf{r} = \int_{\Gamma} \mathbf{F}_c \cdot d\mathbf{r} = g(\mathbf{r}_b, \mathbf{r}_a)
\]

where the subscript \( c \) has been added to \( \mathbf{F} \) as a reminder that the expression applies only to conservative fields. Although the function \( g(\mathbf{r}_b, \mathbf{r}_a) \) is in all cases a scalar, its specific form is determined by the particular force field involved. Equivalently (see P0.21), a conservative force field satisfies

\[
\oint_{\Gamma_c} \mathbf{F}_c \cdot d\mathbf{r} = 0
\]

where \( \Gamma_c \) is an arbitrary closed path and the circle on the integral sign signifies that \( \Gamma_c \) is a closed path starting from \( \mathbf{r}_a \) and proceeding via some route back to \( \mathbf{r}_a \). Thus, a conservative force field may be characterized either by the path independence of the corresponding line integral or by the vanishing of the line integral around an arbitrary closed path.

Conservative force fields (and only conservative force fields) admit the identification of an associated scalar potential energy function, the existence of which can be demonstrated by evaluating the work done on a particle by a conservative force field as the particle moves from \( \mathbf{r}_0 \) to \( \mathbf{r}_a \) and on to \( \mathbf{r}_b \). We can think of the journey as a single step, in which case the work done on the particle is \( g(\mathbf{r}_b, \mathbf{r}_0) \), or as a sequence of two steps, in which case the work done on the particle is \( g(\mathbf{r}_a, \mathbf{r}_0) + g(\mathbf{r}_b, \mathbf{r}_a) \). Since the total work must be independent
of how we view the path, we conclude that these two evaluations must be equal or, with a minor algebraic rearrangement, that

\[ g(r_a, r_b) = g(r_a, r_0) - g(r_a, r_0) \]  

(0.37)

The line integral of a conservative force field between any two points is therefore given by the difference between a function evaluated at one point and the same function evaluated at the other point. To be sure, the function \( g(r, r_0) \) depends on two points, but the point \( r_0 \) is the same in both occurrences; it merely plays the role of a reference point. Relative to the point \( r_0 \), \( g(r, r_0) \) assigns a value to some (scalar) quantity at each point \( r \) in space. This scalar field is intimately associated with the vector field \( F_c(r) \), and it represents physically the work done by \( F_c(r) \) on a particle moved from \( r_b \) to \( r \). It is customary to suppress explicit indication of the reference point and (for reasons that will soon appear) to introduce a minus sign, defining the so-called potential energy field \( U(r) \) by

\[ U(r) = -g(r, r_0) \]  

(0.38)

in terms of which Eq. (0.37) yields the expression

\[ \int_{r_a}^{r_b} F_c \cdot dr = -[U(r_b) - U(r_a)] \]  

(0.39)

for the work done by a conservative force on a particle moved from \( r_a \) to \( r_b \). If \( F_c \) represents the resultant of all forces acting on the particle, Eq. (0.39) combines with Eq. (0.34) to give

\[ \frac{1}{2}m v_a^2 + U(r_a) = \frac{1}{2}m v_b^2 + U(r_b) \]  

(0.40)

which is recognizable as a statement of conservation of mechanical energy. [Had the minus sign not been inserted in Eq. (0.38), it would have appeared preceding each \( U \) in Eq. (0.40).]

The argument of the previous paragraph demonstrates only the existence of a (scalar) potential energy field associated with every (vector) conservative force field. An expression determining the scalar field from the vector field is obtained by setting \( r_b = r \) and \( r_a = r_0 \) in Eq. (0.39) and then rearranging the equation; we find that

\[ U(r) = U(r_0) - \int_{r_0}^{r} F_c \cdot dr \]  

(0.41)

The value of \( U(r_0) \) remains arbitrary; only differences in potential energy are determined by the force field. One way to understand the minus sign in this expression is to recognize that potential energy differences relate to the work that an external agent must do to move the particle from \( r_0 \) to \( r \). In Eq. (0.41), however, the force \( F_c \) is the force exerted on the particle by the force field. To move the particle against this force, the agent must exert a force that is infinitesimally larger than \( F_c \) in magnitude but opposite in direction. The change in potential energy, defined as the work done by the agent, is therefore the work done by the force \( F_{\text{agent}} \), which is \(-F_c\).

Equation (0.41) provides a means for calculating \( U(r) \) when \( F_c(r) \) is known. A solution to the reverse problem—finding \( F_c(r) \) when \( U(r) \) is known—can also be obtained. Since \( U(r) \) is an integral of \( F_c(r) \), one might expect \( F_c(r) \) to be a derivative of \( U(r) \). To see the nature of that derivative, we evaluate the difference in potential energy between two points \( r_a \) and \( r_b = r_a + \Delta r \), where \(|\Delta r|\) is small. Equation (0.39) then gives

\[ U(r_a + \Delta r) - U(r_a) = - \int_{r_a}^{r_a+\Delta r} F_c \cdot dr \]  

(0.42)
To obtain $F_{cx}(r)$, let $\Delta r = \Delta x \hat{i}$. Since the integral is path-independent, we can select a path from $r_a$ to $r_a + \Delta r$ along which $d\mathbf{r} = dx' \hat{i}$, and Eq. (0.42) then gives

$$U(x_a + \Delta x, y_a, z_a) - U(x_a, y_a, z_a) \approx -F_{cx}(x_a, y_a, z_a) \Delta x$$

(0.43)

Dividing Eq. (0.43) by $\Delta x$ and then allowing $\Delta x \to 0$, we find that

$$F_{cx}(x_a, y_a, z_a) = -\frac{\partial U}{\partial x} \bigg|_{x_a,y_a,z_a}$$

(0.44)

Since this expression applies at any point $r_a$, the subscript $a$ can be dropped, and we have finally that,

$$F_{cx}(r) = -\frac{\partial U(r)}{\partial x}$$

(0.45)

Similar expressions for $F_{cy}(r)$ and $F_{cz}(r)$ are obtained by setting $\Delta r = \Delta y \hat{j}$ and $\Delta r = \Delta z \hat{k}$ in Eq. (0.42). In total we find that

$$\mathbf{F}_c = -\left( \hat{i} \frac{\partial U}{\partial x} + \hat{j} \frac{\partial U}{\partial y} + \hat{k} \frac{\partial U}{\partial z} \right)$$

(0.46)

where all functions are understood to be evaluated at argument $r$. Let us now introduce the vector differential operator known as the gradient operator, symbolized by $\nabla$, and represented in Cartesian coordinates by

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

(0.47)

(The operator $\nabla$ is often read ‘del’.) Acting on a scalar field $S(r)$, this operator produces a vector field known as the gradient of $S$ and symbolized by $\nabla S$; more explicitly,

$$\nabla S = \hat{i} \frac{\partial S}{\partial x} + \hat{j} \frac{\partial S}{\partial y} + \hat{k} \frac{\partial S}{\partial z}$$

(0.48)

A physical interpretation of this vector derivative will be explored briefly in Section 0.4; the derivative is introduced here because Eq. (0.46) can then be written in the particularly compact form

$$\mathbf{F}_c = -\nabla U$$

(0.49)

The force exerted on a particle by a conservative force field is therefore given by the negative gradient of the associated potential energy field.

Several equivalent criteria for identifying a conservative force field can now be enumerated. We have already noted (1) that the line integral of a conservative force field is path-independent, (2) that this line integral about any closed path vanishes, and (3) that

$$U(x_a + \Delta x, y_a, z_a) - U(x_a, y_a, z_a) = F_{cx}(x_a + \eta \Delta x, y_a, z_a) \Delta x$$

where $0 \leq \eta \leq 1$, before letting $\Delta x$ approach zero.
a conservative force field can be obtained as the (negative) gradient of an associated scalar (potential energy) field. To obtain a fourth criterion, we use Eq. (0.49), finding that

\[ \mathbf{F}_c \cdot d\mathbf{r} = -\nabla U \cdot d\mathbf{r} \]

\[ = - \left( \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} + \frac{\partial U}{\partial z} \mathbf{k} \right) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) \]

\[ = - \left( \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right) = -dU \]  

(0.50)

Thus, for a conservative force field, \( \mathbf{F}_c \cdot d\mathbf{r} \) is an exact differential. Mathematically, however, exactness of

\[ \mathbf{F}_c \cdot d\mathbf{r} = F_{cx} dx + F_{cy} dy + F_{cz} dz \]  

(0.51)

requires that\(^\text{17}\)

\[ \frac{\partial F_{cz}}{\partial y} - \frac{\partial F_{cy}}{\partial z} = 0, \quad \frac{\partial F_{cx}}{\partial z} - \frac{\partial F_{cz}}{\partial x} = 0, \quad \frac{\partial F_{cy}}{\partial x} - \frac{\partial F_{cx}}{\partial y} = 0 \]  

(0.52)

Let us now define the curl of a vector field \( \mathbf{Q} \) by

\[ \nabla \times \mathbf{Q} = \left( \frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z} \right) \mathbf{i} + \left( \frac{\partial Q_x}{\partial z} - \frac{\partial Q_z}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q_y}{\partial x} - \frac{\partial Q_x}{\partial y} \right) \mathbf{k} \]  

(0.53)

where the notation is suggested by the result of a formal evaluation of \( \nabla \times \mathbf{Q} \) when \( \nabla \) is replaced by the expression in Eq. (0.47). A physical interpretation of this vector derivative is explored briefly in Section 0.4; the derivative is introduced here because the condition in Eq. (0.52) for the exactness of \( \mathbf{F}_c \cdot d\mathbf{r} \)—which is the fourth condition for the “conservativeness” of \( \mathbf{F}_c \)—can then be written in the particularly compact form

\[ \nabla \times \mathbf{F}_c = 0 \]  

(0.54)

Some care must be used, however, in applying this criterion to fields that become infinite at some point in the region of interest.\(^\text{18}\)

While one must, of course, be able to evaluate gradients and curls by hand in several coordinate systems, symbolic manipulating computer programs can also evaluate these quantities. The command \( \text{<<VectorAnalysis}' \)—note that the tick is an opening tick—to \textit{Mathematica}, for example, loads a package of functions for evaluating vector derivatives, including functions for evaluating gradients and curls. After the package \textit{VectorAnalysis}' has been loaded, the commands

\begin{verbatim}
In[1]:= SetCoordinates[Cartesian[x,y,z]] ! Specify coordinates.
In[2]:= S = x*x + y*y + z*z ! Define a scalar field.
In[3]:= Grad[S] ! Evaluate gradient.
In[4]:= V = { x^2*y*z, x*y^2*z, x*y*z^2 } ! Define a vector field.
In[5]:= Curl[V]; ! Evaluate curl.
\end{verbatim}

will evaluate the gradient and the curl of the defined functions of the Cartesian coordinates \( x, y, z \). Note that one of the arguments of the functions \textit{Gradient} and \textit{Curl} specifies the variables with respect to which the gradient and curl are to be evaluated, so any variables can be used and communicated to \textit{Mathematica}'s routines. Among the many available coordinate systems are Cartesian, Cylindrical and Spherical.

\(^{17}\)See exact differentials in the index of your favorite calculus book.

0.3.5 Torque

The torque \( \mathbf{N}(\mathbf{r}) \) applied about the coordinate origin by the force \( \mathbf{F}(\mathbf{r}) \) acting at the point \( \mathbf{r} \) is defined by

\[
\mathbf{N}(\mathbf{r}) = \mathbf{r} \times \mathbf{F}(\mathbf{r}) = |\mathbf{r}| |\mathbf{F}(\mathbf{r})| \sin \theta \hat{\mathbf{n}}
\]

(0.55)

where \( \theta \) is the angle measured from \( \mathbf{r} \) to \( \mathbf{F} \) in the plane of those two vectors and \( \hat{\mathbf{n}} \) is a unit vector directed perpendicular to the plane of \( \mathbf{r} \) and \( \mathbf{F}(\mathbf{r}) \) as conveyed by the right-hand rule that defines the cross product. Geometrically, the magnitude of the torque is most commonly interpreted as the product of the length of \( \mathbf{r} \) times the component of \( \mathbf{F} \) in the plane of \( \mathbf{r} \) and \( \mathbf{F} \) and perpendicular to \( \mathbf{r} \). With this definition, torque is measured in the mks system in newton-meters (N·m), which—recall—is also the unit of energy. The name joule (J), however, is reserved for energy; torque will always be expressed in N·m.

Just as a force whose point of application experiences a displacement does work, so also a torque that acts while its point of application moves does work. To calculate this work, we start by introducing the cylindrical coordinate system of Fig. 0.9 in which the \((r, \phi)\) plane coincides with the plane of \( \mathbf{r} \) and \( \mathbf{F}(\mathbf{r}) \) as conveyed by the right-hand rule that defines the cross product. Geometrically, the magnitude of the torque is most commonly interpreted as the product of the length of \( \mathbf{r} \) times the component of \( \mathbf{F} \) in the plane of \( \mathbf{r} \) and \( \mathbf{F} \) and perpendicular to \( \mathbf{r} \). With this definition, torque is measured in the mks system in newton-meters (N·m), which—recall—is also the unit of energy. The name joule (J), however, is reserved for energy; torque will always be expressed in N·m.

The unit vectors in this system then are \( \hat{\mathbf{r}}, \hat{\phi}, \hat{\mathbf{k}} \). Further, \( \mathbf{r} \) has only a radial component and \( \mathbf{F} \) has only radial and azimuthal components, so the vectors of interest are

\[
\mathbf{r} = r \hat{\mathbf{r}} \quad ; \quad \mathbf{F} = F_\phi \hat{\phi} + F_\phi \hat{\phi}
\]

(0.56)

and

\[
\mathbf{N} = \mathbf{r} \times \mathbf{F} = (r \hat{\mathbf{r}}) \times (F_\phi \hat{\phi} + F_\phi \hat{\phi}) = r F_\phi \hat{\mathbf{k}} = r F \sin \theta \hat{\mathbf{k}}
\]

(0.57)

where \( \theta \) (not \( \phi \), see Fig. 0.9) is the angle between \( \mathbf{F} \) and \( \mathbf{r} \). Now, suppose that, in a short time while the force \( \mathbf{F} \) is acting, the position vector of the point of application of the force changes from \( \mathbf{r} \) to \( \mathbf{r} + \Delta \mathbf{r} \), with

\[
\Delta \mathbf{r} = \Delta r \hat{\mathbf{r}} + \Delta \phi \hat{\phi} + \Delta z \hat{\mathbf{k}}
\]

(0.58)
The work done by the force in that displacement is then given by\(^{19}\)

\[
\Delta W = \mathbf{F}(r) \cdot \Delta r = (F_t \hat{r} + F_\phi \hat{\phi} + F_z \hat{z}) \cdot (\Delta r \hat{r} + \Delta \phi \hat{\phi} + \Delta z \hat{k})
\]

\[
= F_t \Delta r + F_\phi \Delta \phi = F_t \Delta r + F \sin \theta \Delta \phi
\]

\[
= F_t \Delta r + |r \times \mathbf{F}(r)| \Delta \phi = F_t \Delta r + |\mathbf{N}(r)| \Delta \phi
\]  

(0.59)

[See Eqs. (0.26), (0.55), and (0.57).] Here, in the last form, the first term \(F_t \Delta r\) gives the work done by the force in the (infinitesimal) radial displacement of its point of application and the second term \(|\mathbf{N}(r)| \Delta \phi\) gives the work done by the force in the (infinitesimal) azimuthal displacement of its point of application.\(^{20}\) Finally, we introduce a vector to represent the angular displacement \(\Delta \phi\), giving that vector the magnitude \(\Delta \phi\) and the direction perpendicular to the plane in which the angular displacement takes place, i.e. \(\Delta \phi = \Delta \phi \hat{k}\) in the current coordinate system. With that notation, we find that

\[
\Delta W_{\text{torque}} = \mathbf{N}(r) \cdot \Delta \phi = |\mathbf{N}(r)| \hat{k} \cdot \Delta \phi \hat{k} = |\mathbf{N}(r)| \Delta \phi
\]  

(0.60)

for the part of the work attributable to the torque. Integrating over the full displacement, we conclude that the work done by a variable torque as its point of application moves over some displacement is given by

\[
W_{\text{torque}} = \int \mathbf{N} \cdot d\phi
\]  

(0.61)

The total work done in the displacement, of course, must include the work done in any radial displacement, though that contribution will be absent in a pure rotation.

### 0.3.6 Power

By definition, power \(P\) is the rate at which work is done on a system by some force. Suppose the work given by Eq. (0.26) is done in a time period \(\Delta t\). Then, the average power involved in the performance of that work over that time interval is given by

\[
P_{\text{average}} = \frac{\Delta W_i}{\Delta t} = \mathbf{F}(r_i) \cdot \frac{\Delta r_i}{\Delta t}
\]  

(0.62)

and, taking the limit as \(\Delta t \to 0\), we find that the instantaneous power developed by this activity is given by

\[
P(t) = \lim_{\Delta t \to 0} \mathbf{F}(r_i) \cdot \frac{\Delta r_i}{\Delta t} = \mathbf{F}(r_i) \cdot \frac{dr_i}{dt} = \mathbf{F} \cdot \mathbf{v}
\]  

(0.63)

The mks unit of power, work divided by time, is the J/s, a unit called the watt (W).\(^{21}\)

---

\(^{19}\)Strictly, this expression approximates the work done, but the approximation is the better as \(\Delta r\) becomes smaller. Ultimately, we will allow \(\Delta r\) to become infinitesimal, at which point the result is then exact.

\(^{20}\)Since the force has no component in the third (\(\hat{k}\)) coordinate direction, that displacement of the point of application contributes no work.

\(^{21}\)Scottish mechanical engineer James Watt, b. 19 January 1736 in Greenock, Renfrewshire, Scotland; d. 25 August 1819 in Handsworth, Birmingham, England. (Also in footnote 6.)
PROBLEMS

P0.20. A force field in two dimensions is given by the vector \( \mathbf{F}(x,y) = 3x^2y\hat{i} + xy\hat{j} \). Evaluate the work done by this force field on a particle moved along the following paths in the \( xy \)-plane: (a) \((0,0) \rightarrow (1,1)\) along straight line segments; (b) \((0,0) \rightarrow (1,0) \rightarrow (1,1)\) along straight line segments; (c) \((0,0) \rightarrow (1,1)\) along the line \( y = x \); and (d) \((0,0) \rightarrow (1,1)\) along the parabola \( y = x^2 \). Points in the plane are denoted by \((x,y)\).

P0.21. Show that the line integral of a vector field about an arbitrary closed path is zero if and only if the line integral between two arbitrary points is path-independent.

P0.22. Using an argument similar to that presented in the text, show that \( F_{cy} = -\frac{\partial U}{\partial y} \).

P0.23. A force field in three dimensions is given by \( \mathbf{F}(r) = a\frac{r}{r^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}} \)

where \( a \) is a constant. By showing explicitly that \( \mathbf{F} \cdot dr \) is the exact differential of some function, show that \( \mathbf{F} \) is a conservative force field and find the associated potential energy field.

P0.24. Using the operator \( \nabla \) as given in Eq. (0.47), verify that formal evaluation of \( \nabla \times \mathbf{Q} \) leads to Eq. (0.53), and show that \( \nabla \times \mathbf{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q_x & Q_y & Q_z \end{vmatrix} \)

Warning: One must use great care in evaluating \( \nabla \times \mathbf{Q} \) by this means in other coordinate systems; only the Cartesian unit vectors are constants. Compare P0.37.

P0.25. Familiarize yourself with vector identities Eqs. (C.5)–(C.9) in Appendix C, and prove Eqs. (C.5), (C.8), and (C.9).

P0.26. Show that, for a conservative force field, the field lines are perpendicular to the level contours—called here equipotential surfaces—of the associated potential energy field. \( \text{Hint: Let } dr \text{ in Eq. (0.50) represent the displacement between two infinitesimally separated points in the same equipotential surface. Physically, what must be the value of } dU? \)

P0.27. With \( r = (x^2 + y^2 + z^2)^{1/2} \), use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to

(a) Find the force field associated with each of the following potential energies:

\[
V(x, y, z) = \frac{1}{r}; \quad V(x, y, z) = \frac{z}{r^3}; \quad V(x, y, z) = \frac{2z^2 - x^2 - y^2}{r^3}; \quad V(x, y, z) = \frac{e^{-ar}}{r}.
\]

(b) Verify that each field obtained in part (a) is conservative by showing that the curl of each is zero.

(c) Show that \( \nabla \times r = 0 \).
CHAPTER 0. MATHEMATICAL AND PHYSICAL PRELIMINARIES

0.4 Coordinate-Free Definitions for Gradient and Curl

The gradient of a scalar field $S$ and the curl and the divergence (Section 2.3) of a vector field $Q$ play important roles in field theory and we shall subsequently need expressions for these quantities in non-Cartesian coordinates. We can, of course, obtain $\nabla S$ and $\nabla \times Q$ in other coordinates by a tedious but direct transformation of Eqs. (0.48) and (0.53), but the same results can be obtained more easily if we invest a brief preliminary effort in developing coordinate-free definitions for the vector derivatives. Consider first the gradient. From Eq. (0.50) we infer that

$$dS = \nabla S \cdot d\mathbf{r}$$

(0.64)

where $dS$ is the (infinitesimal) change in $S$ that results when the point at which $S$ is evaluated undergoes the (infinitesimal) displacement $d\mathbf{r}$. Let $|d\mathbf{r}| = d\ell$ and let $\mathbf{n}$ be a unit vector in the direction of $d\mathbf{r}$. Then $d\mathbf{r} = \mathbf{n} \ell$ and we find from Eq. (0.64) that

$$\nabla S \cdot \mathbf{n} = \frac{dS}{d\ell}$$

(0.65)

We now define the $\mathbf{n}$-component of $\nabla S$ to be $dS/d\ell$, called the directional derivative of $S$ in the direction $\mathbf{n}$. If we take $\mathbf{n}$ successively to be $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ while simultaneously letting $d\ell$ be the corresponding elements of distance $dx$, $dy$, and $dz$, we recover the components of Eq. (0.48). Equation (0.65), however, is more general. For example, in spherical coordinates, if $\theta$ is increased by $d\theta$ in Fig. 0.5(c), the point identified moves a distance $r d\theta$ in the $\theta$ direction. Taking $\mathbf{n} = \hat{\theta}$ and $d\ell = r d\theta$, we then quickly find from Eq. (0.65) that the $\hat{\theta}$-component of the gradient in spherical coordinates is given by $\nabla S \cdot \hat{\theta} = \frac{\partial S}{\partial \theta}$. Similar additional manipulations with Eq. (0.65) lead to the expressions

$$\nabla S = \hat{\mathbf{r}} \frac{\partial S}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial S}{\partial \phi} + \hat{\mathbf{k}} \frac{\partial S}{\partial z}$$

(0.66)

in cylindrical coordinates and

$$\nabla S = \hat{\mathbf{r}} \frac{\partial S}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial S}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial S}{\partial \phi}$$

(0.67)

in spherical coordinates.

A physical interpretation of $\nabla S$ can also be inferred from Eq. (0.65): the $\mathbf{n}$-component of $\nabla S$ measures the rate at which $S$ changes with respect to distance in the direction $\mathbf{n}$. The more rapidly $S$ changes as an observer (or measuring instrument) moves away from some initial point in some direction $\mathbf{n}$, the bigger will be the component of $\nabla S$ in that direction. In particular, since $\nabla S \cdot \mathbf{n}$ has its maximum value when $\mathbf{n}$ is parallel to $\nabla S$ (Why?), the gradient itself points in the direction of maximum increase of $S$, e.g., directly uphill in Fig. 0.6.

A coordinate-free definition for $\nabla \times Q$ emerges from an evaluation of $\oint Q \cdot d\mathbf{l}$ about a small closed path. Consider, for example, $\oint Q \cdot d\mathbf{l}$ about a small rectangle having sides $\Delta x$ and $\Delta y$ and lying in a plane parallel to and a distance $z$ above the $xy$-plane (Fig. 0.10).

---

22 We switch here to the notation $d\mathbf{l}$ for an infinitesimal element of the path.
Figure 0.10: A small rectangular path used to derive a coordinate-free definition of $\nabla \times \mathbf{Q}$.

Evaluating the integral in the counterclockwise direction as seen from a point above the path, we find that

$$\oint \mathbf{Q} \cdot d\mathbf{l} = \int_{x}^{x+\Delta x} \mathbf{Q}(x', y, z) \cdot dx' \hat{i} + \int_{y}^{y+\Delta y} \mathbf{Q}(x + \Delta x, y', z) \cdot dy' \hat{j}$$

$$+ \int_{x+\Delta x}^{x+\Delta x+y} \mathbf{Q}(x', y + \Delta y, z) \cdot dx' \hat{i} + \int_{y+\Delta y}^{y+\Delta y} \mathbf{Q}(x, y', z) \cdot dy' \hat{j}$$

$$= \int_{y}^{y+\Delta y} [Q_y(x + \Delta x, y', z) - Q_y(x, y', z)] dy'$$

$$+ \int_{x}^{x+\Delta x} [Q_x(x', y, z) - Q_x(x', y + \Delta y, z)] dx'$$

(0.68)

where the integrals in the first form relate, respectively, to the segments labeled 1, 2, 3, and 4 in Fig. 0.10. Assuming that $\Delta x$ and $\Delta y$ are small, we now use the Taylor expansion (Appendix B) to approximate each integrand in Eq. (0.68), obtaining

$$\oint \mathbf{Q} \cdot d\mathbf{l} \approx \Delta x \int_{y}^{y+\Delta y} \left. \frac{\partial Q_y}{\partial x} \right|_{x,y,z} dy' - \Delta y \int_{x}^{x+\Delta x} \left. \frac{\partial Q_x}{\partial y} \right|_{x,y,z} dx'$$

(0.69)

Finally, approximating each integral as we did in Eq. (0.43), we have

$$\oint \mathbf{Q} \cdot d\mathbf{l} \approx \left( \frac{\partial Q_y}{\partial x} - \frac{\partial Q_x}{\partial y} \right) \Delta x \Delta y = (\nabla \times \mathbf{Q}) \cdot \hat{k} \Delta S$$

(0.70)

where the field $\mathbf{Q}$ is understood to be evaluated at the point $(x, y, z)$ and $\Delta S = \Delta x \Delta y$ is the area of the surface bounded by the (small) path. More generally, if $\hat{n}$ is a unit vector perpendicular to the plane of some small area $\Delta S$ and we represent the area by a vector $\hat{n} \Delta S$, then we would find the value

$$\oint \mathbf{Q} \cdot d\mathbf{l} \approx [(\nabla \times \mathbf{Q}) \cdot \hat{n}] \Delta S$$

(0.71)
for the line integral of $Q$ about the path bounding $\Delta S$. From this equation we then infer the coordinate-free definition

$$\langle \nabla \times Q \rangle \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint Q \cdot dl$$

(0.72)

for the $\hat{n}$-component of $\nabla \times Q$. We have above tacitly adopted a convention relating the direction of the vector $\hat{n}$ to the direction in which the line integral is evaluated: If the fingers of the right hand are positioned to point in the direction in which the line integral is evaluated and the palm faces the area bounded by the path, the thumb gives the direction of $\hat{n}$. Equation (0.72) is correct only if this right-hand rule is followed. Equation (0.72) reproduces the components of Eq. (0.53) if the path is taken successively to be the boundary of a small rectangle in a plane perpendicular to the $x$, $y$, and $z$ axes, respectively, and it also yields the expressions

$$\nabla \times Q = \left( \frac{1}{r} \frac{\partial Q_z}{\partial \varphi} - \frac{\partial Q_\varphi}{\partial z} \right) \hat{r} + \left( \frac{\partial Q_z}{\partial r} - \frac{\partial Q_r}{\partial z} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial (rQ_\varphi)}{\partial r} - \frac{\partial Q_r}{\partial \varphi} \right) \hat{k}$$

(0.73)

in cylindrical coordinates and

$$\nabla \times Q = \frac{1}{r \sin \theta} \left( \frac{\partial (\sin \theta Q_\theta)}{\partial \theta} - \frac{\partial Q_\varphi}{\partial \theta} \right) \hat{r} + \left( \frac{1}{r \sin \theta} \frac{\partial Q_\varphi}{\partial \varphi} - \frac{1}{r} \frac{\partial (rQ_\theta)}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial (rQ_\varphi)}{\partial r} - \frac{\partial Q_\varphi}{\partial \theta} \right) \hat{k}$$

(0.74)

in spherical coordinates with nearly equal ease.

A physical interpretation of $\nabla \times Q$ can be inferred from Eq. (0.72): The $\hat{n}$-component of $\nabla \times Q$ measures a particular average, specifically $\oint Q \cdot dl / \Delta S$, of the tangential component of $Q$ computed for a small, closed path lying in a plane perpendicular to $\hat{n}$. In effect, when $(\nabla \times Q) \cdot \hat{n} \neq 0$, $Q$ has a net tangential component about this path. Unfortunately, this statement is considerably less transparent than the corresponding statement for $\nabla S$, and we present also a more specific example even though we risk oversimplification. Let $Q$ represent the (two-dimensional) velocity field describing the motion of points on the surface of a phonograph record revolving about a vertical axis with constant angular velocity $\omega$ [Fig. 0.11(a)]. The linear speed of a point a distance $r$ from the axis is $\omega r$ and the velocity field $Q(r)$ expressed in cylindrical coordinates is given by $Q(r) = \omega r \hat{\phi}$. Direct application of Eq. (0.73) now gives $\nabla \times Q = 2 \omega \hat{k} = 2 \omega$. Thus, for this simple case, $\nabla \times Q$ is everywhere twice the angular velocity. In particular $\nabla \times Q = 0$ when the record is not rotating. Since the field lines of $Q$ in this case are circles centered on the axis of rotation, these results suggest that $\nabla \times Q$ is related to the tendency of field lines to circle (or curl!) about some point in the field, as in Fig. 0.11(b). To be correct, however, this statement must be further interpreted to mean what is said at the beginning of this paragraph. As the field $Q = x^2 \hat{j}$ (for which $\nabla \times Q = 2x \hat{k}$) shows, $\nabla \times Q$ can be nonzero even when the field lines are parallel and straight! One route to developing an intuitive feel for $\nabla \times Q$ might be to sketch the field lines for several fields, e.g., $Q = \omega r \hat{\phi}$, $Q = x^2 \hat{j}$, and $Q = r/r^3$, and then allow the eye to trace around several small closed paths in these fields, mentally estimating the cumulative tangential component of $Q$ for each path.
Figure 0.11: A rotating phonograph record. Part (a) shows the angular velocity of the record and part (b) shows the velocity field in the $xy$-plane.

PROBLEMS

P0.28. (a) Derive Eq. (0.66) by direct transformation of Eq. (0.48). *Hint:* Use Table 0.1. (b) Derive Eqs. (0.66) and (0.67) by the method based on Eq. (0.65).

P0.29. Use the definition of Eq. (0.72) to derive the $\hat{k}$-component of Eq. (0.73).

P0.30. The force field $\mathbf{F}(\mathbf{r})$ is a *central force field* if it has the more explicit form $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\hat{\mathbf{r}}$ in spherical coordinates. This force is a function only of distance from some fixed point conveniently taken to be the origin and is everywhere directed along radial lines from that point. Show that every central force field is conservative and obtain an integral for the associated potential energy field. What is the potential energy field associated with the inverse square force field for which $f(\mathbf{r}) = \frac{a}{r^2}$?

P0.31. Taking $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\mathbf{r} = r\hat{\mathbf{r}} + z\hat{k}$, and $\mathbf{r} = r\hat{\mathbf{r}}$ in turn, evaluate $\nabla \times \mathbf{r}$ and $\nabla (1/r)$ in Cartesian, cylindrical, and spherical coordinates, and show that the results are equivalent.

P0.32. Use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to evaluate the gradient of each of

i. $S(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ (Cartesian coordinates)

ii. $S(\rho, \phi, z) = \frac{1}{\sqrt{\rho^2 + z^2}}$ (cylindrical coordinates)

iii. $S(r, \theta, \phi) = \frac{1}{r}$ (spherical coordinates)

and the curl of each of

iv. $\mathbf{V}(x, y, z) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}$ (Cartesian coordinates)

v. $\mathbf{V}(\rho, \phi, z) = \frac{\rho\hat{\rho} + z\hat{k}}{(\rho^2 + z^2)^{3/2}}$ (cylindrical coordinates)
vi. \( \mathbf{V}(r, \theta, \phi) = \frac{\mathbf{r}}{r^2} \) (spherical coordinates)

Note that the given scalar fields are (essentially) the potential of a point charge at the origin in the three coordinate systems and the given vector fields are (essentially) the electric fields produced by a point charge at the origin in each of the three coordinate systems.

SUPPLEMENTARY PROBLEMS

**P0.33.** The Kronecker delta \( \delta_{ij} \) and the three-index symbol \( \epsilon_{ijk} \) are defined by

\[
\delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j
\end{cases}
\]

\[
\epsilon_{ijk} = \begin{cases} 
+1, & (i, j, k) = (1, 2, 3), (3, 1, 2), (2, 3, 1) \\
-1, & (i, j, k) = (3, 2, 1), (1, 3, 2), (2, 1, 3) \\
0, & \text{anything else}
\end{cases}
\]

With all the indices assuming the values 1, 2, 3, show that

(a) \( \hat{e}_i \cdot \hat{e}_j = \delta_{ij} \),

(b) \( \mathbf{A} \cdot \mathbf{B} = \sum_i \sum_j \delta_{ij} A_i B_j \),

(c) interchange of any two indices changes the sign of \( \epsilon_{ijk} \),

(d) \( \hat{e}_i \times \hat{e}_j = \sum_k \epsilon_{ijk} \hat{e}_k \),

(e) \( (\mathbf{A} \times \mathbf{B})_i = \sum_j \sum_k \epsilon_{ijk} A_j B_k \), and

(f) \( \sum_k \epsilon_{ijk} \epsilon_{rsk} = \delta_{ir} \delta_{js} - \delta_{is} \delta_{jr} \).

Finally, (g) derive Eq. (0.2) by manipulating with these expressions. *Hint:* Begin by evaluating \( [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_k \).

**P0.34.** (a) A vector \( \mathbf{A} \) *fixed in space* is described by its components \((A_x, A_y, A_z)\) and \((A_{x'}, A_{y'}, A_{z'})\) in two Cartesian coordinate systems related as shown in Fig. 0.12. Show that the components must be related by

\[
A_{x'} = A_x \cos \phi + A_y \sin \phi \\
A_{y'} = -A_x \sin \phi + A_y \cos \phi \\
A_{z'} = A_z
\]

if the orientation and magnitude of \( \mathbf{A} \) are to be unchanged by the rotation. These are the rules for the transformation of the components of a vector under this rotation, and a three-component entity transforming in this way is said to be invariant to the transformation.
Table 0.2: Pseudocode algorithm for evaluating a sum.

INPUT N and DETERMINE coordinates of points on path
INITIALIZE S (for accumulating sum) to zero
INITIALIZE I (counter) to one

LOOP
  CALCULATE I-th contribution to sum and INCREMENT S
  EXIT LOOP WHEN I = N
  INCREMENT I to I + 1
END LOOP
PRINT S
STOP

(b) The prototype vector (think of displacement) is invariant to reflection of coordinates, i.e., to the transformation \( x'' = -x, \, y'' = -y, \, z'' = z \), which means that \( A_{x''} = -A_x, \, A_{y''} = -A_y, \, A_{z''} = -A_z \). (Why?) Suppose \( \mathbf{A} \) and \( \mathbf{B} \) are invariant to reflection. How does the “vector” \( \mathbf{C} = \mathbf{A} \times \mathbf{B} \) behave under reflection? Is \( \mathbf{C} \) a vector?

P0.35. Equation (0.27) can be used to effect a numerical evaluation of \( \int_{\Gamma} \mathbf{F} \cdot d\mathbf{r} \) on some path \( \Gamma \). For simplicity consider a two-dimensional field, \( \mathbf{F}(x, y) = F_x(x, y) \hat{i} + F_y(x, y) \hat{j} \), and let the path \( \Gamma \) lie in the \( xy \)-plane. Further, let the \( i \)-th segment of \( \Gamma \) begin at \((x_i, y_i)\) and end at \((x_{i+1}, y_{i+1})\), \( 1 \leq i \leq N \). Equation (0.27) then becomes

\[
\sum_{i=1}^{N} \mathbf{F}(\tau_i, \bar{y}_i) \Delta x_i + F_y(\tau_i, \bar{y}_i) \Delta y_i 
\]

where \( \tau_i = \frac{1}{2}(x_{i+1} + x_i), \, \bar{y}_i = \frac{1}{2}(y_{i+1} + y_i), \, \Delta x_i = x_{i+1} - x_i, \, \Delta y_i = y_{i+1} - y_i, \) and \( N \) is the number of segments into which \( \Gamma \) is divided. (a) Write a computer program to evaluate this integral for the force field in P0.20. A possible general strategy is illustrated in Table 0.2. (b) Run your program for the paths in P0.20 and for several progressively larger values of \( N \). Optional: (1) Try other force fields of your choosing and try several paths, including some closed paths. (2) Develop techniques and write programs for the numerical evaluation of other two- and three-dimensional line integrals, e.g., \( \int_S d\mathbf{r} \) and \( \int Q \times d\mathbf{r} \), where \( S \) is a scalar field and \( Q \) a vector field.

P0.36. Let \( \Phi \) be a scalar function of position. Evaluate \( \oint \Phi \, d\mathbf{l} \) about the path shown in Fig. 0.10 to show that

\[
\oint \Phi \, d\mathbf{l} = (\hat{k} \times \nabla \Phi) \Delta x \Delta y
\]

and then infer that

\[
\hat{n} \times \nabla \Phi = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint \Phi \, d\mathbf{l}
\]

where \( \hat{n} \Delta S \) is the vector representing the area of the (plane) surface bounded by the (small) path about which the integral extends.

P0.37. Consider a general curvilinear coordinate system with coordinates \((q_1, q_2, q_3)\) defined in terms of the Cartesian coordinates \((x_1, x_2, x_3)\) by the functions \( q_i = q_i(x_1, x_2, x_3), \, i = 1, 2, 3 \). We assume these equations can be inverted to give \( x_i \) as some function of the \( q \)’s, \( x_j = x_j(q_1, q_2, q_3), \, j = 1, 2, 3 \). Further, we introduce three quantities \( h_i, \, i = 1, 2, 3 \)—which may be functions of the \( q \)’s—such that \( ds_i = h_i \, dq_i \), where \( ds_i \) is the physical distance that a point moves when its \( i \)-th coordinate alone is changed from \( q_i \) to \( q_i + dq_i \). Finally, we confine our attention to orthogonal systems, which in general means that the three
families of surfaces defined by \( q_i = \text{constant}, \ i = 1, 2, 3, \) are mutually orthogonal at every point in space and more specifically means (1) that the distance \( ds \) that a point moves when all three coordinates are simultaneously incremented is given by a sum of squares

\[
ds^2 = h_1^2 (dq_1)^2 + h_2^2 (dq_2)^2 + h_3^2 (dq_3)^2
\]

with no cross terms (e.g., \( dq_1 dq_2 \)) and (2) that the unit vectors \( \hat{e}_i \) in the direction of increasing \( q_i \) are mutually orthogonal. Let \( q_1, q_2, q_3 \) be ordered so that \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) form a right-handed set and let \( S \) and \( Q \) be a scalar and a vector field, respectively. Show that

(a) \( dv = \) volume element = \( h_1 h_2 h_3 dq_1 dq_2 dq_3. \)

(b) \( \nabla S = \sum_i (1/h_i) (\partial S/\partial q_i) \hat{e}_i. \) Hint: Use Eq. (0.65).

(c) \( \nabla \times Q = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial & \partial & \partial \\ h_1 Q_1 & h_2 Q_2 & h_3 Q_3 \end{vmatrix} \)

Hint: Use Eq. (0.72) and note that the derivatives in row 2 are understood to operate only on the entries in row 3 and not on the entries in row 1. Finally, (d) determine the \( h \)'s for cylindrical coordinates \( (q_1, q_2, q_3) = (r, \phi, z) \) and for spherical coordinates \( (q_1, q_2, q_3) = (r, \theta, \phi) \) and show that the expressions in parts (b) and (c) reduce to the expressions given in the text.

REFERENCES


Chapter 1

Quantifying Charge and Current

Every physical theory contains some (small) number of fundamental physical quantities, each of which is measured by following a carefully prescribed but arbitrarily selected procedure for comparing a given amount of the quantity to an arbitrarily selected standard amount. In contrast, derived quantities in a theory are defined in terms of the fundamental quantities without introducing any further arbitrary standards. In classical mechanics, length, time, and mass are almost universally chosen as the most convenient fundamental quantities. No such unanimity exists in electricity and magnetism. This theory is sometimes constructed without introducing any fundamental quantities beyond the three mechanical quantities, and it is sometimes constructed on a base obtained by supplementing the mechanical quantities with one or more fundamental and specifically electromagnetic quantities. The primary objectives of this chapter are (1) to provide a background into which most of the common approaches to electricity and magnetism can be placed and (2) to expose the system of units adopted in the remainder of this book. To these ends, we imagine—at least initially—that we are discovering (electric) charge and (electric) current anew.\footnote{Strictly, we should also include permanent magnetism in this list, but approaches based on this phenomenon are less common than those based on charge and current and, in any case, the essential ideas can be made clear from a consideration of charge and current alone.} Emphasis is placed deliberately on the logical aspects of a development from the qualitative properties of each phenomenon to quantitative definitions both for charge and for current. The reader is cautioned, first, to draw no close parallels between the actual historical development of these ideas and the development implied in this chapter and, second, to realize that the experiments by which the properties quoted in this chapter are in fact most accurately confirmed are much more sophisticated than the experiments here employed.

1.1 The Phenomenon of Electric Charge

The first step in our study of electricity is the recognition that electric charge (hereafter simply charge) exists. Experimentally, rubbing a hard rubber rod with rabbit’s fur or a glass rod with silk modifies the properties of the rod so that it is capable of deflecting the leaves of an electroscope (Fig. 1.1). The rubbed rod—and more generally any object capable of deflecting the leaves of an electroscope—is said to be charged, and we attribute
this property to a physical entity called charge that we view to reside on the object. The electroscope itself is a primitive detector of charge.

Qualitative experiments on objects charged by rubbing suggest the existence of two kinds of charge. Let one charged object be brought up to an (initially uncharged) electroscope. A deflection of the leaves occurs. Now, leaving the first object near the electroscope, bring up a second charged object. A priori we expect that the deflection of the leaves may either decrease or increase. Experimentally, we find that nature admits both possibilities.\footnote{In the analogous gravitational experiment, only one of the two a priori possibilities is realized in the physical world.} Thus, relative to a particular object, an array of second objects can be divided into two groups. Members of the first group increase the deflection and are said to have the same kind of charge as the first object; members of the second group decrease the deflection and are said to have a kind of charge opposite to that of the first object. Furthermore, we find experimentally that the specific objects included in each group are the same, regardless of which object is chosen as the first object. We conclude that any charged object (hereafter simply any charge) can be placed unambiguously in one of two groups, that any two members of either group have like charges (each increases the deflection produced by the other), and that two charges drawn one from each group have unlike or opposite charges (each decreases the deflection produced by the other). To reflect algebraically the tendency of charges drawn from one group to cancel deflections produced by charges drawn from the other group, we designate the charge on members of one group as positive and that on members of the other group as negative. Convention takes the charge appearing on a hard rubber rod rubbed with rabbit’s fur to be negative and that appearing on a glass rod rubbed with silk to be positive, but this firmly established choice is mere convention and might have been made the other way. In view of contemporary models of matter, which take macroscopic objects to be composed microscopically of very large amounts of charge of both signs, we must now slightly modify the above definition of a charged macroscopic object: A macroscopic object is charged when microscopically it has more charge of one sign than of the other. In reverse, an uncharged object is an object in which positive and negative charges exactly cancel, not an object containing no charge at all.
1.2 The Interaction of Point Charges

Although the recognition that charge exists is the first step, the specification of a quantitative definition for charge must precede the development of a quantitative science of charge. We could, of course, define charge as a fundamental quantity, say by picking a standard electroscope, supplying it with an arbitrarily marked scale to convert deflection into charge units, and specifying some standard way to present a charge to this instrument for measurement. The consequences of this approach, however, might be awkward expressions for physical laws involving charge (P1.10). We begin, instead, by exploring quantitatively the forces of interaction among charged objects, being careful not to make premature restrictive assumptions about the quantitative nature of charge. The experimental properties of these forces, which we intuitively expect to be conditioned by the amount of charge present, can then be used as a guide to the selection of a convenient definition for charge. We assume that the fundamental quantities—length, time, and (inertial) mass—and the derived quantity—force—are already measurable.

Turning then to quantitative aspects (but omitting descriptions of specific experiments), we discover first that the force of interaction between two arbitrary objects arbitrarily charged is not related in any simple way to the relative positions of the objects. When the objects are far apart, however, the interaction is more easily described. We therefore introduce the idealization of a point charge, realized physically when a charged object is viewed from a distance large compared to its own dimensions (so that it looks like a point), and at least initially we concentrate on the interactions between (idealized) point charges.

If we further constrain these charges to be at rest relative to one another, we then discover the following experimental properties of the force of interaction between two point charges in vacuum:

1. The force between two point charges at rest is inversely proportional to the square of the separation of the charges. Experimental tests of this statement are usually quoted by giving an upper limit on the deviation of \( n \) from 2 when the statement is written in the form force \( \propto 1/(\text{separation})^n \). Let \( n - 2 = \eta \). Contemporary experiments (as of 1986) show that \( \eta < (1.0 \pm 1.2) \times 10^{-16} \) over distances ranging from the macroscopic (tens of centimeters, cm) to the atomic (10\(^{-8}\) cm), that an inverse square law remains valid at nuclear distances (10\(^{-13}\) cm), but that this law may fail at the subnuclear level (< 10\(^{-14}\) cm).\(^3\)

2. The force between two point charges at rest is proportional to the magnitude of both charges. Much as it might seem otherwise, experimental evidence supporting this observation can be obtained without presupposing the ability to measure charge. It is certainly intuitive to require that equal charges produce equal deflection when presented in some standard way to an electroscope. Thus, even though we cannot yet...

---

\(^3\) L. P. Fletcher, *Phys. Rev. A* **33** 759 (1986); E. R. Williams, J. E. Fuller, and H. A. Hill, *Phys. Rev. Letters* **26**, 721 (1971); R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley Publishing Company, Inc. Reading, Mass. 1964), Volume II, Section 5-8. The first of these references contains a table showing how the upper limit on \( \eta \)—at least for macroscopic distances—has slowly been reduced since the measurements of Cavendish (1773; \( |\eta| < 2 \times 10^{-2} \)) and Coulomb (1785; \( |\eta| < 4 \times 10^{-2} \)).

British scientist Henry Cavendish was born on 10 October 1731 in Nice, Kingdom of Sardinia, and died on 24 February 1810 in London, England; French physicist Charles-Augustin de Coulomb was born on 14 June 1736 in Angiulème, Angoinois, France, and died on 23 August 1806 in Paris, France.
assign a number to any charge, we can nonetheless produce an array of equal charges. (Recall that we already are able to determine the sign of a charge.) Let us now require of our ultimate definition of charge the (intuitive) property that charge be additive, i.e., that several charges placed simultaneously in the same position be equivalent to a single object having a charge equal to the (algebraic) sum of the component charges. Then two equal charges placed close together and observed from a distance large compared both to their separation and to their dimensions will be equivalent to a point charge having twice the charge of either object separately. Thus, starting with a point charge having any magnitude, we can produce a point charge having twice (or three times or . . . ) that magnitude, even though we do not know the magnitude of the first charge. We then discover that if one of two charges is replaced by a charge of twice (or three times or . . . ) its magnitude, the force of interaction is doubled (or tripled or . . . ).

3. The force between two point charges at rest acts along the line joining the two charges and is repulsive for charges of like sign and attractive for charges of opposite sign. If one assumes space to be isotropic—the same in all directions—the force between two isolated charges can have no other direction, for the direction of the line joining the point charges is the only uniquely distinguishable direction in space.

4. The force between two point charges at rest satisfies Newton's third law.

In this chapter, we need primarily properties (1) and (2), which are expressed symbolically by the equation

\[ F_{qq'} = k_1 \frac{qq'}{d^2} \]  

(1.1)

where \( F_{qq'} \) is the magnitude of the force between two charges\(^4\) of magnitude \( q \) and \( q' \) separated by a distance \( d \) and \( k_1 \) is a proportionality constant. The minor additions to Eq. (1.1) that incorporate properties (3) and (4) will be introduced in Chapter 4, the full statement being known as Coulomb’s Law.\(^5\)

Equation (1.1) has been experimentally inferred without having a unit of charge available; we needed only an unambiguous means to produce multiples of a given charge. The constant \( k_1 \), however, remains unspecified and provides the freedom to adapt Eq. (1.1) to whatever unit of charge is selected. We have assumed force and distance to be measurable. If a unit of charge consistent with but independent of Eq. (1.1) is also available, then all physical quantities in Eq. (1.1) are measurable, and \( k_1 \) must be found experimentally. If we choose to, however, we can use Eq. (1.1) to define the unit of charge by stipulating \( k_1 \) arbitrarily. If, for example, \( k_1 \) is given the dimensionless value unity, we find that

\[ F_{qq'} = \frac{qq'}{d^2} \]  

(1.2)

and the procedure for measuring an unknown charge \( Q \) involves the following steps:

1. Make a second charge of equal magnitude \( Q \) using the procedure already outlined.
2. Place this second charge a measured distance \( R \) from the first charge.

---

\( ^4 \)We can, of course, give a symbol to a charge without knowing a specific number for the amount of charge represented by the symbol.

\( ^5 \)See footnote 3 in Chapter 1 on page 31.
3. *Measure* the force $F_{QQ}$ between these two charges.

4. Calculate the magnitude of $Q$ from the expression $F_{QQ} = Q^2/R^2$, or $Q = \sqrt{F_{QQ}/R^2}$.

5. Determine the sign of the original charge by noting whether it is repelled by or attracted to a charge of known sign.

In this example, charge is a *derived* quantity having the (mechanical) dimensions of $\sqrt{\text{force} \cdot \text{(length)}^2}$. The cgs-electrostatic system of units (cgs-esu), in which force is measured in dynes (dyn), length in centimeters (cm), and time in seconds (s), is based on exactly this procedure. The constant $k_1$ is set equal to 1 to define a unit of charge officially called the *statcoulomb* (statC) but more commonly referred to as the esu of charge. Other units of charge in common use will be discussed in Section 1.6; not all of these are defined by Coulomb’s Law. However the unit of charge is defined, we shall use the term *magnitude* for an amount of charge when its sign is ignored and the term *strength* for an amount of charge when its sign is included, i.e., a charge of strength $-4$ statC has a magnitude of $4$ statC.

Once a procedure for measuring charge is selected, we can then produce an array of known charges which in turn can be used to calibrate the scale on an electroscope. The resulting instrument is, of course, considerably more convenient for laboratory measurements of charge than the primary procedure outlined in this section.

### 1.3 The Phenomenon of Current

Although it is well known that current and charge are related (and we shall soon incorporate this relationship), let us consider current for the moment as a separate phenomenon. The first step in a study of current is then the recognition of its existence. Experimentally, when a wire made of a suitable material (a conductor) is connected across the poles of the right sort of device (a battery or its equivalent), a small compass (or permanent magnet) placed near the wire manifests an interaction with the wire. Since an electroscope placed near the wire exhibits no effect, this new interaction is different from an electrostatic interaction and we shall attribute it to a *current* in the wire (even though we have at this point in the argument no convincing reason to think of the current as flowing in the wire rather than residing on the wire). We shall now examine the properties of this current, using the compass as a primitive detector.

To make this simple detector more useful, let one pole of a small compass needle be weighted slightly so that the needle assumes a vertical equilibrium position when supported from its center on a horizontal axis. (To avoid conceptually irrelevant complications, we assume that the magnetic effect of the earth on the compass is negligible compared to the effects we seek to examine.) Further introduce a scale with its zero position at the point of equilibrium. The resulting device is illustrated in Fig. 1.2 and, for want of a standard name, will be called a *currentscope*. This instrument is used by placing it behind a long vertical wire as shown in Fig. 1.3. When the poles of a battery are connected to the two ends of the wire, the compass needle will deflect to one side, manifesting the interaction we are attributing to a current in the wire. Experiment quickly reveals that the reading of the currentscope diminishes as the instrument is moved away from the wire. Thus, we adopt a standard distance from the wire to the instrument.
More detailed (but still qualitative) experiments with the currentscope result in the following observations:

1. The reading of the currentscope is independent of where along the wire it is placed. The current is therefore the same at all points along the wire, and we can speak unambiguously of the current in the wire.

2. If the connections to the battery are reversed, the currentscope deflects in the opposite direction, and we therefore adopt a convention taking one of the currents to be positive and the other to be negative. (The common convention will be introduced later in a different context; for now, we simply pick either convention.)

Together these two observations support the view that current flows along a wire.

1.4 The Interaction of Parallel Currents

Let us now develop a quantitative definition for current. The approach adopted parallels the approach used in determining a quantitative definition for charge but is entirely independent of that development. Just as two objects that an electroscope reveals to be charged experience an interaction with one another, two wires that a currentscope reveals to carry currents experience mutual forces. Experimentally, however, the force between two wires is not easily determined if the wires are arbitrarily oriented. Although we shall later consider the more general case, for the moment we shall restrict our attention to the interactions between long, thin, parallel wires and to points remote from either end of the wires. Re-
peating experiments first performed by Ampere,⁶ we then find experimentally that the force per unit length between two stationary, parallel wires carrying (steady) currents

1. is inversely proportional to the separation of the wires.
2. is proportional to the magnitude of each current.
3. is directed along the common perpendicular to the wires and is attractive for currents of like sign and repulsive for currents of opposite sign.
4. satisfies Newton’s third law.

In this chapter, we need primarily properties (1) and (2), which are expressed symbolically by the equation

\[ F_{II'} = k_2 \frac{II'}{s} \]  \hspace{1cm} (1.3)

where \( F_{II'} \) is the magnitude of the force per unit length between two long parallel wires carrying currents of magnitude \( I \) and \( I' \), \( s \) is the (perpendicular) distance between the wires, and \( k_2 \) is a proportionality constant. The reformulation of Eq. (1.3) to include properties (3) and (4) and to include also interactions between currents in nonparallel wires is presented in Chapter 5.

As with the constant \( k_1 \) in Eq. (1.1), the constant \( k_2 \) permits Eq. (1.3) to be adapted to whatever unit of current is selected. Again we assume force and distance to be measurable. If a unit of current is also available independently of Eq. (1.3), then \( k_2 \) in Eq. (1.3) must be found experimentally. If we choose to, however, we can use Eq. (1.3) to define the unit of current by stipulating \( k_2 \) arbitrarily. If, for example, \( k_2 \) is given the (dimensionless) value 2, we find that

\[ F_{II'} = \frac{2II'}{s} \]  \hspace{1cm} (1.4)

and the procedure for measuring an unknown current \( i \) involves the following steps:

1. Construct a second wire carrying a current of the same strength.
2. Place this second wire parallel to and a measured distance \( R \) from the first current.
3. Measure the force per unit length \( F_{ii} \) experienced by either wire.
4. Calculate the magnitude of the current from the expression \( F_{ii} = 2i^2/R \), or \( i = \sqrt{F_{ii}R/2} \).
5. Determine the sign of \( i \) by noting whether it is repelled by or attracted to a current of known sign.

In this example, current is a derived quantity having the (mechanical) dimensions of \( \sqrt{\text{force}} \). (Remember that \( F_{ii} \) is a force per unit length.) The cgs-electromagnetic system of units (cgs-emu), in which force is measured in dynes, length in centimeters, and time in seconds, is based on exactly this procedure. The constant \( k_2 \) is set equal to 2 to define a unit of current officially called the abampere (abA) but more commonly referred to as the emu of current. Other units of current will be discussed in Section 1.6; not all of these are defined

---

⁶French physicist and mathematician André Marie Ampère was born on 20 January 1775 in Lyon, France, and died on 10 June 1836 in Marsielle, France.
by Eq. (1.3). We shall apply the terms *magnitude* and *strength* to currents with meanings analogous to those given at the end of Section 1.2 for charges.

Once a procedure for measuring current is selected, we can then produce an array of known currents which in turn can be used to calibrate a currentscope.

### 1.5 Current as Charge in Motion

Consider now the following experiment. Suppose a wire runs from a suitable charged object properly placed near a calibrated electroscope past a calibrated currentscope to ground as shown in Fig. 1.4. When the switch is closed, not only do the leaves on the electroscope fall, implying that the charge on the object is getting smaller, but also that the currentscope deflects as long as there is any charge remaining on the object. We therefore conclude that charge flows off the object through the wire and that current is the same as charge in motion. By recording the readings of the electroscope and the currentscope as functions of time $t$, we further discover that the current $I(t)$ in the wire and the charge $q(t)$ measured by the electroscope are related by

$$I(t) = k_3 \frac{dq(t)}{dt}$$

(1.5)

where in the present context $k_3$ is an empirically determined proportionality constant whose value depends on the units in which $I(t)$ and $q(t)$ (and indeed $t$) are measured. Although the connection expressed in Eq. (1.5) leads us to view moving charge wherever it occurs—possibly in wires but equally well, for example, within the body of an ionized gas or in the beam of a particle accelerator—as a current, we shall postpone exploring this broader interpretation until Chapter 2.
1.6 Units in Electricity and Magnetism

We now have three different expressions involving charge and current: (1) Coulomb’s Law for the force between two point charges,

\[ F_{qq'} = k_1 \frac{qq'}{d^2} \quad (1.6) \]

(2) the analogous law for the force per unit length between two parallel wires,

\[ F_{II'} = k_2 \frac{II'}{s} \quad (1.7) \]

and (3) the relationship

\[ I = k_3 \frac{dq(t)}{dt} \quad (1.8) \]

between charge and current. Unless we are willing to define \( t \) as well as \( I \) and \( q \) by these equations, there are at most\(^7\) two quantities (\( I \) and \( q \)) to be defined. Therefore, at most two of the three constants \( k_1 \), \( k_2 \), and \( k_3 \) can be assigned arbitrary values. The third constant is fixed by the other two even though it takes an experiment to determine its numerical value. Several different systems of units, all in reasonably common use, will now be described.

1.6.1 CGS-ESU

The cgs [centimeter-gram-second]-electrostatic system of units adopts the cgs units of length, mass, and time and defines a unit of charge (the statcoulomb, statC) by setting \( k_{\text{esu}}^1 = 1 \) and a unit of current (the statampere, statA) by setting \( k_{\text{esu}}^3 = 1 \). Both of these units are mechanical, the statcoulomb having the dimensions of \( \sqrt{\text{dyn} \cdot \text{cm}^2} \) and the statampere having the dimensions of \( \text{statC/s} = \sqrt{\text{dyn} \cdot \text{cm}^2/\text{s}^2} \). Although it is at times convenient to regard \( k_{\text{esu}}^1 \) to have the dimensions \( \sqrt{\text{dyn} \cdot \text{cm}^2/\text{statC}^2} \), this combination is actually dimensionless. In cgs-esu, the remaining constant \( k_{\text{esu}}^2 \) is determined empirically. For subsequent elegance, we introduce another constant \( c \) by

\[ k_{\text{esu}}^2 = \frac{2}{c^2} \quad (1.9) \]

and write Eq. (1.7) in the form

\[ F_{II'} = \frac{2}{c^2} \frac{II'}{s} \quad (1.10) \]

The dimensions of \( c \) are demonstrably those of velocity (see P1.2). Measurement of \( c \), in principle by measuring \( F_{II'} \) when known currents are separated by a measured amount \( s \) but in practice by any of several more refined techniques, leads to the value\(^8\)

\[ c = (2.997925 \pm 0.000003) \times 10^{10} \text{ cm/s} \quad (1.11) \]

for which the approximate value \( 3 \times 10^{10} \text{ cm/s} \) is often satisfactory. Although unanticipated and accidental at this point, this value is numerically equal to the speed of light, which

\(^7\)Remember that it is logically possible for \( i \) and \( q \) to be defined independently of these equations.

\(^8\)See Section 1.7 for a discussion of the impact of recent changes in the status of the speed of light and other units.
explains why the 2 was inserted in Eqs. (1.4) and (1.9). This appearance of the speed of light in the context of electricity and magnetism also suggests strongly that light and electromagnetism are closely connected, but the intimacy of that connection will not emerge more fully until Chapter 7.

1.6.2 CGS-EMU

The cgs [centimeter-gram-second]-electromagnetic system of units adopts the cgs units of length, mass, and time and defines a unit of current (the abampere, abA) by setting $k_{2}^{\text{emu}} = 2$ and a unit of charge (the abcoulomb, abC) by setting $k_{3}^{\text{emu}} = 1$. Both of these units are mechanical, the abampere having the dimensions of $\sqrt{\text{dyn}}$ and the abcoulomb having the dimensions of $\text{abA} \cdot \text{s} = \sqrt{\text{dyn} \cdot \text{s}^2}$. Although it is at times convenient to regard $k_{2}^{\text{emu}}$ to have the dimensions $(\text{dyn}/\text{abA}^2)$, this combination is actually dimensionless. In cgs-emu, the remaining constant $k_{1}^{\text{emu}}$ is determined empirically. For subsequent elegance, we set\(^9\)

$$k_{1}^{\text{emu}} = c^2$$

(1.12)

and write Eq. (1.6) in the form

$$F_{qq'} = c^2 \frac{qq'}{d^2}$$

(1.13)

The dimensions of this $c$ are also demonstrably those of velocity (P1.2). When measured by measuring the force between two known point charges at known separation (or by some equivalent more refined technique), the $c$ here turns out to have the value given in Eq. (1.11), and our use of the same symbol is justified.

1.6.3 Rationalized MKS Units

The rationalized mks [meter-kilogram-second] system of units adopts the mks units of length, mass, and time and defines a unit of current (the ampere, A) by setting $k_{2}^{\text{mks}} = 2 \times 10^{-7}$ and a unit of charge (the coulomb, C) by setting $k_{3}^{\text{mks}} = 1$. Both of these units are mechanical, the ampere having the dimensions of $\sqrt{\text{N}}$, where N abbreviates newton, and the coulomb having the dimensions of $\text{A} \cdot \text{s} = \sqrt{\text{N} \cdot \text{s}^2}$. Although it is at times convenient to regard $k_{2}^{\text{mks}}$ to have the dimensions $(\text{N}/\text{A}^2)$, in our approach this combination is actually dimensionless. The constant $k_{2}^{\text{mks}}$ is often given the expression

$$k_{2}^{\text{mks}} = \frac{\mu_0}{2\pi} = 2 \times \frac{\mu_0}{4\pi}$$

(1.14)

in which $\mu_0$ is defined to have the value $4\pi \times 10^{-7} \text{ N/A}^2$ and is called the permeability of free space. In mks units, the force per unit length between two parallel wires then has the expression

$$F_{II'} = \frac{\mu_0}{4\pi} \frac{2II'}{s}$$

(1.15)

The remaining constant $k_{1}^{\text{mks}}$ is determined empirically and has the value

$$k_{1}^{\text{mks}} = (8.98755 \pm 0.00002) \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$$

(1.16)

\(^9\)At the moment this $c$ is different from the $c$ in Eq. (1.9), though the identity of the two $c$’s will be established in the last sentence of this paragraph.
The value $9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$ is commonly employed unless extreme accuracy is needed. It is also common to write $k_{\text{mks}}^1$ in terms of another constant $\epsilon_0$ as follows:

$$k_{\text{mks}}^1 = \frac{1}{4\pi \epsilon_0}$$  \hspace{1cm} (1.17)

in which $\epsilon_0 = (8.85418 \pm 0.00002) \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$ is called the permittivity of free space. Coulomb’s Law then assumes the form

$$F_{qq'} = \frac{1}{4\pi \epsilon_0} \frac{qq'}{d^2}$$  \hspace{1cm} (1.18)

The word rationalized in the name of this system of units refers to the deliberate placement of an explicit factor of $4\pi$ in Coulomb’s Law. By this means, the $4\pi$ is suppressed in several subsequent relationships that are more commonly used than Coulomb’s Law.

### 1.6.4 Other Systems of Units

Three other systems of units deserve brief mention. In the Gaussian system, the cgs units of length, mass, and time are supplemented with the esu of charge ($k_G^1 = 1$) and the esu of current ($k_G^2 = 1$); $k_G^2$ is then determined empirically to have the value $2/c^2$. Although Gaussian units therefore are at this stage identical with cgs-esu, differences will subsequently appear in the way in which other quantities, particularly the magnetic field, are defined in the two systems. The rationalized counterpart of the Gaussian system is called the Heaviside-Lorentz system;\textsuperscript{10,11} it uses cgs units of length, mass, and time and the values $k_{\text{HL}}^1 = 1/4\pi$ and $k_{\text{HL}}^3 = 1$ to define units of charge and current; $k_{\text{HL}}^2$ is then determined experimentally to have the value $1/2\pi c^2$. Finally, in the natural system, adopted principally by nuclear physicists, the velocity of light and a few other fundamental constants are given the value unity.

### 1.6.5 Summary

The forms assumed by Eqs. (1.6)–(1.8) in the several systems of units described above are summarized in Table 1.1. Although we shall continue to indicate the form of subsequent definitions in each of these sets of units, this text will use the rationalized mks system of units.

### 1.6.6 Conversion of Units

The need to convert from one system of units to another accompanies the common use of more than one system. We shall illustrate the technique for obtaining conversion factors by working out the factors relating cgs-esu and cgs-emu. Consider two point charges separated by a distance $d$. In cgs-esu, the force between these two charges is given by

$$F_{qq'}(\text{dyn}) = \frac{q_{\text{esu}} q'_{\text{esu}}}{d_{\text{cm}}^2}$$  \hspace{1cm} (1.19)

\textsuperscript{10} English electrical engineer, mathematician, and physicist Oliver Heaviside, b. 18 May 1850 in Camden Town, London, United Kingdom; d. 3 February 1925 in Torquay, United Kingdom.

\textsuperscript{11} Dutch physicist Hendrik Antoon Lorentz, b. 18 July 1853 in Arnhem, Netherlands; d 4 February 1928 in Haarlem, Netherlands.
Table 1.1: The form of Eqs. (1.6)–(1.8) in Several Common Systems of Units. In this table \( c = (2.997925 \pm 0.000003) \times 10^{10} \text{ cm/s} \) and must be expressed numerically in these units wherever it occurs in the table. Further,

\[
\begin{align*}
\mu_0 &= 4\pi \times 10^{-7} \text{ N/A}^2 \\
\epsilon_0 &= (8.85418 \pm 0.00002) \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2 \\
\frac{1}{4\pi\epsilon_0} &= (8.98755 \pm 0.00002) \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>System</th>
<th>Coulomb’s Parallel Current Interaction</th>
<th>Current-Charge Connection</th>
</tr>
</thead>
<tbody>
<tr>
<td>cgs-esu</td>
<td>( \frac{q'q}{d^2} ), ( \frac{1}{c^2} \frac{2II'}{s} )</td>
<td>( I = \frac{dq}{dt} )</td>
</tr>
<tr>
<td>cgs-emu</td>
<td>( \frac{c^2q'q}{d^2} ), ( \frac{2II'}{s} )</td>
<td>( I = \frac{dq}{dt} )</td>
</tr>
<tr>
<td>Rationalized-mks</td>
<td>( \frac{1}{4\pi\epsilon_0} \frac{q'q}{d^2} ), ( \frac{\mu_0 2II'}{4\pi s} )</td>
<td>( I = \frac{dq}{dt} )</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( \frac{q'q}{d^2} ), ( \frac{1}{c^2} \frac{2II'}{s} )</td>
<td>( I = \frac{dq}{dt} )</td>
</tr>
<tr>
<td>Heaviside-Lorentz</td>
<td>( \frac{1}{4\pi} \frac{q'q}{d^2} ), ( \frac{1}{4\pi c^2} \frac{2II'}{s} )</td>
<td>( I = \frac{dq}{dt} )</td>
</tr>
</tbody>
</table>

and in cgs-emu this same force is given by

\[
F_{qq'}(\text{dyn}) = c_{\text{cm}}^2 \frac{q_{\text{emu}}q'_{\text{emu}}}{d_{\text{cm}}^2} 
\]  

(1.20)

where units have been indicated explicitly and \( q_{\text{esu}} \) and \( q_{\text{emu}} \), for example, express the magnitude of the same physical charge in the two sets of units. Since the forces and the distances in Eqs. (1.19) and (1.20) are numerically the same, we infer the numerical equality

\[
q_{\text{esu}}q'_{\text{esu}} = c_{\text{cm}}^2/s q_{\text{emu}}q'_{\text{emu}} 
\]  

(1.21)

or, since \( q_{\text{esu}} \) is proportional to \( q_{\text{emu}} \), we find that

\[
q_{\text{esu}} = c_{\text{cm}}/s q_{\text{emu}} 
\]  

(1.22)

Similarly, starting with two currents and the equations

\[
F_{II'}(\text{dyn/cm}) = \frac{2I_{\text{esu}}I'_{\text{esu}}}{c_{\text{cm}}^2 s_{\text{cm}}} = \frac{2I_{\text{emu}}I'_{\text{emu}}}{s_{\text{cm}}} 
\]  

(1.23)

we find that

\[
I_{\text{esu}} = c_{\text{cm}}/s I_{\text{emu}} 
\]  

(1.24)
Table 1.2: Conversion Factors. In this table, the number 3 arises from the speed of light and, in accurate work, should be replaced by 2.997925.

\[
\begin{align*}
1 \ \{ \text{statcoulomb} \} & = \frac{1}{3} \times 10^{-10} \ \{ \text{abcoulomb} \} = \frac{1}{3} \times 10^{-9} \ \{ \text{coulomb} \} \\
1 \ \{ \text{abampere} \} & = 10 \ \{ \text{coulomb} \} = 3 \times 10^{10} \ \{ \text{statcoulomb} \} \\
1 \ \{ \text{ampere} \} & = 3 \times 10^9 \ \{ \text{statcoulomb} \} = 0.1 \ \{ \text{abampere} \}
\end{align*}
\]

Equivalently, we obtain \(I_{\text{esu}}\) by the chain of argument expressed in the equation

\[
I_{\text{esu}} = \frac{dq_{\text{esu}}}{dt_s} = c_{\text{cm/s}} \frac{dq_{\text{emu}}}{dt_s} = c_{\text{cm/s}} I_{\text{emu}}
\]  

(1.25)

Equations such as Eqs. (1.22) and (1.24) frequently generate confusion. These equations are numerical equalities. Equation (1.22), for example, states that the numerical value of some particular charge in esu is equal to \(c_{\text{cm/s}} \approx 3 \times 10^{10}\) in numerical value times the numerical value of the same charge in emu. Thus, a charge of 1 abC is physically the same charge as a charge of about \(3 \times 10^{10}\) statC. The temptation to read this and similar equations the other way—i.e., \(1 \ \text{esu} = 3 \times 10^{10} \ \text{emu}\)—must be resisted; it is incorrect.

Conversion factors relating esu, emu, and mks units of charge and current are summarized in Table 1.2.

### 1.7 Recent Changes in Definitions of Units

At the time the first edition of this book was published in 1975, the meter was officially but arbitrarily defined as 1,650,763.73 times the wavelength in vacuum of the orange-red radiation of Kr\(^{86}\), and the second was officially but arbitrarily defined as the duration of exactly 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of Cs\(^{133}\) at 0 K.\(^{12}\) Presupposing these definitions in Section 1.6, we defined the relationship between charge and current by arbitrarily—in all sets of units—setting \(k_3 = 1\) in Eq. (1.8). Then, depending on the particular set of units to be defined, we arbitrarily assigned a convenient value either to \(k_1\) in Eq. (1.6) or to \(k_2\) in Eq. (1.7). Finally (because no more flexibility remained), we determined the one remaining constant \((k_2\ or\ k_1)\) experimentally, somewhat to our surprise finding that the measured speed of light was embedded somehow in one or the other—or perhaps jointly in both—of these constants.

\(^{12}\)This definition of the meter was officially adopted in 1960; this definition of the second was officially adopted in 1967, though the stipulation of the temperature was not added until 1997. Both of these definitions were chosen so that the resulting sizes of the quantities defined coincided with their sizes under the definitions that the new superseded.
In 1983, eight years after the publication of the first edition of this book, the meter was officially redefined to be the distance light travels in exactly $1/299,792,458$ s. (The second retained the definition described in the previous paragraph.) At that time, the speed of light became an arbitrarily defined and exact number—$299,792,458$ m/s. Because the (now defined) speed of light is embedded directly or indirectly in $k_1$ and $k_2$ (P1.8), we must take a different perspective on the flexibilities in the definition of the units. Specifically,

- For cgs-esu, we accept the value $k_{esu}^2 = 2/c^2$ in Eq. (1.7) as dictated by the defined speed of light $c$ to define current, and we set $k_{esu}^3 = 1$ in Eq. (1.8) to define charge from current and time. In this case, $k_{esu}^1$ in Eq. (1.6) is no longer arbitrary (because charge, force, and distance are already defined) and must be measured, with the expectation that its value will turn out to be 1.

- For cgs-emu, we accept the value $k_{emu}^1 = c^2$ in Eq. (1.6) as dictated by the defined speed of light $c$ to define charge, and we set $k_{emu}^3 = 1$ in Eq. (1.8) to define current from charge and time. In this case, $k_{emu}^2$ in Eq. (1.7) is no longer arbitrary (because current, force, and distance are already defined) and must be measured, with the expectation that its value will turn out to be 1.

- For rationalized mks units, we arbitrarily set $k_{mks}^2 = \mu_0/2\pi$ in Eq. (1.7) to $2 \times 10^{-7}$ to define current, and we set $k_{mks}^3 = 1$ in Eq. (1.8) to define charge from current and time. In this case, $k_{mks}^1 = 1/4\pi\epsilon_0$ in Eq. (1.6) is no longer arbitrary (because current, force, and distance are already defined) and must be measured, with the expectation that its value will turn out to be such $\mu_0\epsilon_0 = 1/c^2$.

We must also mention, however, that further adjustments in the definition of the seven fundamental units in the SI system are under consideration, the aim being to make all fundamental quantities defined in terms of natural constants. Presently (2015), the second and the meter (and the candela) are already so defined. A proposal that is heading for adoption at the next conference of the International Committee for Weights and Measures in 2018 is based on setting the seven values shown in Table 1.3. Once these definitions have been adopted, we will set $k_3$ in Eq. (1.8) to 1 to define the statcoulomb (cgs-esu), the abampere (cgs-emu), and the coulomb (rationalized mks units), but both $k_1$ and $k_2$ will need to be measured in all three cases.

PROBLEMS

P1.1. Write a paragraph outlining a procedure for confirming property (2) of Section 1.4 before a quantitative measure of current is available.

P1.2. Show that the constant $c$ in Eq. (1.10) and the constant $c$ in Eq. (1.13) both have the dimensions of velocity.

P1.3. At one time, the ampere was defined by an experiment in electrolysis that involved measuring the mass of silver deposited per unit time in a standard silver voltameter. If we had adopted this definition, we would properly view the ampere as a fundamental quantity. Explain how this change affects the status of Eq. (1.15) and of the constant $k_{mks}^2$. The value assigned to $k_{mks}^2$ in the text was, of course, chosen to make the present definition of

---

13 The fundamental units are second, meter, kilogram, ampere, kelvin, mole, and candela, though we here need be concerned only about the first four.
1.7. RECENT CHANGES IN DEFINITIONS OF UNITS

Table 1.3: Proposed declared values for several physical constants. The values in this table are copied from the document dated 16 December 2013 that comes up when the link “Draft Chapter 2” near the bottom of the page at the URL http://www.npl.co.uk/reference/measurement-units/proposed-si-changes/ is selected. Subsequent discussion of the pieces remaining unsettled as of the quoted date can be found at links at the URL http://www.npl.co.uk/reference/measurement-units/.

- Set the unperturbed ground state hyperfine splitting frequency of $^{133}$Cs to exactly 9,192,631,770 Hz, which has already been adopted to define the second, s—the SI unit of time.
- Set the speed of light to exactly 299,792,458 m/s, which defines the meter, m—the SI unit of length.
- Set Planck’s constant (Max Planck, see footnote 2 in Section E.1) to exactly $6.62606957 \times 10^{-34}$ kg m$^2$ s$^{-1}$, which is proposed—though, as of 14 February 2014, not yet adopted—to define the kilogram, kg—the SI unit of mass.
- Set the charge on the electron to exactly $1.602176565 \times 10^{-19}$ A s, which is proposed as the definition of the ampere, A—the SI unit of current.
- Set Boltzmann’s constant (Ludwig Boltzmann, see footnote 17, Chapter 8, page 239) to $1.3806488 \times 10^{23}$ kg m$^2$ K$^{-1}$ s$^{-2}$, which is proposed as the definition of the kelvin, K—the SI unit of (thermodynamic) temperature.
- Set Avogadro’s number (Amadeo Avogadro, see footnote 3 in Section E.1) to exactly $6.02214129 \times 10^{23}$ mol$^{-1}$, which is proposed as the definition of the mole, mol—the SI unit of the amount of substance of an elementary entity (atom, molecule, ion, electron, ...).
- Set the luminous efficacy of monochromatic radiation of frequency $540 \times 10^{12}$ Hz to exactly 683 cd sr s$^{-3}$ kg$^{-1}$ m$^{-2}$, which has already been adopted to define the candela, cd—the SI unit of luminous intensity.

...the ampere agree with the earlier definition, so the numerical value of $k_{\text{mks}}^2$ is not changed by the change in definition.

P1.4. An alternative way to state the definition of the ampere is as follows: The ampere is that constant current which, when present in two parallel wires separated by 1 m, gives rise to a force of $2 \times 10^{-7}$ N/m between the wires. Formulate similar definitions for the statcoulomb and the abampere, which are defined by Eqs. (1.6) and (1.7), respectively, and for the coulomb, the statampere, and the abcoulomb, which are defined by Eq. (1.8).

P1.5. Show that the ampere is exactly one-tenth the abampere.

P1.6. (a) Derive the conversion factor(s) between esu and mks units of charge and current. (b) Derive the conversion factor(s) between emu and mks units of charge and current. (c) The charge on the proton is $1.60210 \times 10^{-19}$ C. Determine this charge in statcoulombs and in abcoulombs.

P1.7. In what are called modified Gaussian units, charges are measured in statcoulombs ($k_1 = 1$) and currents are measured in abamperes ($k_2 = 1$). Determine the value of $k_3$ in modified Gaussian units and write out the corresponding entries in Table 1.1.

P1.8. Let an unused set of units—call them Cookian units—be defined as follows:
CHAPTER 1. QUANTIFYING CHARGE AND CURRENT

Length: 1 cookmeter = \( b \) cm

Time: 1 cooksecond = \( f \) s

Force: 1 cookyne = \( a \) dyn

Charge: Set \( k_1 = K_1 \) in Eq. (1.6), defining the cookcoulomb

Current: Set \( k_2 = 2K_2 \) in Eq. (1.7), defining the coookampere

where \( a, b, f, K_1, \) and \( K_2 \) are arbitrary but fixed. (Note, for example, that Cookian units reduce to cgs-esu if \( a = b = f = K_1 = 1 \) and \( K_2 = 1/c^2 \), with \( c \) in cm/s.) With everything (including the speed of light) expressed in Cookian units, show that

\[
\frac{dq}{dt} = \sqrt{\frac{K_2c^2}{K_1}} I
\]

and infer (a) that any system of units in which \( I = dq/dt \) must have \( K_1/K_2 = c^2 \) and (b) that, in mks units, \( \mu_0\varepsilon_0 = 1/c^2 \).

P1.9. What (numerically) is the ratio of the electrostatic force of repulsion to the gravitational force of attraction between two protons?

P1.10. (a) Given two particles with mass \( m \) and charge \( q \) suspended from a single point by strings of length \( \ell \), show that the angle \( \theta \) between each string and the vertical satisfies

\[
tan \theta \sin^2 \theta = \left(\frac{q}{q_0}\right)^2,
\]

where (in mks units) \( q_0^2 = 16\pi\mu_0\varepsilon_0mg\ell^2 \) and \( g \) is the acceleration of gravity. (b) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to obtain a graph of \( q/q_0 \) versus \( \theta \). (c) What deflection is produced by a charge of strength \( q_0 \)? Hint: Estimate the angle from your graph but then, using a short computer program in a familiar language (FORTRAN, C, ...) that inputs an angle and then calculates and prints out the value of \( q/q_0 \), guess the solution, try it, and then refine your guess until you have found the solution to, say, four significant figures. (d) What (numerically) is \( q_0 \) for a reasonable apparatus, say \( m = 0.5 \) g and \( \ell = 20 \) cm? Compare your answer with the charge on the proton. Optional: Suppose a charge has strength \( q \) in coulombs and strength \( Q \) in foolcoulombs, the latter value being measured by making a second, equal charge, placing each charge on a separate small sphere of some standard mass \( m \), suspending the spheres from strings of some standard length \( \ell \), and calculating \( Q \) from the measured angle \( \theta \) (in radians) by the defining equation \( Q = 2\theta/\pi \). Show that no charge can have a magnitude greater than 1 foolcoulomb, and find the (very awkward) form assumed by Coulomb’s Law if charges are measured in foolcoulombs and everything else is measured in mks units.

P1.11. Two uncharged aluminum spheres, each having a mass of 1 g, are suspended so their centers are 10 cm apart. By observing the parallelism of the supporting strings, an experimenter deduces that any forces of interaction between these spheres are weaker than about \( 10^{-5} \) N—actually a fairly coarse measurement. What maximum fraction of the total number of electrons in one sphere could be transferred to the other without upsetting this result? What does your answer imply about the precision with which apparently neutral matter in fact contains equal amounts of positive and negative charges? Note: The atomic number of aluminum is 13; its atomic weight is 27.

P1.12. A wire is free to slide without friction on rails that make an angle \( \theta \) with the horizontal. It carries a current \( I \) opposite in direction but equal in magnitude to the current carried by an infinitely long parallel wire at the bottom of the rails. If the sliding wire has length \( \ell \) and mass \( m \), find the separation \( s \) of the two wires at which the sliding wire will be in equilibrium. (See Fig. 1.5.)
REFERENCES


Chapter 2

Specifying Arbitrary Distributions: Charge and Current Densities

Much of our subsequent development is concerned with the properties and the behavior of various spatial distributions of (possibly moving) charge. We therefore need a means to specify the “state” of a general charge distribution. For the simplest distributions, which contain a small number of discrete and widely separated particles, the state is conveniently and satisfactorily specified by giving the charge, position, and velocity of each particle. Many of the distributions with which we must deal, however, “fill” some region of space by containing a very large number—say $10^{23}$—of very small, densely packed, erratically moving, charged particles. Merely printing the charges, positions, and velocities of this many particles would keep a high-speed line printer on a computer busy for about $10^{14}$ years! Clearly, for describing space-filling distributions, we must replace charges, positions, and velocities of individual particles with more appropriate new concepts. The objectives of this chapter are (1) to define these concepts, called the charge density and the current density, and (2) to explore the mathematical apparatus that is useful for manipulating these concepts analytically.

We shall introduce the macroscopic description in terms of charge and current densities by relating it to the underlying microscopic description in terms of individual particles. The success of this transition from a microscopic to a macroscopic model is a consequence of the large differences that exist between the microscopic and the macroscopic scales of time and distance. On the microscopic scale, for example, macroscopic observations are extremely sluggish; at best they are able to resolve time intervals (say $\approx 10^{-4}$ s) that are something like $10^{11}$ times as long as the period ($\approx 10^{-15}$ s) of the thermal oscillations of the microscopic particles. Again, on the microscopic scale, macroscopic observations are extremely coarse; at best they are able to resolve spatial separations (say $\approx 10^{-2}$ cm) that are something like $10^6$ times as large as the dimensions ($\approx 10^{-8}$ cm) of the space occupied by a single particle. Thus, what a macroscopic measurement detects as an instant of time is still long enough to include very many—perhaps $10^9$—microscopic fluctuations; what a macroscopic measurement detects as a point in space still occupies sufficient volume to contain very many—perhaps $10^{15}$—microscopic particles. Instead of responding directly to the microscopic details, macroscopic measurements therefore average these microscopic details over macroscopically small temporal and spatial intervals that are still large enough macroscopically to include many fluctuations and many particles. The theoretical concepts
appropriate to the description of macroscopic charge distributions are therefore averages of the microscopic concepts, and we shall make these averages more explicit in the next two sections.

The transition from microscopic to macroscopic descriptive concepts carries with it a change in our mental image of these charge distributions. In effect, the microscopic model of erratically moving, small particles is replaced by a more sedate macroscopic model in which we ignore not only the erratic motion of each particle but also the elemental discreteness of matter and of charge. Insofar as we think of them at all, particles are effectively at rest or perhaps in motion with some smoothly varying drift velocity. Further, matter and charge are spread continuously throughout the region occupied by the distribution. The result is a model of a macroscopic charge distribution that, except for the charge it carries with it, corresponds exactly with the classical model of a fluid. We therefore precede the definition of charge and current densities by introducing two fields that have been found useful for the macroscopic description of fluids, whether charged or not. Classify each microscopic particle in the fluid according to its mass and charge, particles of type \( a \) having mass \( m_a \) and charge \( q_a \). The first field, called the particle density field \( n^{(a)}(r, t) \), measures at time \( t \) the number of particles of type \( a \) per unit volume in a small volume \( \Delta V \) centered on the point \( r \); it is defined formally by

\[
n^{(a)}(r, t) = \lim_{\Delta V \to 0} \frac{\Delta N^{(a)}(t)}{\Delta V}
\]

where \( \Delta N^{(a)}(t) \) is the number of particles of type \( a \) in the volume \( \Delta V \) at time \( t \), and (both here and hereafter) the limit \( \Delta V \to 0 \) carries the proviso that \( \Delta V \) becomes macroscopically small but remains microscopically large. The second field, called the velocity field, gives the typical drift velocity of a particle of type \( a \) that happens at time \( t \) to be in the volume \( \Delta V \); it is defined formally by

\[
v^{(a)}(r, t) = \lim_{\Delta V \to 0} \frac{1}{\Delta N^{(a)}(t)} \sum_i v^{(a)}_i(t)
\]

where \( v^{(a)}_i(t) \) is the drift velocity of the \( i \)-th particle of type \( a \) in the volume \( \Delta V \) at time \( t \). Because these definitions involve macroscopically small but microscopically large temporal and spatial intervals, both \( n^{(a)}(r, t) \) and \( v^{(a)}(r, t) \) are smoothly and slowly varying functions of \( r \) and \( t \) and, in particular, may even be constant in time and/or in space. Although the state of a charged fluid is more directly specified by the charge and current densities to be introduced in the next sections, we shall occasionally use \( n^{(a)} \) and \( v^{(a)} \) to link charge and current densities to the underlying particulate model.

\[1\] Strictly, \( \Delta N^{(a)}(t) \) is the (time) average of the number of particles of type \( a \) in the volume \( \Delta V \), the average being evaluated over a macroscopically small but microscopically large time interval centered at time \( t \). Subsequent statements similar to the one to which this footnote applies must be similarly interpreted.
2.1. CHARGE DENSITY

PROBLEMS

P2.1. A high-speed line printer may print on the order of 1200 lines per minute. Suppose each line can contain the charge, position, and velocity for two particles. (Remember that position and velocity are vectors.) (a) How long would it take to print out the state of a system containing \(10^{23}\) particles? (b) How thick would be the resulting pile of paper?

P2.2. Estimate the period associated with the erratic oscillations of a microscopic particle. Hint: A typical thermal speed, say at room temperature, can be estimated from kinetic theory. How long does it take an electron (proton) to travel a typical microscopic distance, say \(10^{-8}\) cm, at this speed?

2.1 Charge Density

Consistent with the macroscopic model developed in the above introductory paragraphs, we ignore the elemental discreteness of electric charges. In addition to being concentrated on small objects (point charges), charge may then be distributed more or less smoothly throughout volumes, over surfaces, and along lines. To facilitate a macroscopic description of each type of distribution, we therefore introduce three macroscopic charge densities: (1) a volume charge density, having mks units of coulombs per cubic meter (C/m\(^3\)) and defined by

\[
\rho(r, t) = \lim_{\Delta V \to 0} \frac{\Delta q(t)}{\Delta V}
\]

(2.3)

(2) a surface charge density, having mks units of coulombs per square meter (C/m\(^2\)) and defined by

\[
\sigma(r, t) = \lim_{\Delta S \to 0} \frac{\Delta q(t)}{\Delta S}
\]

(2.4)

and (3) a linear charge density, having mks units of coulombs per meter (C/m) and defined by

\[
\lambda(r, t) = \lim_{\Delta l \to 0} \frac{\Delta q(t)}{\Delta l}
\]

(2.5)

where \(\Delta q(t)\) is successively the charge in the volume \(\Delta V\), on the surface of area \(\Delta S\), and on the line of length \(\Delta l\) at time \(t\). In each case, the point \(r\) must lie in the volume, surface, or line element involved, and the element must become small compared to macroscopic dimensions while remaining large compared to microscopic dimensions. When these charge densities can be meaningfully defined (i.e., for space-filling macroscopic distributions), all three are smooth functions of \(r\) and \(t\). A distribution is said to be static if the corresponding charge densities do not depend on time and uniform if the charge densities do not depend on the spatial coordinates.

Among other things, knowledge of the appropriate charge densities permits a calculation of how much charge \(Q(t)\) is present at time \(t\) in any portion of the distribution. We
simply break the portion of the distribution of interest into appropriate small elements, distinguished by an index \( i \), and add up the contributions \( \Delta q_i \) from each element, passing to the limit as the extent of all elements becomes small and, correspondingly, the total number of elements becomes large. If the distribution is described by a volume charge density, for example, we find that

\[
Q(t) = \lim_{\Delta V_i \to 0} \sum_i \Delta q_i(t) = \lim_{\Delta V_i \to 0} \sum_i \rho(r, t) \Delta V_i = \int \rho(r, t) \, dv
\]

(2.6)

where \( \Delta V_i \) is the volume of the \( i \)-th element, \( r_i \) is a point in the \( i \)-th element, \( dv \) is an infinitesimal volume element in a convenient coordinate system [see Eqs. (0.22)–(0.24)], and the volume integral extends over the portion of the charge distribution of interest (which may, of course, be the entire distribution). By a similar limiting process, we find for surface and line distributions that

\[
Q(t) = \int \sigma(r, t) \, dS \quad ; \quad Q(t) = \int \lambda(r, t) \, dl
\]

(2.7)

where \( dS \) and \( dl \) are the area and length of infinitesimal surface and line elements, respectively, and the integrals again extend over a portion (or perhaps over all) of the distribution. Here and subsequently, we shall combine integrals like those in Eqs. (2.6)–(2.7) by introducing an infinitesimal element of charge \( dq \) to stand for \( \rho \, dv \), \( \sigma \, dS \), or \( \lambda \, dl \) as appropriate. Then, the integrals all have the form

\[
Q = \int dq
\]

(2.8)

where explicit indication of spatial and temporal dependencies has been suppressed. Although we shall make no use of it, it is in fact possible to introduce a mathematical function, known as the \textit{Dirac delta function},\(^2\) in terms of which all types of charge distribution, including point charges, can be formally described with volume charge densities only. With this function, Eq. (2.6) alone is sufficient to cover all cases.

Finally, we relate the volume charge density to the particle density field defined in Eq. (2.1). Suppose the distribution of interest contains particles of several different types and let particles of type \( a \) with charge \( q_a \) be distributed in accordance with the particle density field \( n^{(a)}(r, t) \). Then the total charge \( \Delta q(t) \) at time \( t \) in a volume \( \Delta V \) centered at the point \( r \) is

\[
\Delta q(t) = \sum_a q_a \, n^{(a)}(r, t) \, \Delta V
\]

(2.9)

and Eq. (2.3) gives

\[
\rho(r, t) = \sum_a q_a \, n^{(a)}(r, t)
\]

(2.10)

The \( a \)-th term in this sum, of course, expresses the contribution of particles of type \( a \) to the total charge density.

---

\(^2\)English physicist Paul Adrien Maurice Dirac, b. 8 August 1902 in Bristol, Gloucestershire, England; d. 20 October 1984 in Tallahassee, Florida.
2.2 CURRENT DENSITY

Figure 2.1: View of a region of space filled with moving charged particles. An imaginary
surface is shown in the region.

PROBLEMS

P2.3. Let a total charge $Q$ be uniformly distributed in succession throughout the volume of a
sphere of radius $R$, on the surface of the same sphere, and along the perimeter of a circle
of radius $R$. What is the charge density resulting in each case?

P2.4. According to quantum mechanics, the electron in the ground state of the hydrogen atom
is characterized by a charge density

$$\rho(r) = -\frac{q}{\pi a_0^3} e^{-2r/a_0}$$

where $a_0$ is the Bohr radius. (a) Show that the total charge in this distribution is $-q$.
(b) Using tools like IDL, MATLAB, OCTAVE, PYTHON, MAXIMA, MAPLE, and/or
Mathematica, determine the fraction of this charge that lies within a sphere of radius $R$
centered on the nucleus and obtain a careful graph of this fraction as a function of $R/a_0$.

2.2 Current Density

To define a current density for specifying the motion of a charge distribution, we must
first extend Eq. (1.8) so that the meaning of current is unambiguous even when the flow
of charge does not follow the path defined by a wire. Imagine that a region of space is
occupied by a large number of charged particles, each particle moving about within this
region in some smoothly varying way. (We ignore again the erratic microscopic motion.)
Let some of these particles be positively charged and others negatively charged. Now, place
an imaginary surface somewhere in the region of space occupied by these charges (Fig. 2.1)
and let this surface be fixed in position. Charges of both signs move in both directions
across this surface. In any given small (but macroscopic) interval of time $\Delta T$ there may be
a net transfer of charge through the surface from one side to the other. Let the net charge
transported across the surface be evaluated as follows: Choose (arbitrarily) one direction
through the surface as the positive direction. Then let positive charge moving in the positive
direction and negative charge moving in the negative direction make positive contributions
to the charge transported, and let positive charge moving in the negative direction and
negative charge moving in the positive direction make negative contributions. Finally, identify the net charge transported $\Delta Q$ as the algebraic sum of these several contributions. Thus, a positive charge transport results when a net positive charge is transported in the positive direction across the surface; a negative charge transport results when a net positive charge is transported across the surface in the negative direction; etc. If the net charge $\Delta Q$ is transported across the surface in an elapsed time $\Delta t$, then [consistent with Eq. (1.8)] the average current $\bar{I}$ flowing across the surface in this time interval is given by

$$\bar{I} = \frac{\Delta Q}{\Delta t} \tag{2.11}$$

where we have set $k_3 = 1$, thereby fixing at least a part of the system of units. The instantaneous current $I(t)$ flowing across the surface at a given time $t$ is then defined by allowing $\Delta t$ to shrink to zero about the time instant $t$, i.e., by

$$I(t) = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt} \tag{2.12}$$

where the expression as a derivative is appropriate only if $Q$ is interpreted as the total charge that has arrived at time $t$ on the positive side of the surface by a route that passes through the surface. As always, the limit means more specifically that $\Delta t$ must be made macroscopically small while simultaneously remaining microscopically large. With the sign conventions established above for measuring $\Delta Q$, the current across the surface will be positive if positive charge is transported in the positive direction across the surface, and so on. Convention takes the direction of current flow across the surface to be the direction of the equivalent positive charge transport.

PROBLEMS

P2.5. Suppose a current in a wire is carried solely by electrons. How many electrons are transported in 1 s past a point in this wire if the current is 0.1 A? What conclusion do you draw about the legitimacy of regarding a (macroscopic) current as a flowing charged fluid?

P2.6. (a) A total charge $Q$ is distributed uniformly on the perimeter of a circular ring of radius $R$. The ring is then set into rotation about its axis with angular speed $\omega$ (rad/s). Determine the current represented by the rotating ring. (b) Suppose now the charge $Q$ is concentrated on a particle that moves in a circle of radius $\tilde{R}$ with angular speed $\omega$. Under what conditions will the result of part (a) also give the current represented by the circulating particle?

Although the current across a surface is very often the answer to practical questions, a more convenient description of moving charge is formulated about a quantity known as the current density. We shall consider only the current density appropriate to distributions occupying some three-dimensional volume; a current density suited to the description of surface distributions of charge is explored in P2.26.

To define the volume current density, we begin by examining the state of affairs at time $t$ within a small volume $\Delta V$ surrounding the point $\mathbf{r}$ in the arbitrary charge distribution of Fig. 2.1. A view of this volume is shown in Fig. 2.2. If there is any net transport of charge past the point $\mathbf{r}$, that transport, of course, occurs in some well-defined direction and we
Figure 2.2: View of a small volume about the point $r$ in Fig. 2.1. Individual charges are not shown.

introduce a unit vector $\hat{n}(r,t)$, which may in general depend on $r$ and $t$, having the direction of this transport. Further, let us introduce a small plane surface of area $\Delta S$ oriented with its plane perpendicular to $\hat{n}$ and positioned so that some point on $\Delta S$ is at $r$. Finally, let $\Delta I(t)$, determined as described in an earlier paragraph, be the current crossing $\Delta S$ at time $t$. The current density $\mathbf{J}(r,t)$ at point $r$ and time $t$ in the illustrated distribution is then defined by

$$\mathbf{J}(r,t) = \lim_{\Delta S \to 0} \frac{\Delta I(t)}{\Delta S} \hat{n}(r,t)$$

and is a vector quantity whose direction coincides with the direction of net (positive) charge transport and whose magnitude is the rate at which charge is transported (i.e., the current) across a surface of unit area oriented perpendicular to the direction of charge transport. The mks units of $\mathbf{J}$ are therefore amperes per square meter ($\text{A/m}^2$). This current density characterizes the macroscopic motion of the charge distribution and knowledge of both $\mathbf{J}(r,t)$ and the (volume) charge density $\rho(r,t)$ at some time $t$ and at all points $r$ in an arbitrary (volume) charge distribution fully determines the state of that charge distribution at time $t$.

PROBLEMS

P2.7. (a) Show that the current density in a wire carrying a total current $I$ uniformly distributed over a cross-sectional area $S$ is $\mathbf{J} = (I/S)\hat{t}$, where $\hat{t}$ is a unit vector tangent to the wire and in the direction of the current. (b) The product $I\,dl$, where $dl = d\ell \hat{t}$ is a vector representing an infinitesimal element of the wire having length $d\ell$, will occur in our study of magnetism. Show that $I\,dl = J \,dv$, where $dv$ is the volume of the element represented by $dl$.

P2.8. A point radioactive source located at the origin emits $N$ particles per second, each having charge $q$. If these particles are emitted uniformly in all directions, determine the current density at the point $r$. Hint: Express the answer in spherical coordinates.
Figure 2.3: Geometry for determining the current across a surface whose plane is not perpendicular to the current density.

P2.9. A total charge $Q$ is distributed uniformly throughout the volume of a sphere of radius $a$. The sphere is then set into rotation with (constant) angular velocity $\omega$ (rad/s) about a diameter, which it is convenient to take as the axis of a cylindrical coordinate system $(r, \phi, z)$. Express the current density in the region $|r| \leq a$ in terms of cylindrical coordinates and unit vectors.

Among other things, knowledge of the current density permits a calculation of the current flowing across an arbitrary surface placed somewhere in the charge distribution. We simply break the surface into appropriate small elements, calculate the current across each element, sum the individual contributions, and then let the size of the elements become macroscopically small while remaining microscopically large. First, however, we must find the current across a small plane surface, e.g., the surface of area $\Delta S$ in Fig. 2.3, whose plane is not perpendicular to $\mathbf{J}$. Let this surface be small enough to fit within a small volume throughout which at any instant of time $t$ the current density can be regarded to have the constant value $\mathbf{J}(r, t)$, where $r$ locates some fixed point on the surface. Then, because $\mathbf{J}(r, t)$ gives the direction of the net transport of charge, no (net) charge is transported across any surface whose plane is parallel to $\mathbf{J}$, and the current across $\Delta S$ is the same as the current across the auxiliary surface of area $\Delta S'$ obtained by projecting $\Delta S$ onto a plane perpendicular to $\mathbf{J}$ and passing through $r$. But the current across the auxiliary surface is just $|\mathbf{J}| \Delta S'$. Thus, since $\Delta S' = \Delta S \cos \theta$, where $\theta$ is the angle between $\mathbf{J}$ and the normal to the surface $\Delta S$, we conclude that the current $\Delta I$ across $\Delta S$ is given by

$$\Delta I = |\mathbf{J}| \Delta S \cos \theta = \mathbf{J} \cdot \Delta \mathbf{S}$$

(2.14)

where the vector $\Delta \mathbf{S}$ represents the surface of area $\Delta S$ and by convention is assigned a direction perpendicular to the plane of the surface and a magnitude equal to the area of the surface represented. Whichever of the two possible directions is assigned to $\Delta S$, Eq. (2.14) will give a positive value for $\Delta I$ if a net positive charge is transported across the surface in
2.2. CURRENT DENSITY

the direction of $\Delta S$ and a negative value if a net positive charge is transported across the surface in a direction opposite to $\Delta S$. In principle, either direction may be assigned to $\Delta S$; in practice—at least in some special cases—particular choices have become conventional, and we shall introduce these conventions as we encounter them.

We are now ready to determine the current flowing at time $t$ across an arbitrary surface placed in the charge distribution described by the current density $J(r, t)$. Let the surface be divided into small elements, each of which can be regarded as plane and over each of which the current density is approximately constant, and let the $i$-th element of the surface be represented by a vector $\Delta S_i$ defined as in the previous paragraph with the added proviso that all elements $\Delta S_i$ be assigned vectors in the “same” direction. One element of this surface is shown in Fig. 2.4, which also shows the current density $J(r_i, t)$ at the $i$-th element of the surface at time $t$. In accordance with Eq. (2.14), the current across this element is given by $J(r_i, t) \cdot \Delta S_i$, and the total current $I(t)$ across the arbitrary surface, call it $\Sigma$, is given by

$$I(t) = \lim_{||\Delta S_i|| \to 0} \sum_i J(r_i, t) \cdot \Delta S_i = \int_{\Sigma} J(r, t) \cdot dS$$

(2.15)

which, in addition to expressing the total current across the surface, defines formally what is called a surface integral. In particular, if the surface over which the integral extends is a plane surface lying in the $xy$-plane, $dS = \pm dx \, dy \, k$ and the surface integral becomes an ordinary two-dimensional integral on the variables $x, y$. If $I$ as given by Eq. (2.15) is positive, the conventional current is flowing across the surface in the direction of the vectors $dS$; if $I < 0$, the conventional current is opposite to the vectors $dS$.

---

3The meaning of this proviso is difficult to state for surfaces that may be curved. In essence the requirement might be stated by imagining some small creature walking around on one side of the surface. The creature must always look either up away from the surface or down through the surface to find the head of the nearest vector $\Delta S_i$. If the creature must look sometimes up and sometimes down, the several vectors $\Delta S_i$ do not point in the “same” direction. Fortunately, difficulties rarely arise on this question.
CHAPTER 2. SPECIFYING ARBITRARY DISTRIBUTIONS

Figure 2.5: A portion of a system of charged particles. Only particles of type $a$ are shown, and the surface $\Delta S$ is centered at the point $\mathbf{r}$. All of the particles shown have charge $q_a$ and are moving with velocity $\mathbf{v}^{(a)}(\mathbf{r}, t)$.

PROBLEMS

P2.10. Evaluate $\int \mathbf{J} \cdot d\mathbf{S}$ over a plane surface of area $S$ if $\mathbf{J}$ is constant.

P2.11. Determine the current crossing the plane surface bounded by a square of side $2a$ positioned with its plane parallel to the $xz$-plane and its center at the point $(x, y, z) = (0, b, 0)$ if (a) $\mathbf{J}(\mathbf{r}) = \alpha \mathbf{r}$ and (b) $\mathbf{J}(\mathbf{r}) = \beta (xy^2 z \mathbf{i} + x^2 y z \mathbf{j} + xyz^2 \mathbf{k})$, where $a$, $b$, $\alpha$, and $\beta$ are constants.

We shall conclude this section by obtaining an expression for the current density in terms of the particle density fields and velocity fields introduced in Eqs. (2.1) and (2.2). To this end, we examine first the contribution of particles of type $a$ to the current across a small surface of area $\Delta S$ placed in an arbitrary (macroscopic) charge distribution. Let $|\Delta S|$ be small enough so that the surface itself can be entirely contained within a volume $\Delta V$ throughout which, at any fixed time $t$, $n^{(a)}(\mathbf{r}, t)$ and $\mathbf{v}^{(a)}(\mathbf{r}, t)$ can be regarded as constant. Then, at any time $t$, the portion of the charge distribution composed of particles of type $a$ has the appearance shown in Fig. 2.5. Now, let $\Delta t$ be a time interval sufficiently small that (1) the density and velocity fields do not change appreciably in the interval from $t$ to $t + \Delta t$ and (2) the volume $\Delta V'$ outlined with the broken line, whose slant height is $|\mathbf{v}^{(a)}(\mathbf{r}, t)| \Delta t$, lies wholly within $\Delta V$. Under these conditions, the total charge $\Delta Q^{(a)}$ transported by particles of type $a$ across $\Delta S$ in time $\Delta t$ is given by $q_a$ times the number of particles in the volume $\Delta V'$, or by

$$
\Delta Q^{(a)} = q_a n^{(a)}(\mathbf{r}, t) \Delta V' = q_a n^{(a)}(\mathbf{r}, t) |\mathbf{v}^{(a)}(\mathbf{r}, t)| |\Delta S| \cos \theta^{(a)} \Delta t
= q_a n^{(a)}(\mathbf{r}, t) \mathbf{v}^{(a)}(\mathbf{r}, t) \cdot \Delta S \Delta t
$$

(2.16)
where $\theta^{(a)}$ is the angle between $\Delta S$ and $\mathbf{v}^{(a)}(\mathbf{r}, t)$. and $\Delta V'$ is evaluated as the area of the base, $|\Delta S|$, times the altitude $d = |\mathbf{v}^{(a)}(\mathbf{r}, t)| \Delta t \cos \theta^{(a)}$. The total charge transported across $\Delta S$ by charges of all types is now obtained by summing Eq. (2.16) over $a$, and the total current is then found by dividing that result by $\Delta t$; we find that

$$\Delta I(t) = \left( \sum_a q_n^{(a)}(\mathbf{r}, t) \mathbf{v}^{(a)}(\mathbf{r}, t) \right) \cdot \Delta S \quad (2.17)$$

from which, by comparison with Eq. (2.14), we find that

$$\mathbf{J}(\mathbf{r}, t) = \sum_a q_n^{(a)}(\mathbf{r}, t) \mathbf{v}^{(a)}(\mathbf{r}, t) \quad (2.18)$$

The $a$-th term in this sum expresses the contribution of particles of type $a$ to the total current density.

### 2.3 Digression I: Stokes’ and Divergence Theorems

The surface integral defined in Eq. (2.15) shares with the line integral defined in Eq. (0.28) a very prominent role in the classical theory of fields. Although it has arisen in a context for which its value has a very direct physical connection with something—namely charge—that actually moves across a surface, the surface integral $\int_{\Sigma} \mathbf{Q} \cdot d\mathbf{S}$ of a (continuous) vector field $\mathbf{Q}(\mathbf{r})$ is itself a well-defined mathematical entity quite apart from the existence of any direct physical interpretation. Even though it rarely represents something that is physically moving across $\Sigma$, $\int_{\Sigma} \mathbf{Q} \cdot d\mathbf{S}$ is called the flux of $\mathbf{Q}$ across $\Sigma$. This integral has a number of mathematical properties that we digress to develop because of their subsequent utility.

#### 2.3.1 Stokes’ Theorem

Let us first derive an identity that relates the line integral of a (suitably continuous) vector field $\mathbf{Q}$ about a closed path to a particular surface integral. Consider, for example, $\oint_{\Gamma} \mathbf{Q} \cdot d\mathbf{l}$ about the path $\Gamma$ shown by the heavy line in Fig. 2.6. As illustrated, any (open) surface $\Sigma$ bounded by this path can be divided into small elements. Further, the sum of the line integrals about each element is approximately the line integral about the original perimeter, because each internal line occurs twice in that sum, traversed once in each direction. The approximation improves as the size of all elements is reduced and we conclude that

$$\oint_{\Gamma} \mathbf{Q} \cdot d\mathbf{l} = \lim_{|\Delta S| \to 0} \sum_i \oint_i \mathbf{Q} \cdot d\mathbf{l} \quad (2.19)$$

where the integral on the right is taken around the perimeter of the $i$-th element. Once the elements are small enough, however, we can use Eq. (0.71) to evaluate each integral under the sum in Eq. (2.19); we find that

$$\oint_{\Gamma} \mathbf{Q} \cdot d\mathbf{l} = \lim_{|\Delta S| \to 0} \sum_i (\nabla \times \mathbf{Q})^i \cdot d\mathbf{S}_i \quad (2.20)$$

where $(\nabla \times \mathbf{Q})^i$ is the value of $\nabla \times \mathbf{Q}$ at some point on the $i$-th surface element and $\Delta S_i$ is the vector representing that surface element. The direction of $\Delta S_i$ must, of course, be
Figure 2.6: Division of a large area into small elements. The line integral about the perimeter of the area is approximately the sum of the line integrals about each separate element.

chosen to be consistent with the condition imposed on Eq. (0.71); that is, $\Delta S_i$ must be directed as the thumb of the right hand when the fingers point in the direction of $dl$ and the palm faces the area $\Delta S_i$. The right-hand side of Eq. (2.20) now exactly defines the surface integral over the surface $\Sigma$ and we have what is called Stokes’ theorem.\footnote{Irish mathematician and physicist George Stokes, b. 13 August 1819 in Skreen, County Sligo, Ireland; d. 1 February 1903 in Cambridge, England.}

\[
\oint_{\Gamma} Q \cdot dl = \int (\nabla \times Q) \cdot dS \tag{2.21}
\]

Thus, the line integral of a (suitably continuous) vector field about a closed path\footnote{Note that the circle is on only one of these integrals; the path is closed but the surface is open.} is equal to the integral of the curl of the field over any open surface bounded by the path, provided the direction assigned to the surface vector is related to the direction of traverse of the path by the right-hand rule described above. Although we have derived Eq. (2.21) assuming that $Q$ depends only on $r$, the theorem also applies if $Q$ depends on other variables (e.g., $t$) provided these other variables can be treated as constants within the context of the theorem itself.

### 2.3.2 The Divergence Theorem

To obtain a starting point for deriving another identity involving a surface integral, let us evaluate the surface integral of a (suitably continuous) vector field $Q$ over the closed surface that bounds a small rectangular parallelepiped of sides $\Delta x$, $\Delta y$, and $\Delta z$. Let the lower back corner of the surface be at the point $(x, y, z)$. Figure 2.7 illustrates the geometry. The surface integral consists of six contributions, one from each of the six faces. Denoting an integral over a closed surface by a circle superimposed on the integral sign, taking $dS$ everywhere in the direction of the outward normal, and writing explicitly only the contributions from...
2.3. DIGRESSION I: STOKES’ AND DIVERGENCE THEOREMS

Figure 2.7: A small rectangular parallelopiped whose surface is used in deriving the divergence theorem.

\[ \text{the two faces parallel to the } xz\text{-plane, we find that} \]
\[
\oint Q \cdot dS = \int_1 Q(x', y + \Delta y, z') \cdot dx' dz' \hat{j} + \int_2 Q(x', y, z') \cdot (-dx' dz' \hat{j}) + \cdots
\]
\[
= \int_{x}^{x+\Delta x} \int_{z}^{z+\Delta z} dx' dz' [Q_y(x', y + \Delta y, z') - Q_y(x', y, z')] + \cdots
\]
\[
\approx \Delta y \int_{x}^{x+\Delta x} dx' \int_{z}^{z+\Delta z} dz' \frac{\partial Q_y(x', y, z')}{\partial y} + \cdots
\]
\[
\approx \Delta x \Delta y \Delta z \frac{\partial Q_y}{\partial y} + \cdots \tag{2.22}
\]

where evaluation of \( Q_y \) at argument \((x, y, z)\) has been suppressed in the final form. The remaining terms in this approximation combine to yield the total result

\[
\oint Q \cdot dS \approx (\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z}) \Delta x \Delta y \Delta z \tag{2.23}
\]

for the integral over the rectangular parallelopiped. Let us now define the divergence of the vector field \( Q \) by

\[
\nabla \cdot Q = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \tag{2.24}
\]

where the notation is suggested by a formal evaluation of \( \nabla \cdot Q \) when \( \nabla \) is replaced by the expression in Eq. (0.47). We then have from Eq. (2.23) the relationship

\[
\oint Q \cdot dS \approx \nabla \cdot Q \Delta V \tag{2.25}
\]

where \( \Delta V = \Delta x \Delta y \Delta z \) is the volume enclosed by the small surface over which the integral extends. Because of the convention adopted above, Eq. (2.23) is correct only when \( dS \) has
the direction of the outward normal to the surface. [If the convention were reversed, a minus sign would appear in Eq. (2.25).]

Equation (2.25) now can be used to derive a second theorem similar to Stokes’ theorem. Note first that any closed surface bounds a volume that can be divided into small elements of the sort considered in the previous paragraph. Let the index $i$ number these small elements. Then, from Eq. (2.25) we find that

$$
\sum_i \oint_{\text{1-\text{th element}}} \mathbf{Q} \cdot d\mathbf{S} \approx \sum_i (\nabla \cdot \mathbf{Q})^i \Delta V_i
$$

(2.26)

where $\Delta V_i$ is the volume of the $i$-th element and $(\nabla \cdot \mathbf{Q})^i$ denotes the divergence of $\mathbf{Q}$ evaluated at some point within the $i$-th element. Now, except for those surfaces that coincide with the surface of the finite volume, every surface element on the left in Eq. (2.26) occurs twice; these “inner” surface elements bound two different volume elements. Since the outward normal to each such surface element is oppositely directed for the two occurrences, these inner surface elements make no net contribution to the left-hand side of Eq. (2.26) and the sum reduces to an integral over the surface bounding the finite volume. In the limit as all $\Delta V_i \to 0$, the right-hand side of Eq. (2.26) becomes a volume integral and the equation itself becomes exact; we have derived what is called the divergence theorem,

$$
\oint \mathbf{Q} \cdot d\mathbf{S} = \int \nabla \cdot \mathbf{Q} \, dv
$$

(2.27)

In words, the flux of a (suitably continuous) vector field $\mathbf{Q}$ out of a closed surface is equal to the integral of the divergence of the field over the volume bounded by the surface, provided the surface vectors are all assigned the direction of the outward normal to the surface. As with Stokes’ theorem, the divergence theorem can also be applied to fields that depend on more variables than the three spatial coordinates by regarding the other variables to be constants insofar as the theorem itself is concerned.

Equation (2.25) also provides a coordinate-free definition of the divergence of a vector field, namely

$$
\nabla \cdot \mathbf{Q} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint \mathbf{Q} \cdot d\mathbf{S}
$$

(2.28)

Equation (2.28) certainly reduces to Eq. (2.24) in Cartesian coordinates; it takes very little labor to verify that Eq. (2.28) leads to

$$
\nabla \cdot \mathbf{Q} = \frac{1}{r} \frac{\partial (rzQ_z)}{\partial r} + \frac{1}{r} \frac{\partial Q_\phi}{\partial \phi} + \frac{\partial Q_z}{\partial z}
$$

(2.29)

in cylindrical coordinates and to

$$
\nabla \cdot \mathbf{Q} = \frac{1}{r^2} \frac{\partial (r^2Q_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta Q_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial Q_\phi}{\partial \phi}
$$

(2.30)

in spherical coordinates.

A geometric interpretation for $\nabla \cdot \mathbf{Q}$ can be inferred if we recognize that $\oint \mathbf{Q} \cdot d\mathbf{S}$ in effect counts the (net) number of field lines of $\mathbf{Q}$ that cross the closed surface. (Remember

---

$^6$ Again, note that the circle is on only one of the integrals; the surface is closed. Indeed, the surface must be closed for the volume it bounds to be at all unambiguously defined.
Figure 2.8: A point \( \mathbf{r}_0 \) lying (a) outside of and (b) inside of a closed surface \( \Sigma \).

that, by convention, the number of field lines crossing a unit surface perpendicular to the field is proportional to the magnitude of the field.) In this reckoning, lines passing from inside to outside are regarded as positive; those going from outside to inside are negative. Thus, when \( \int \mathbf{Q} \cdot d\mathbf{S} \) over some surface differs from zero, field lines either emerge from or terminate within the volume bounded by the surface. Arguing from Eq. (2.28), we conclude that a point at which \( \nabla \cdot \mathbf{Q} \neq 0 \) is a point at which at least some of the field lines of \( \mathbf{Q} \) either start or stop. In reverse, if \( \nabla \cdot \mathbf{Q} = 0 \) at some point, field lines neither begin nor terminate at that point, and, even more generally, if \( \nabla \cdot \mathbf{Q} = 0 \) everywhere, the field lines of \( \mathbf{Q} \) have neither beginning nor ending anywhere and can only close on themselves. The velocity fields of points on a rotating phonograph record [Fig. 0.11(b)], for which \( \mathbf{Q} = \mathbf{\omega} \hat{\phi} \) and \( \nabla \cdot \mathbf{Q} = 0 \), is a simple example of this latter type.

2.3.3 A Useful Integral: Solid Angle

To illustrate a typical use of the divergence theorem, we now evaluate an integral that will be important to our development of Gauss’s law in Section 4.4. Let \( \mathbf{r}_0 \) be some point in space and \( \Sigma \) be a closed surface (Fig. 2.8). Consider the integral

\[
G = \oint_{\Sigma} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot d\mathbf{S} \tag{2.31}
\]

where the integration variables are the components of \( \mathbf{r} \) suitably constrained so that \( \mathbf{r} \) always lies on \( \Sigma \). Using the divergence theorem we can rewrite Eq. (2.31) as the volume integral

\[
G = \int_V \nabla \cdot \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \, dv \tag{2.32}
\]

where \( V \) is the volume bounded by \( \Sigma \), and \( \nabla \) involves derivatives with respect to the components of \( \mathbf{r} \). Direct evaluation shows that the integrand in Eq. (2.32) is zero except when \( \mathbf{r} = \mathbf{r}_0 \). Thus, if the point \( \mathbf{r}_0 \) lies outside the volume \( V \), Eq. (2.32) gives \( G = 0 \). When \( \mathbf{r}_0 \) lies inside \( V \), however, the evaluation of \( G \) is a bit more involved. We begin by dividing \( V \) into two regions by inserting a small sphere centered on \( \mathbf{r}_0 \) and denoted by \( \Sigma' \).
CHAPTER 2. SPECIFYING ARBITRARY DISTRIBUTIONS

[Fig. 2.8(b)]. Now, let $V''$ be the volume between $\Sigma$ and $\Sigma'$. The point $r_0$ lies outside $V''$ and, consequently, we have that

$$\int_{V''} \nabla \cdot \frac{r-r_0}{|r-r_0|^3} \, dv = 0 \quad (2.33)$$

We now use the divergence theorem to express Eq. (2.33) as a surface integral, the surface involved having two parts; we find that

$$\int_{\Sigma} \frac{r-r_0}{|r-r_0|^3} \cdot dS + \int_{\Sigma'} \frac{r-r_0}{|r-r_0|^3} \cdot dS = 0 \quad (2.34)$$

In both integrals $dS$ stands for an outward normal, where the direction outward is reckoned from an observation point within $V''$. Thus, $dS$ on $\Sigma'$ in fact points toward $r_0$. Since the first integral in Eq. (2.34) is $G$, we have that

$$G = \int_{\Sigma'} \frac{r-r_0}{|r-r_0|^3} \cdot dS \quad (2.35)$$

The integral to which we have reduced $G$, however, can now be quickly evaluated. Let $R$ be the radius of the sphere $\Sigma'$. Then $|r-r_0|^3 = R^3$ at all points on the sphere and $(r-r_0) \cdot dS = -R |dS|$. (Remember that $dS$ points toward $r_0$ on $\Sigma'$.) Thus Eq. (2.35) reduces to

$$G = \frac{1}{R^2} \oint_{\Sigma'} |dS| = 4\pi \quad (2.36)$$

since $\int |dS|$ is merely the surface area of the sphere ($4\pi R^2$). Combining our two results, we have finally that

$$G = \oint_{\Sigma} \frac{r-r_0}{|r-r_0|^3} \cdot dS = \begin{cases} 4\pi, & r_0 \text{ inside } \Sigma \\ 0, & r_0 \text{ outside } \Sigma \end{cases} \quad (2.37)$$

The integral in Eq. (2.31) is directly related to the geometric concept of solid angle, which is the three-dimensional analog of the familiar angle in the plane and measures the three-dimensional “opening” at the vertex of a general cone. The contribution $d\Omega$ of a small element $dS$ of a broad surface $\Sigma$ to the solid angle subtended by $\Sigma$ from an observation point $r_0$ is defined to be the quotient of the area of $dS$ projected onto a plane normal to the line of sight and the square of the distance of the area from the observation point; that is

$$d\Omega = \left( \frac{r-r_0}{|r-r_0|^3} \cdot dS \right) \div |r-r_0|^2 \quad (2.38)$$

where the element $dS$ is located at $r$ and $r-r_0$ is a vector from the observation point to $dS$ (Fig. 2.9). The solid angle subtended by the entire surface $\Sigma$ is than given by

$$\Omega = \int_{\Sigma} d\Omega = \int_{\Sigma} \frac{r-r_0}{|r-r_0|^3} \cdot dS \quad (2.39)$$

which in particular coincides with Eq. (2.31) if $\Sigma$ is a closed surface. Interpreted geometrically, Eq. (2.37) states that the solid angle subtended by a closed surface $\Sigma$ from an observation point inside (outside) $\Sigma$ is $4\pi$ ($0$). (Visualization of this general statement may be aided by composing the analogous statement involving ordinary angles and closed curves in a plane. See P2.21.) Although solid angle is a dimensionless concept, it is common to quote solid angles with “units” of steradians.
2.4 THE EQUATION OF CONTINUITY

During the conduct of electrostatic experiments, it is observed empirically that, whenever a neutral object is charged by rubbing, equal amounts of positive and negative charge appear.
The net charge in the universe appears to be conserved, that is

$$\frac{d}{dt} \int_{\text{entire universe}} \rho(r, t) \, dv = 0 \quad (2.40)$$

where $\rho(r, t)$ is the charge density at point $r$ at time $t$. This expression of charge conservation, however, is less useful than one that applies to a more local region of space. An alternative expression is obtained by inquiring about the total charge in a closed (but not necessarily isolated) region of space. The charge $Q(t)$ at time $t$ in a volume $V$ enclosed by a surface $\Sigma$ is given by

$$Q(t) = \int_V \rho(r, t) \, dv \quad (2.41)$$

If we assume that there are no sources or sinks of charge within $V$, then the only way that the charge within $V$ can change is by transport of charge across the bounding surface $\Sigma$. With $d\mathbf{S}$ being the conventional outward normal, the current flowing into $V$ across $\Sigma$ is given by

$$I_{\text{in}}(t) = -\oint_{\Sigma} \mathbf{J}(r, t) \cdot d\mathbf{S} \quad (2.42)$$

where the minus sign appears because of the convention on the direction of $d\mathbf{S}$. [Compare Eq. (2.15).] Because $I_{\text{in}}(t)$ is also the rate at which the charge inside $V$ is changing, we have that $I_{\text{in}} = dQ/dt$, or

$$-\oint_{\Sigma} \mathbf{J}(r, t) \cdot d\mathbf{S} = \frac{d}{dt} \int_V \rho(r, t) \, dv \quad (2.43)$$

Since $\int_V \rho(r, t) \, dv$ is dependent only on $t$ and not on $r$, we can write $\partial / \partial t$ instead of $d/dt$ in Eq. (2.43). Then, since the volume $V$ is fixed in space, we can take the partial time derivative under the integral sign. With some further rearrangement of the resulting terms and with suppression of the explicit arguments $r$ and $t$, we then find the integral form of the equation of continuity,

$$\oint_{\Sigma} \mathbf{J} \cdot d\mathbf{S} + \int_V \frac{\partial \rho}{\partial t} \, dv = 0 \quad (2.44)$$

This equation expresses charge conservation as applied to nonisolated regions of space. In effect, it states that any current flowing into some closed volume must change the charge in that volume by precisely the net amount transported into the volume.

An equivalent and extremely useful alternative form of the equation of continuity may be obtained by applying Eq. (2.44) specifically to a small volume surrounding the point $r$. We can then rewrite the first integral in Eq. (2.44) by using Eq. (2.25) and we can express the second integral more simply by assuming $\partial \rho / \partial t$ to be (approximately) constant throughout $V$—now more appropriately denoted by $\Delta V$; Eq. (2.44) then becomes

$$\nabla \cdot \mathbf{J} \Delta V + \frac{\partial \rho}{\partial t} \Delta V \approx 0 \quad (2.45)$$

We now divide by $\Delta V$ and then allow $\Delta V$ to become arbitrarily small; the result is the differential form of the equation of continuity,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (2.46)$$

In particular, if the currents are steady so charge does not accumulate anywhere ($\rho$ independent of $t$), then

$$\nabla \cdot \mathbf{J} = 0 \quad \text{(steady currents)} \quad (2.47)$$
We shall need these results occasionally in the next three chapters but we postpone detailed exploration of the differential forms of the basic equations until Chapter 6 and later chapters.

**PROBLEM**

**P2.20.** The current density in a region of space is given by

\[ J(r) = \alpha [r \cdot r - (r \cdot \hat{k})^2] e^{-\beta r \cdot \hat{k}} \]

where \( \alpha \) and \( \beta \) are constants. Imagine a volume bounded by a cylindrical surface having its axis coincident with the \( z \)-axis and its lower face in the \( xy \)-plane. Let the cylinder have radius \( a \) and altitude \( b \). If at time zero there is no net charge within this volume, determine how much charge it contains at a later time \( t \).

---

### 2.5 Digression II: Operators Involving \( \nabla \)

The differential operator \( \nabla \) occurs in a variety of important contexts beyond the three first-order derivatives—\( \nabla S \), \( \nabla \times Q \), and \( \nabla \cdot Q \)—that have already been introduced. One additional first-order derivative is the operator \( Q \cdot \nabla \), where \( Q \) may be a constant vector or a spatially and temporally dependent vector field. This operator is a scalar operator, and it may act either on a scalar field \( S \), giving \( Q \cdot \nabla S \), where \( \nabla S \) is the ordinary gradient, or on another vector field \( R \), giving \( (Q \cdot \nabla)R \), which is a vector quantity whose Cartesian components are \( Q \cdot \nabla R_x \), \( Q \cdot \nabla R_y \), and \( Q \cdot \nabla R_z \), but whose components in other coordinate systems are much less simple because of the spatial dependence of the unit vectors in non-Cartesian coordinates. Acting on a scalar, this operator usually assumes one of the forms

\[
Q \cdot \nabla = Q_x \frac{\partial}{\partial x} + Q_y \frac{\partial}{\partial y} + Q_z \frac{\partial}{\partial z} \quad (2.48)
\]

\[
= Q_r \frac{\partial}{\partial r} + \frac{Q_\phi}{r} \frac{\partial}{\partial \phi} + Q_z \frac{\partial}{\partial z} \quad (2.49)
\]

\[
= Q_r \frac{\partial}{\partial r} + \frac{Q_\theta}{r \sin \theta} \frac{\partial}{\partial \theta} + \frac{Q_\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \quad (2.50)
\]

Note the property \( (Q \cdot \nabla)r = Q \) for any vector field \( Q \).

The three quantities \( \nabla \cdot Q \), \( \nabla S \), and \( \nabla \times Q \) can be the operand of the operator \( \nabla \) in at least five different ways. The first quantity is a scalar and possesses a gradient; the second and third quantities are vectors and possess both a divergence and a curl. We shall comment on each of these second-order derivatives:

1. \( \nabla(\nabla \cdot Q) \) occurs in some vector identities but (as far as the author knows) has no special significance.

2. \( \nabla \cdot \nabla S \), commonly symbolized \( \nabla^2 S \), is called the Laplacian of \( S \) and plays a very important role in potential theory. Expressions for the operator \( \nabla^2 \) in the three common coordinate systems can be obtained by direct substitution of the components
of the gradient [Eqs. (0.48), (0.66), and (0.67)] into the expressions for the divergence of a vector [Eqs. (2.24), (2.29), and (2.30)]; we find

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (2.51)
\]

\[
= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \quad (2.52)
\]

\[
= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (2.53)
\]

Occasionally, the Laplacian \(\nabla^2 Q\) of a vector arises; in Cartesian coordinates the components of this vector are \(\nabla^2 Q_x\), \(\nabla^2 Q_y\), and \(\nabla^2 Q_z\), but expressions for its components in other coordinate systems are much less simple because of the spatial dependence of the unit vectors in non-Cartesian coordinates.

3. \(\nabla \times (\nabla S)\) is always zero if \(S\) is well behaved. This property is directly related to two properties of a conservative force field \(F_c\): \(\nabla \times F_c = 0\) and \(F_c = -\nabla U\). (See Section 0.3.) Either property, in fact, implies the other and our earlier discussion in effect proved the theorem: The curl of a vector field \(Q\) is zero if and only if that field is the gradient of an associated scalar field \(S\). That is, \(\nabla \times Q = 0 \iff Q = \nabla S\). This theorem, of course, merely states the existence of \(S\); if does not provide a means to find \(S\) if \(Q\) is known. A vector field with zero curl is said to be irrotational.

4. \(\nabla \cdot \nabla \times Q\) is also always zero if \(Q\) is well behaved. This property suggests, but does not prove all aspects of, the theorem: The divergence of a vector field \(R\) is zero if and only if that field is the curl of another vector field \(Q\). That is, \(\nabla \cdot R = 0 \iff R = \nabla \times Q\). As with the analogous theorem in (3), this theorem merely states the existence of \(Q\); it does not provide a means to find \(Q\) if \(R\) is given. A vector field with zero divergence is said to be solenoidal.

5. \(\nabla \times (\nabla \times Q)\) occurs frequently and is usually simplified to the equivalent form \(\nabla (\nabla \cdot Q) - \nabla^2 Q\), interpretation of the term \(\nabla^2 Q\) requiring special care in non-Cartesian coordinates.

**PROBLEM**

P2.21. Familiarize yourself with identities (C.15)–(C.19), (C.26), and (C.27) in Appendix C, and prove (C.18) and (C.19).

**SUPPLEMENTARY PROBLEMS**

P2.22. Let \(\Gamma\) be some path lying wholly in the \(xy\)-plane and let \(r_0\) be a point also in the \(xy\)-plane but not on \(\Gamma\). Show that the (ordinary) angle \(\theta\) subtended by \(\Gamma\) from the point \(r_0\) is given by

\[
\theta = \hat{j} \cdot \int_{\Gamma} \frac{r - r_0}{|r - r_0|^2} \times dl
\]

where the integration extends over points \(r\) on \(\Gamma\), and show in particular that, if \(\Gamma\) is a closed path, \(\theta\) is either \(2\pi\) or 0, depending on whether \(r_0\) lies inside of outside \(\Gamma\).
P2.23. Referring to the general orthogonal coordinate system described in P0.37, use the definition in Eq. (2.28) to show that
\[
\nabla \cdot \mathbf{Q} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q_i} \left( \frac{h_1 h_2 h_3}{h_i} Q_i \right)
\]
and combine this result with the expression in P0.37 for \( \nabla S \) to show that
\[
\nabla^2 = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q_i} \left( \frac{h_1 h_2 h_3}{h_i^2} \frac{\partial}{\partial q_i} \right)
\]
Finally, show that these results reduce to those given in the text for \( \nabla \cdot \mathbf{Q} \) and \( \nabla^2 \) in cylindrical and in spherical coordinates.

P2.24. Determine the form of the equation of continuity first in Cookian units (P1.8) and then in modified Gaussian units (P1.7).

P2.25. Let \( \psi(r,t) \) be the wave function describing a quantum mechanical particle of mass \( m \). The probability for finding this particle in a volume \( V \) is given by \( \int_V \psi^* \psi \, dv \). Given that \( \psi \) satisfies the Schrödinger equation
\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U \psi
\]
where \( \hbar \) is Planck’s constant divided by \( 2\pi \) and \( U \) is the potential energy of the particle, show that
\[
\frac{\partial}{\partial t} \int_V \psi^* \psi \, dv = - \oint_{\Sigma} \mathbf{J} \cdot d\mathbf{S}
\]
where \( \Sigma \) bounds \( V \), and find an explicit expression for \( \mathbf{J} \). Evidently, quantum mechanical probabilities satisfy a continuity equation if the probability current \( \mathbf{J} \) is properly identified. 

Hint: Use Green’s theorem, Eq. (C.27).

P2.26. Currents flowing in a surface can be described by a current density of a slightly different sort than the one introduced in the text. Suppose that a current is flowing in the plane of the paper as indicated in Fig. 2.10. Then charge is transported across any line in this surface. The appropriate current density at any point in the surface is defined to have a direction determined by the direction in which positive charge is being transported at that point and a magnitude given by the rate at which charge is transported across a line of unit length lying in the surface and oriented perpendicular to the direction of the current density. (a) What are the dimensions of this current density? (b) Show that the current flowing across a line in the surface is given by
\[
I = \int \mathbf{j} \cdot d\mathbf{l} \times \hat{n}
\]
where \( j \) is the surface current density, \( dl \) is an element of the line, \( \hat{n} \) is a unit vector normal to the surface, and the positive direction of current flow across the line is determined by the direction of \( dl \times \hat{n} \). (c) Derive a continuity equation applicable to currents in a surface.

P2.33. The command \texttt{<<VectorAnalysis"} to \textit{Mathematica} loads a package of functions for doing vector calculus, including functions for evaluating the Laplacian of a scalar, creating a vector field, and evaluating the divergence of a vector. After the package \texttt{VectorAnalysis"} has been loaded, the commands

\begin{verbatim}
s := x*x + y*y + z*z;
    ! Define a scalar field.
Laplacian( s, [x,y,z] );
    ! Evaluate Laplacian.
V := VectorField(< x*y, y*z, z*x >,
    'cartesian'[x,y,z] );
    ! Define a vector field.
Divergence( v, [x,y,z] );
    ! Evaluate divergence.
\end{verbatim}

will evaluate the Laplacian and the divergence of the defined functions of \( x, y, z \). Taking the vector \( r \) to be \( x \hat{i} + y \hat{j} + z \hat{k} \) and \( r \) to be \( (x^2 + y^2 + z^2)^{1/2} \),

(a) show that \( \nabla \cdot r = 3 \).
(b) show that \( \nabla^2 (1/r) = 0 \). \textit{(Note: Except at } r = 0).\textit{)}
(c) evaluate the Laplacian of the scalar potential \( e^{-ar}/r \).

P2.34. Study the manuals or on-line help for \textit{Mathematica}, seeking to discover the way to evaluate the divergence, gradient, curl, and Laplacian of scalar and vector functions expressed in cylindrical coordinates \( (r, \phi, z) \). Then, noting that \( r = z \hat{r} + z \hat{k} \) and \( r = (r^2 + z^2)^{1/2} \) in cylindrical coordinates, test your definitions by evaluating \( \nabla (1/r), \nabla \cdot r, \nabla \times r, \) and \( \nabla^2 (e^{-ar}/r) \). \textit{Hint: See the discussion of gradient and curl with } Mathemat}
Chapter 3

The Electromagnetic Field: Its Definition and Its Effect on General Charge Distributions

We shall next examine the interaction between two general charge distributions. We could consider this interaction as an indivisible whole, regarding each distribution to exert forces directly on the other. It is significantly more fruitful, however, to replace this concept of action at a distance with an alternative view in which the force exerted by one distribution on another is regarded as the end result of two successive effects. In brief, the first distribution, called the source distribution, produces an electromagnetic field in some region of space, and the second distribution then experiences forces by virtue of its interaction with this field rather than by virtue of a direct interaction with the first distribution. In effect, the field communicates forces of interaction from one distribution to another. To make this conceptual division of the interaction physically useful, however, we need detailed quantitative answers to the following questions:

1. How is the electromagnetic field defined?
2. What forces and torques are experienced by an arbitrary distribution placed in a given electromagnetic field?
3. What electromagnetic field is established by a given source distribution?

Questions (1) and (2) are considered in this chapter; question (3) is treated in Chapters 4, 5, and 6.

3.1 Forces on Point Charges: A Definition of the Electromagnetic Field

The electromagnetic field at point \( r \) and time \( t \) is defined most directly by relating it to the force on a (point) test charge that happens at time \( t \) to be at point \( r \). Since we are concerned only with the part of the total force that depends specifically on the charge of
the test particle, such forces as the gravitational force, if present at all, must be subtracted from the total force in order to isolate the electromagnetic force. Even the electromagnetic force turns out experimentally to have two parts. The electric force is independent of the velocity of the particle and, in particular, is experienced by test charges at rest; the magnetic force depends on the velocity of the particle and is experienced only by test charges in motion. Experimentally, only the position and velocity of the test charge are important; accelerated particles experience no forces beyond those on particles moving with constant velocity.\footnote{We ignore the very small forces resulting from the reaction of an accelerated particle to its own electromagnetic radiation. (See Section 14.6.)} Thus, the division of the electromagnetic force into velocity-independent and velocity-dependent parts is exhaustive and we write

\[
\mathbf{F}_q(\mathbf{v}, \mathbf{r}, t) = \mathbf{F}^{\text{elec}}_q(\mathbf{r}, t) + \mathbf{F}^{\text{mag}}_q(\mathbf{v}, \mathbf{r}, t)
\]

where $\mathbf{v}$ and $\mathbf{r}$ are the velocity and position at time $t$ of a test particle with charge $q$, $\mathbf{F}_q$ is the electromagnetic force, $\mathbf{F}^{\text{elec}}_q$ is the electric part, and $\mathbf{F}^{\text{mag}}_q$ is the magnetic part. The objective in the remainder of this section is to introduce position- and time-dependent (but not velocity-dependent) electromagnetic fields from which each part of the electromagnetic force can be determined.

The electric field at the point $\mathbf{r}$, $t$ in space-time is measured (and hence defined) by the following operations: Place a test charge of strength $q$ at rest at the point $\mathbf{r}$ at time $t$, and measure the force $\mathbf{F}^{\text{elec}}_q(\mathbf{r}, t)$ on this charge. The electric field $\mathbf{E}(\mathbf{r}, t)$ is then defined by the ratio $\mathbf{F}^{\text{elec}}_q(\mathbf{r}, t)/q$. Equivalently,\footnote{In the spirit of Chapter 1, we might relate $\mathbf{E}$ to $\mathbf{F}^{\text{elec}}_q$ by setting $\mathbf{F}^{\text{elec}}_q = k_4 q \mathbf{E}$, where $k_4$ assumes an arbitrary value if this expression defines $\mathbf{E}$ and an empirically determined value if $\mathbf{E}$ is defined independently of this expression. Further, $k_4$ may or may not have dimensions. All systems of units of which the author is aware define $\mathbf{E}$ by setting $k_4$ equal to the dimensionless constant unity, i.e., by Eq. (3.2).} we set

\[
\mathbf{F}^{\text{elec}}_q(\mathbf{r}, t) = q \mathbf{E}(\mathbf{r}, t)
\]

Strictly, of course, Eq. (3.2) defines the electric field at $\mathbf{r}$, $t$ \textit{when the test charge is present}. Since in general the test charge gives rise to new forces on whatever source distribution is establishing the field, its presence may cause a redistribution of the source charges so that the field with which the test charge interacts may differ from the field present at $\mathbf{r}$ before the test charge was introduced. To minimize the modifying influence of the test charge on the field, we make the test charge small. Indeed, all uncertainty in the definition of $\mathbf{E}(\mathbf{r}, t)$ is removed if we allow $q$ to become arbitrarily small, defining $\mathbf{E}(\mathbf{r}, t)$ by

\[
\mathbf{E}(\mathbf{r}, t) = \lim_{q \to 0} \frac{\mathbf{F}^{\text{elec}}_q(\mathbf{r}, t)}{q}
\]

If Eq. (3.3) is adopted, then Eq. (3.2) still gives the force on a small test charge placed at $\mathbf{r}$, $t$ provided the influence of the test charge on the source can be neglected. By either equation, the electric field is a vector quantity whose direction at the point $\mathbf{r}$ is the same as the direction of the force on a positive charge placed at that point. The dimensions of $\mathbf{E}$ are those of force per charge and, in mks units, $\mathbf{E}$ is expressed in newtons per coulomb (N/C).

The magnetic force $\mathbf{F}^{\text{mag}}_q(\mathbf{v}, \mathbf{r}, t)$, which is experienced only by a moving test charge, is attributed to the interaction of the charge with a second field $\mathbf{B}(\mathbf{r}, t)$, called the magnetic
induction field (or more briefly the magnetic induction). To guide our definition of \( B(r, t) \), we explore the properties of \( F_q^{\text{mag}}(v, r, t) \) by projecting test charges in different directions and with different velocities through the point \( r \) at time \( t \), measuring the magnetic force on the particle in each experiment.\(^3\) If all of the test charges are small enough so as not to disturb whatever sources establish \( B(r, t) \), we find experimentally that the magnitude of the magnetic force is proportional both to the speed of and to the charge on the test particle and that the direction of the magnetic force is always perpendicular to the velocity of the particle. All of these experimental properties follow if we assign to the point \( r \), \( t \) a magnetic induction field \( B(r, t) \) that determines the magnetic force by

\[
F_q^{\text{mag}}(v, r, t) = qv \times B(r, t) \tag{3.4}
\]

Strictly, this expression defines \( B(r, t) \) in mks units, and the mks unit of \( B \) therefore has the abbreviation \( \text{N} \cdot \text{s} / \text{C} \cdot \text{m} = \text{N} / \text{A} \cdot \text{m} \)—a combination that has long been called the weber/m² but is now officially called the tesla (T).\(^4,5,6\)

Although it is readily verified that the force determined by Eq. (3.4) exhibits the properties required by experiment, it is not as easily seen that Eq. (3.4) indeed leads to a means to measure the magnetic induction at some point \( r \), \( t \) in space-time. We shall describe such a procedure in this paragraph.\(^7\) Unfortunately, Eq. (3.4) cannot be solved explicitly for \( B \); the best we can do is compute the cross product of Eq. (3.4) with \( v \) and exploit the identity in Eq. (C.1) for expanding the triple vector product to find that

\[
B = -\frac{v \times F_q^{\text{mag}}}{qv^2} + \frac{(v \cdot B)v}{v^2} \tag{3.5}
\]

where all arguments have been suppressed. Now, project a particular test charge through the point in question with two different velocities \( v_1 \) and \( v_2 \), and measure the (magnetic) force \( F_{q1}^{\text{mag}} \) and \( F_{q2}^{\text{mag}} \) experienced in each case. We can then use Eq. (3.5), which gives

\[
B = -\frac{v_1 \times F_{q1}^{\text{mag}}}{qv_1^2} + \frac{(v_1 \cdot B)v_1}{v_1^2} \tag{3.6}
\]

and also

\[
B = -\frac{v_2 \times F_{q2}^{\text{mag}}}{qv_2^2} + \frac{(v_2 \cdot B)v_2}{v_2^2} \tag{3.7}
\]

\(^3\)The experimental difficulty of making several different measurements all at a single time does not preclude our using such measurements to define \( B \) in principle. Actual measurement of both \( E \) and \( B \) is almost always accomplished by means other than those envisioned in the definitions.

\(^4\) Again in the spirit of Chapter 1, we might relate \( B \) to \( F_{qk}^{\text{mag}} \) by \( k_5v \times B \), where \( k_5 \) assumes an arbitrary value if this expression defines \( B \) and an empirically determined value if \( B \) is defined independently of this expression. Further, \( k_5 \) may or may not have dimensions. Two different choices for \( k_5 \) are in common use. The Gaussian and Heaviside-Lorentz systems set \( k_5 \) equal to the dimensional constant \( 1/c \), where \( c \) is the speed of light; cgs-esu, cgs-emu, and rationalized mks units all assign to \( k_5 \) the dimensionless value unity, as in Eq. (3.4). The Gaussian unit of \( B \) is called the gauss (G); 1 T is \( 10^4 \) G.

\(^5\) German physicist Wilhelm Eduard Weber, b. 24 October 1804 in Wittenberg, Saxony, Holy Roman Empire; d. 23 June 1891 in Gottingen, Hanover, Prussia.

\(^6\) Serbian American inventor and physicist Nikola Tesla, b. 10 July 1856 in Smiljan, Austrian Empire (now Croatia), d. 7 January 1943 in New York, New York.

\(^7\)The procedure here described is patterned after the procedure described in Foundations of Electromagnetic Theory by J. R. Reitz and F. J. Milford (Addison-Wesley Publishing Company, Inc., Reading, MA, 1967), Chapter 8, and is used here by permission of Addison-Wesley Publishing Company.
CHAPTER 3. DEFINITION OF THE ELECTROMAGNETIC FIELD

to determine the magnetic induction field in the following way. From Eq. (3.7), we find that

$$v_1 \cdot B = -\frac{v_1 \cdot (v_2 \times F_{mag}^{q_2})}{qv_2^2} + \frac{(v_2 \cdot B)}{v_2^2} (v_1 \cdot v_2)$$  \hspace{1cm} (3.8)

Hence, if we select \(v_2\) perpendicular to \(v_1\) (so that \(v_1 \cdot v_2 = 0\)), we find that \(v_1 \cdot B\) is determined by the measurable first term in Eq. (3.8). Substituting that term into Eq. (3.6), we find an expression for \(B\) that involves only measurable quantities, viz.,

$$B = -\frac{v_1 \times F_{mag}^{q_1}}{qv_1^2} - \frac{v_1 \cdot (v_2 \times F_{mag}^{q_2})}{qv_1^2 v_2^2} v_1, \quad v_1 \perp v_2$$  \hspace{1cm} (3.9)

Although this expression, perhaps with the limit \(q \to 0\), might have been taken instead of Eq. (3.4) as a definition of \(B\), it is unlikely that someone might arrive at Eq. (3.9) without first recognizing something like Eq. (3.4).

Combining Eqs. (3.2) and (3.4), we find finally that the electromagnetic force \(F_q\) on a charged particle in the electromagnetic field \(E, B\) is given (in mks units) by

$$F_q(v, r, t) = qE(r, t) + qv \times B(r, t)$$  \hspace{1cm} (3.10)

an expression often called the Lorentz force. Relative to the coordinate origin, this particle also experiences a torque \(N_q(v, r, t)\) given by

$$N_q(v, r, t) = r \times F_q(v, r, t) = r \times [qE(r, t) + qv \times B(r, t)]$$  \hspace{1cm} (3.11)

We conclude this section by pointing out that the electromagnetic field here defined is the field as observed in a specific frame of reference. We have used particles at rest and particles in motion with specific velocities in defining the fields and have therefore tacitly selected a frame of reference. The procedures we have described for measuring \(E\) and \(B\) can, of course, be carried out in any frame of reference, so the fields are meaningful to any observer. The relationship between the fields as measured by two different observers in uniform relative motion is one aspect of special relativity and will be examined in Chapter 15.

PROBLEM

P3.1. Show that 1 T is exactly \(10^4\) G.

3.2 Trajectories of Particles in Prescribed Fields

Finding the trajectory of a charged particle in a prescribed electromagnetic field is important in designing particle accelerators and other apparatus for guiding and focusing beams of charged particles, in studying the behavior of ionized gases, and in investigating many other physical phenomena that involve charged particles. The starting point for calculating such
a trajectory is Newton’s second law, Eq. (0.26)), in which the force on the particle is given by Eq. (3.10). The equation of motion of the particle in a given field $\mathbf{E}$, $\mathbf{B}$ thus is

$$m \frac{d^2 \mathbf{r}}{dt^2} = q \mathbf{E}(\mathbf{r}, t) + q \frac{d\mathbf{r}}{dt} \times \mathbf{B}(\mathbf{r}, t)$$

(3.12)

where $m$ and $q$ are the mass and charge of the particle and $\mathbf{r} = \mathbf{r}(t)$ is now the position of the particle as a function of time. Initial values of $\mathbf{r}$ and $d\mathbf{r}/dt$ must, of course, be given before the solution to Eq. (3.12) is unique.

Very few analytic solutions to Eq. (3.12) exist. When the fields are constant in both time and space, Eq. (3.12) is linear and can be solved. If, for example, the electric field is constant and the magnetic field is zero, a single particle follows a parabolic trajectory (P3.2, P3.3). In a constant magnetic field when the electric field is zero, a single particle follows a circular or helical path (P3.4, P3.5). If both fields are constant and non-zero, the motion is more complicated and includes a drift velocity in a direction perpendicular to both the electric and the magnetic fields (P3.6, P3.16, P3.17). Finally, the motion of a single charged particle in the time-independent electric field produced by a fixed point charge (Coulomb field) is essentially identical to that of a planet moving around a fixed sun (both force fields are inverse square), except that in the electric case both attractive and repulsive interactions exist in nature. Beyond these simple cases, trajectories in electric and magnetic fields can be quite complicated.

**PROBLEMS**

**P3.2.** Show that a particle of mass $m$ and charge $q$ moving in a constant electric field, $\mathbf{E} = \mathbf{E}_0 = \text{constant}$ and $\mathbf{B} = 0$, follows a trajectory given by

$$\mathbf{r}(t) = \frac{qt^2}{2m} \mathbf{E}_0 + \mathbf{v}_0 t + \mathbf{r}_0$$

where $\mathbf{r}_0$ and $\mathbf{v}_0$ are the initial position and velocity. Describe the trajectory in words and note the similarity between this result and the trajectory of a particle moving in a uniform gravitational field.

**P3.3.** A constant electric field $\mathbf{E} = E \hat{\mathbf{j}}$, with $E > 0$, exists in the region of space $0 < x < a$. A particle of charge $q$ and mass $m$ is projected into this region from the origin with an initial speed $v$ along the $x$-axis. Show that the particle follows a trajectory that lies in the $xy$-plane and determine the $y$-coordinate of the point at which the particle strikes the plane $x = b$, $b > a$. The situation described in this problem is a crude model of the beam in a cathode ray oscilloscope or television tube.

**P3.4.** Without solving Eq. (3.12), show that a particle projected into a constant magnetic induction, $\mathbf{B} = \mathbf{B}_0 = \text{constant}$ and $\mathbf{E} = 0$, with an initial velocity $\mathbf{v}_0 \perp \mathbf{B}_0$ moves with a constant speed in a circle of radius $a = mv_0/qB_0$, where $q$ and $m$ are the charge and mass of the particle. Show also that the angular velocity of the particle is independent of its speed—a property crucially important to the functioning of the cyclotron. Finally, describe the trajectory if the initial velocity is not perpendicular to the field.

**P3.5.** Show that a particle of mass $m$ and charge $q$ moving in a constant magnetic induction, $\mathbf{B} = \mathbf{B}_0 = \text{constant}$ and $\mathbf{E} = 0$, follows a trajectory given by

$$\mathbf{r}(t) = \mathbf{r}_0 + (\mathbf{v}_0 \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} t + \frac{\sin \omega t}{\omega} [\hat{\mathbf{b}} \times (\mathbf{v}_0 \times \hat{\mathbf{b}})] + \frac{1 - \cos \omega t}{\omega} \mathbf{v}_0 \times \hat{\mathbf{b}}$$

where $\mathbf{r}_0$ and $\mathbf{v}_0$ are the initial position and velocity, $\hat{\mathbf{b}}$ is a unit vector parallel to $\mathbf{B}_0$, and $\omega = qB_0/m$. Describe the trajectory in words. *Hint:* Start by choosing a coordinate
system whose $z$-axis coincides with the magnetic field, so that $\mathbf{B} = B_0 \hat{k}$, find the equations of motion in that coordinate system, and show that the particle follows a trajectory given by

\[
x(t) = x_0 + \frac{v_{0y}}{\omega} \left( 1 - \cos \omega t \right) + \frac{v_{0x}}{\omega} \sin \omega t
\]
\[
y(t) = y_0 - \frac{v_{0x}}{\omega} \left( 1 - \cos \omega t \right) + \frac{v_{0y}}{\omega} \sin \omega t
\]
\[
z(t) = z_0 + v_{0z} t
\]

Then (1) combine these coordinates to produce $\mathbf{r} = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}$, (2) recognize that $\hat{k} = \hat{b}$, and (3) recast everything in a full vector formulation. An available symbol manipulating program like MAXIMA, MAPLE or Mathematica may be helpful in solving the differential equations symbolically.

**P3.6.** A region of space contains crossed $\mathbf{E}$ and $\mathbf{B}$ fields, i.e., a constant electric field $\mathbf{E}$ and a constant magnetic induction $\mathbf{B}$ with $\mathbf{E}$ perpendicular to $\mathbf{B}$. A particle having charge $q$ and mass $m$ is projected with an initial velocity $\mathbf{v}$ in the direction of $\mathbf{E} \times \mathbf{B}$. Find the speed $v_0$ for which the particle moves through the region undeflected and describe qualitatively what happens if $v \neq v_0$. This simple arrangement is the basis of one type of velocity selector, used to separate particles of a specific velocity from a beam containing a wide spread of velocities.

When simple analytic solutions to Eq. (3.12) cannot be found, numerical solutions must be sought. Many numerical procedures of varying sophistication and accuracy have been developed. Essentially, these procedures prescribe means to use Eq. (3.12) and the given initial conditions at $t = 0$ to estimate the position and velocity at a slightly later time $t = \Delta t$. The same method can then be applied to generate position and velocity at $t = 2\Delta t$ from those at $t = \Delta t$, and then at $t = 3\Delta t$ from those at $t = 2\Delta t$, and so on until (estimated) solutions have been obtained over some prespecified time interval. Underlying many of the common methods is a pair of equations obtained by formal integration of the definitions of velocity and of acceleration. For example, integrating $\dot{\mathbf{r}} = \mathbf{v}$ (dots symbolize time derivatives) over the interval $t \leq t' \leq t + \Delta t$, we find that

\[
\mathbf{r}(t + \Delta t) = \mathbf{r}(t) + \int_t^{t+\Delta t} \mathbf{v}(t') \, dt' = \mathbf{r}(t) + \Delta t \langle \mathbf{v} \rangle_t^{t+\Delta t}
\]  

(3.13)

where $\langle \mathbf{v} \rangle_t^{t+\Delta t}$ is the average value of $\mathbf{v}$ over the interval of integration. Similarly, we obtain

\[
\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \Delta t \langle \mathbf{a} \rangle_t^{t+\Delta t}
\]  

(3.14)

from the definition $\dot{\mathbf{v}} = \mathbf{a}$, where $\mathbf{a}$ is the acceleration of the particle. Thus, if we know the average velocity over each time interval, we can use Eq. (3.13) to step progressively from $\mathbf{r}(t)$ to $\mathbf{r}(t + \Delta t)$ to $\mathbf{r}(t + 2\Delta t)$ to $\ldots$; similarly, if we know the average acceleration over each time interval, Eq. (3.14) can be used to obtain $\mathbf{v}$ at a sequence of times. The exact average velocities and accelerations, however, are in general not available. We are forced to approximate, and the differences among various numerical methods of solution frequently lie in the manner adopted to approximate the needed averages. The simplest approach, known as the *Euler method*,\(^8\) approximates the averages by their values at the lower end of

\(^8\)Swiss mathematician and physicist Leonard Euler, b. 15 April 1707 in Basel, Switzerland; d. 18 September 1783 in St. Petersburg, Russia.
3.2. TRAJECTORIES OF PARTICLES IN PRESCRIBED FIELDS

Figure 3.1: The Euler method for solving ordinary differential equations numerically.

PROGRAM EULER
    INPUT initial position R0
    INPUT initial velocity V0
    INPUT time increment DT
    T <-- 0 ! Initialize time variable
    PRINT headings for table to come

    LOOP
        A0 <-- a(R0, V0, T) ! Calculate current acceleration
        PRINT T, R0, V0, A0 ! Display current values
        EXIT LOOP WHEN time to stop
        V1 <-- V0 + DT*A0 ! Calculate next velocity
        R1 <-- R0 + DT*V0 ! Calculate next position
        T <-- T + DT ! Increment T to next time
        R0 <-- R1 ! Move next values to current values
        V0 <-- V1
    END LOOP
END PROGRAM

the interval, i.e.,

\[
\langle a \rangle_{t+\Delta t}^t \approx a(t) \quad \langle v \rangle_{t+\Delta t}^t \approx v(t)
\]

where, since \( r(t) \) and \( v(t) \) are known when these averages are needed, \( a(t) = a(r, v, t) \) is calculable directly from the equation of motion—Eq. (3.12) in the present context. Figure 3.1 shows a flow diagram that could be used as a guide to solving Eq. (3.12) either manually or automatically by the Euler method. (For compactness, vector quantities are referred to as vectors; an actual computation might involve separate evaluation of the three components.)

Several more accurate but computationally more involved methods for solving ordinary differential equations have been developed. The improved Euler method, one of the so-called predictor-corrector schemes, involves regarding the Euler solution at each step as a prediction of the solution and then improving (or correcting) that prediction by using it to obtain a better estimate of the average acceleration and velocity over each time interval. The approach is presented in Fig. 3.2. Still more elaborate adaptive methods take each step tentatively, assess the error in the result, and redo the step using either a larger or a smaller time interval based on a comparison with the targeted accuracy of the solution to be generated. Euler’s method, the improved Euler method, and several other methods generate solutions at regularly spaced instants in time and invest computational effort equally, regardless of how rapidly or slowly the solution changes; adaptive methods generate solutions at irregularly spaced instants in time, thereby concentrating the computational effort in temporal regions where the solution changes rapidly and spending less effort in regions where the solution changes more slowly. These methods and the many others
Figure 3.2: A predictor-corrector scheme (improved Euler method) for solving ordinary differential equations numerically.

PROGRAM IMPROVED EULER
INPUT initial position R0
INPUT initial velocity V0
INPUT time increment DT
T <-- 0 ! Initialize time variable

PRINT headings for table to come

LOOP
  A0 <-- a(R0, V0, T) ! Calculate current acceleration
  PRINT T, R0, V0, A0 ! Display current values
  EXIT LOOP WHEN time to stop
  V1 <-- V0 + DT*A0 ! Calculate predicted next velocity
  R1 <-- R0 + DT*V0 ! Calculate predicted next position
  T <-- T + DT ! Increment T to next time
  A1 <-- a(R1, V1, T) ! Calculate predicted next acceleration
  A2 <-- 0.5*(A0 + A1) ! Calculate corrected avg acceleration
  V2 <-- V0 + DT*A2 ! Calculate corrected next velocity
  V3 <-- 0.5*(V0 + V2) ! Calculate corrected avg velocity
  R2 <-- R0 + DT*V3 ! Calculate corrected next position
  R0 <-- R2 ! Move corrected next values
  V0 <-- V2 ! to current values
END LOOP
END PROGRAM

described in books on numerical analysis all give solutions of improving accuracy as $\Delta t$ is made smaller, provided only that $\Delta t$ is not made so small that intrinsic roundoff errors in the computer dominate the accuracy of the solution.

It is, of course, not always necessary for someone needing a solution to a difficult set of differential equations to write programs from scratch. Numerous “canned” programs can be found. Some—e.g., IDL, MATLAB, OCTAVE, and PYTHON—not only have built-in procedures for solving ordinary differential equations but can also generate graphical displays of the resulting solutions. Alternatively, for more elaborate systems and/or faster solution, one might use the FORTRAN program LSODE or procedures from the FORTRAN or C Numerical Recipes library, which, however, will require either supplementary FORTRAN or C routines or exporting of numerical results into a program like IDL, MATLAB, OCTAVE, or PYTHON to display the resulting solutions graphically.  

9In the Lawrence Computational Physics Laboratory, the built-in IDL routines rk4 and lsode, the Lawrence-generated IDL routines ludiffeq_23, and ludiffeq_45, the built-in MATLAB routines ode23, ode45, and ode113, the built-in OCTAVE routine lsode, and the built-in PYTHON routine odeint in the scipy.integrate module can be invoked. In each case, the user needs to create a file defining the differential equations and then invoke the appropriate routine. (The two routines named lu* may have been installed
PROBLEMS

P3.7. A particle having charge $q$ and mass $m$ is projected from the origin in a uniform magnetic induction $\mathbf{B} = B_0 \hat{k}$ with an initial velocity $\mathbf{v}_0 = \hat{i}$. Let $qB_0/m = \omega$. (a) Show that the equation of motion reduces to $\ddot{x} = \omega \dot{y}$, $\ddot{y} = -\omega \dot{x}$, $\ddot{z} = 0$. (b) Show that the particle moves in the $xy$-plane, i.e., that $z(t) = 0$. (c) Use Euler’s method by hand with $\Delta t = 0.1$ and $\omega = 1$ to obtain the first four points on the trajectory of the particle in the $xy$-plane. (d) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to explore the behavior of this particle for various initial conditions and for different (both positive and negative) values of $\omega$. (The sign of $\omega$ reflects the sign of the charge.)

P3.8. (a) For Eq. (3.12) show that the three components of the acceleration are

$$a_x = \frac{q}{m} (E_x + \dot{y}B_z - \dot{z}B_y)$$
$$a_y = \frac{q}{m} (E_y + \dot{z}B_x - \dot{x}B_z)$$
$$a_z = \frac{q}{m} (E_z + \dot{x}B_y - \dot{y}B_x)$$

(b) Using an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON, explore the trajectories of particles moving in several different fields with several different initial conditions. Try

- the Coulomb force $\mathbf{E} = r/r^3$, $\mathbf{B} = 0$;
- a slightly perturbed Coulomb force, $\mathbf{E} = r/r^{3.1}$, $\mathbf{B} = 0$;
- a constant magnetic field, $\mathbf{E} = 0$, $\mathbf{B} = B_0 \hat{k}$;
- crossed constant electric and magnetic fields $\mathbf{E} = E_0 \hat{j}$, $\mathbf{B} = B_0 \hat{k}$;

and any others that occur to you.

P3.9. Do the previous problem but use the predictor-corrector scheme of Fig. 3.2. Optional: Look up the fourth-order Runge-Kutta method in an available book on numerical analysis, write a program using this method, and test the program.

Yet another means to obtain solutions to Eq. (3.12) involves constructing an electronic circuit in such a way that voltages within the circuit vary with time exactly as the position and velocity vary with time. The solution is then obtained by measuring or plotting voltages as functions of time. Circuits of this sort are said to simulate the physical system described by Eq. (3.12). Before the development of digital plotters, an analog computer equipped with a pen plotter provided a valuable way to obtain graphical output conveying these solutions. Indeed, analog computers, on which the values of parameters were set by turning a potentiometer, provided a very convenient instrument on which to explore the dependence of a solution on the values of parameters. A detailed discussion of these now obsolete analog computers, which facilitate the construction of circuits to simulate Eq. (3.12), would take us far afield, and we leave this discussion to other authors.\(^\text{10}\)

---

3.3 Forces and Torques on General Distributions in Prescribed Fields

In this section we shall consider the forces and torques on a general charge distribution when it is placed in a known electromagnetic field. For a distribution consisting of point charges, the $i$-th of which has charge $q_i$ at time $t$ and is located at $r_i$ with velocity $v_i$, the net (external) electromagnetic force $F(t)$ and the net (external) torque $N(t)$ are obtained by summing individual contributions, i.e.,

$$F(t) = \sum_i q_i [E(r_i, t) + v_i \times B(r_i, t)] \quad (3.16)$$

$$N(t) = \sum_i r_i \times q_i [E(r_i, t) + v_i \times B(r_i, t)] \quad (3.17)$$

When the charge distribution is more easily described by particle densities $n^{(a)}(r, t)$ and velocity fields $v^{(a)}(r, t)$ for each of several types of particle, the sum over all particles in Eq. (3.16), for example, is more appropriately expressed as

$$F(t) = \sum_j \Delta F_j(t) \quad (3.18)$$

where $\Delta F_j$ is the force experienced by an element of the distribution occupying a volume $\Delta V_j$ centered at the point $r_j$. Within $\Delta V_j$ at time $t$, however, there are $n^{(a)}(r_j, t) \Delta V_j$ particles of type $a$, each of which experiences the force $q_a [E(r_j, t) + v^{(a)}(r_j, t) \times B(r_j, t)]$. Thus,

$$\Delta F_j(t) = \sum_a n^{(a)}(r_j, t) \Delta V_j \left( q_a [E(r_j, t) + v^{(a)}(r_j, t) \times B(r_j, t)] \right)$$

$$= [\rho(r_j, t)E(r_j, t) + J(r_j, t) \times B(r_j, t)] \Delta V_j \quad (3.19)$$

where we have recognized the charge density [Eq. (2.10)] and the current density [Eq. (2.18)] in the sum over types of particle. Finally, summing Eq. (3.19) over all volume elements and allowing $|\Delta V_j| \to 0$ for all $j$,\(^{11}\) we find that

$$F(t) = \int [\rho(r, t)E(r, t) + J(r, t) \times B(r, t)] dv \quad (3.20)$$

where the integral extends over the region occupied by the distribution at time $t$. The expression

$$F = \int (\rho E + J \times B) \, dv \quad (3.21)$$

obtained by suppressing all arguments is easier to remember and also supports more obviously the interpretation of $\rho E$ as an electric force density, $J \times B$ as a magnetic force density, and the combination $\rho E + J \times B$ as an electromagnetic force density. Similar arguments lead from Eq. (3.17) to the expression

$$N = \int r \times (\rho E + J \times B) \, dv \quad (3.22)$$

for the torque at time $t$ on a general distribution placed in the fields $E, B$.

\(^{11}\)As always, keeping $|\Delta V_j|$ microscopically large even as it becomes macroscopically small.
3.3. FORCES AND TORQUES ON GENERAL DISTRIBUTIONS

Although Eqs. (3.21) and (3.22) answer the primary question of this section, some alternative forms are frequently useful. If, for example, the charge distribution is more appropriately described by a surface or line charge density, it will be convenient to recognize in $\rho \, dv$ a charge element $dq$ and to write the electric force in the form

$$ F_{\text{elec}} = \int E \, dq $$

(3.23)

and the electric torque in the form

$$ N_{\text{elec}} = \int r \times E \, dq $$

(3.24)

Here $dq$ can be interpreted as $\rho \, dv$, $\sigma \, dS$, or $\lambda \, d\ell$, as appropriate. Indeed, it is often convenient to understand these integrals to include sums over point charges in the distribution as well as integrals over distributed charge.

Other useful alternative forms are expressions for the magnetic parts of Eqs. (3.21) and (3.22) that apply more specifically to currents in wires. In that special case, the entity $I \, dl$ replaces $J \, dv$ (P2.7) and line integrals replace volume integrals. Thus, the magnetic force on a wire carrying current $I$ is given by

$$ F_{\text{mag}} = I \int dl \times B $$

(3.25)

and the magnetic torque by

$$ N_{\text{mag}} = I \int r \times (dl \times B) $$

(3.26)

Both integrals extend over the portion of the wire on which the force or torque is desired and may—indeed most frequently will—be extended over a closed circuit. In these integrals, the direction of $dl$ coincides with the direction of $J$ if $I$ is to be a positive number and the wire must be supposed to have a small cross section so that the current element can be adequately treated as having zero cross section.

Equations (3.23) and (3.24) have particularly simple evaluations if $E$ is constant. Under those conditions $E$ can be taken outside the integral and we find for a general charge distribution in a constant $E$-field that

$$ F_{\text{elec}} = E \int dq = QE $$

(3.27)

where $Q$ is the net charge in the distribution, and that

$$ N_{\text{elec}} = \left( \int r \, dq \right) \times E = p \times E $$

(3.28)

where $p$, defined by

$$ p = \int r \, dq $$

(3.29)

is called the (electric) dipole moment of the general distribution.\(^{12}\) Both $Q$ and $p$ are characteristics of the charge distribution; they do not depend in any way on the externally applied field.

\(^{12}\)Equation (3.29) defines the electric dipole moment of a charge distribution in all systems of units with which the author is familiar.
The general expressions for the magnetic force and torque on a complete circuit (closed line integrals) in a constant magnetic induction field can also be fully evaluated. If $\mathbf{B}$ is constant, the magnetic force on a complete circuit is given by

$$
\mathbf{F}^{\text{mag}} = I \oint d\mathbf{l} \times \mathbf{B} = I \left( \oint d\mathbf{l} \right) \times \mathbf{B} = 0
$$  \hspace{1cm} (3.30)

regardless of the shape of the circuit. ($\oint d\mathbf{l}$ is the vector from the starting point to the end point of the path, which is the zero vector if the path is closed.) The magnetic torque on a circuit in a uniform magnetic induction is more difficult to evaluate, since it is more difficult to remove $\mathbf{B}$ from under the integral sign in Eq. (3.26). Recognizing that $d\mathbf{l}$ in Eq. (3.26) is in fact an increment in $\mathbf{r}$, we can replace $d\mathbf{l}$ with $d\mathbf{r}$. Then

$$
\mathbf{N}^{\text{mag}} = I \oint \mathbf{r} \times (d\mathbf{r} \times \mathbf{B}) = I \oint (\mathbf{r} \cdot \mathbf{B}) d\mathbf{r} - I \left( \oint \mathbf{r} \cdot d\mathbf{r} \right) \mathbf{B} = I \oint (\mathbf{r} \cdot \mathbf{B}) d\mathbf{r}
$$  \hspace{1cm} (3.31)

the final form following because $\mathbf{r} \cdot d\mathbf{r} = x \, dx + y \, dy + z \, dz = d(x^2 + y^2 + z^2)/2$ is an exact differential and $\oint \mathbf{r} \cdot d\mathbf{r}$, which extends over a closed path, is therefore zero. We now remove $\mathbf{B}$ from under the integral sign in Eq. (3.31). First, Eq. (C.20) with $d\mathbf{l}$ replaced by $d\mathbf{r}$ converts Eq. (3.31) to

$$
\mathbf{N}^{\text{mag}} = I \int d\mathbf{S} \times \nabla (\mathbf{r} \cdot \mathbf{B})
$$  \hspace{1cm} (3.32)

where the integral now extends over the surface bounded by the circuit and $d\mathbf{S}$ assumes the direction of the thumb of the right hand when the fingers point around the circuit in the direction in which the line integral is traversed and the palm faces towards the surface enclosed by the circuit. Remembering that $\mathbf{B}$ is constant and applying Eq. (C.15), we next find that

$$
\mathbf{N}^{\text{mag}} = I \int d\mathbf{S} \times [\mathbf{B} \cdot \nabla \mathbf{r} + \mathbf{B} \times (\nabla \times \mathbf{r})]
$$  \hspace{1cm} (3.33)

Finally, recognizing that $(\mathbf{B} \cdot \nabla)\mathbf{r} = \mathbf{B}$ and $\nabla \times \mathbf{r} = 0$, we find that the torque on a current loop in a constant $\mathbf{B}$-field is given by

$$
\mathbf{N}^{\text{mag}} = \mathbf{m} \times \mathbf{B}
$$  \hspace{1cm} (3.34)

where

$$
\mathbf{m} = I \int d\mathbf{S}
$$  \hspace{1cm} (3.35)

is called the (magnetic) dipole moment of the loop. If in particular the loop is plane and bounds a plane surface with unit normal $\mathbf{n}$ and area $S$, $d\mathbf{S} = |d\mathbf{S}| \mathbf{n}$ and $\mathbf{m} = I S \mathbf{n}$. Thus, in this simple case $\mathbf{m}$ is directed normal to the plane of the loop and has a magnitude given by the product of the current in the loop and the area of the loop.\(^\text{13}\) As with $\mathbf{Q}$ and $\mathbf{p}$ in the previous paragraph, $\mathbf{m}$ is a characteristic of the current distribution; it does not depend in any way on the externally applied field. It is shown in P3.20 that Eq. (3.35) can alternatively be written as a line integral\(^\text{14}\)

$$
\mathbf{m} = \frac{1}{2} I \oint \mathbf{r} \times d\mathbf{r}
$$  \hspace{1cm} (3.36)

\(^{\text{13}}\)The usual right-hand rule, which entered when we converted the line integral in Eq. (3.31) to the surface integral in Eq. (3.32), relates the direction of $I$ to that of $\mathbf{n}$.

\(^{\text{14}}\)In all systems of units with which the author is familiar, the magnetic dipole moment of a current distribution is defined so that Eq. (3.34) gives the torque experienced by the dipole in a constant magnetic induction. In cgs-esu, cgs-emu, and mks units, Eqs. (3.35) and (3.36) can be used to calculate the magnetic moment. In Gaussian and Heaviside-Lorentz units, however, the factor of $c$ incorporated explicitly in the
3.3. FORCES AND TORQUES ON GENERAL DISTRIBUTIONS

The power input to a general distribution by an external electromagnetic field is the final quantity of interest in this section. For a collection of discrete particles distinguished by an index $i$, the power input $P(t)$ at time $t$ is given by

$$P(t) = \sum_i v_i \cdot F_i = \sum_i q_i v_i \cdot E(r_i, t)$$

(3.37)

where $F_i$ is the electromagnetic force on the $i$-th particle and the magnetic force ultimately contributes nothing to the power because it is always perpendicular to $v$ [$v \cdot (v \times B) = 0$]. When the charge distribution is described by particle density and velocity fields $n(a)(r, t)$ and $v(a)(r, t)$, we follow an argument similar to that leading from Eq. (3.16) to Eq. (3.20). In the notation of that argument, the power input $\Delta P_j(t)$ to a volume element $\Delta V_j$ centered at $r_j$ then is given by

$$\Delta P_j(t) = \sum_a q_an(a)(r_j, t) v(a)(r_j, t) \cdot E(r_j, t) \Delta V_j$$

(3.38)

Summing over all volume elements and allowing $|\Delta V_j| \rightarrow 0$ for all $j$, we find the expression

$$P(t) = \int J(r, t) \cdot E(r, t) \, dv$$

(3.39)

for the power input to the entire distribution.

PROBLEMS

P3.10. (a) Interpreting the integral in Eq. (3.29) to include a sum over any point charges present, show that the dipole moment of a distribution consisting of two (rigidly connected) charges of strength $q$ and $-q$ located at $r_+$ and $r_-$ is given by $p = qa$, where $a$ is the vector from charge $-q$ to charge $q$. Qualitatively, what characteristic of a charge distribution is measured by the dipole moment? (b) Let this charge distribution be placed in a constant electric field $E$ and oriented with $p$ at an angle $\theta$ to $E$. Draw a diagram showing the external force experienced by each charge and, arguing from this diagram, find the net force and the net torque on the distribution. Verify that your results agree with Eqs. (3.27) and (3.28). (c) Suppose this distribution, initially oriented as in part (b), is released from rest. Describe its subsequent motion qualitatively. Are there any positions of static equilibrium? If so, are these positions stable or unstable? (d) Let $r_{\pm} = (x_0 \pm \frac{1}{2}a)\hat{i} + y_0 \hat{j} + z_0 \hat{k}$ and let the field now be the (nonuniform) field $E(r) = E(x)\hat{i}$. Assuming $a$ to be sufficiently small, show that this distribution experiences the force $F = p \frac{dE}{dx}|_0 \hat{i}$.

P3.11. The response of an arbitrary charge distribution to a constant external field is observed experimentally. Discuss what can be learned about the distribution from these observations.

The definition of $B$ (see footnote 4) will also appear dividing $J$ in Eq. (3.21). Thus, Eq. (3.26) becomes

$$N^{mag} = \frac{I}{c} \int \mathbf{r} \times (d\mathbf{l} \times \mathbf{B})$$

and, in Gaussian and Heaviside-Lorentz units, the magnetic dipole moment must be defined by

$$m = \frac{I}{2c} \int d\mathbf{S} = \frac{I}{2c} \oint \mathbf{r} \times d\mathbf{r}$$

if Eq. (3.34) is to be preserved.
P3.12. The electron cloud of one of the stationary states of the hydrogen atom in a constant external electric field along the $z$-axis is described quantum mechanically by the charge density

$$\rho(r) = -\frac{q}{16\pi a_0^3} \left( 1 - \frac{r}{a_0} \sin^2 \frac{\theta}{2} \right)^2 e^{-r/a_0}$$

where $q$ is the proton charge, $a_0$ is the Bohr radius, and spherical coordinates are employed. Calculate the electric dipole moment of this atom about the nucleus. *Hints*: (1) $r = r(\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k})$ in spherical coordinates. (2) Don’t be afraid of integral tables. Better yet, use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica.

P3.13. A rectangular circuit of length $a$ and width $b$ is placed in a constant magnetic induction $B$ such that the normal to its plane makes an angle $\theta$ with the direction of $B$. The circuit carries a current $I$ that goes into the paper at the point marked $\times$ in Fig. 3.3 and emerges at the point marked $\bullet$. (a) Draw a diagram showing the external force experienced by each side of the rectangle and, arguing from this diagram, find the net force and the net torque on the rectangle. Express the torque in terms of the magnetic moment of the current loop and show that your result can be written in the form of Eq. (3.34). (b) Let the rectangle now be released from rest. Describe its subsequent motion qualitatively. Are there any positions of static equilibrium? If so, are these positions stable or unstable?

P3.14. Calculate the magnetic dipole moment of a sphere of radius $R$ carrying charge $Q$ distributed uniformly over its surface and spinning about a diameter with angular velocity $\omega$. *Hint*: Break the sphere into small current loops and sum the contribution of each.

SUPPLEMENTARY PROBLEMS

P3.15. A particle having charge $q$ and mass $m$ is projected from the origin with an initial velocity $v_0 = v_{x0} \hat{i} + v_{z0} \hat{k}$ in a constant magnetic induction $B = B_0 \hat{k}$. Find the trajectory of the particle and in particular determine the coordinates $x_d$ and $y_d$ of the point $P$ at which the particle strikes a screen in the plane $z = d$. Describe how those coordinates depend on $B_0$. The analysis of this problem is essentially the analysis of the Busch method for measuring the charge-to-mass ratio of a particle. *Hint*: Express the coordinates of the point $P$ in terms of the angle $\psi = qdB_0/2mv_{z0}$ and then interpret the angle geometrically.

P3.16. Use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to solve the equation of motion for a particle of charge $q$ and mass $m$ moving in the combination of a constant magnetic induction $B = B_0 \hat{k}$ and a constant electric field $E = E_0 \hat{j}$. Let the particle start at the origin but have an arbitrary initial velocity in...
3.3. FORCES AND TORQUES ON GENERAL DISTRIBUTIONS

P3.17. Solve the equation of motion identified in P3.16 by showing first that, when \( r \) satisfies Eq. (3.12) with these fields, \( r' = r - [(E \times B)/B^2]t \) satisfies \( \ddot{r}' = q r' \times B \), which is the equation solved in P3.5. Thus, the trajectory in crossed \( E \) and \( B \) fields is a circle on which is superimposed the constant drift velocity \( v_d = E \times B/B^2 \).

P3.18. Let the charge distribution to which Eq. (3.23) applies be confined to a small enough region of space about the point \( r_0 \) that the terms

\[
E(r) = E_0 + (x - x_0) \left. \frac{\partial E}{\partial x} \right|_0 + (y - y_0) \left. \frac{\partial E}{\partial y} \right|_0 + (z - z_0) \left. \frac{\partial E}{\partial z} \right|_0
\]

of the Taylor expansion provide an adequate approximation to \( E \) over the region occupied by the charge. Show that

\[
F_{\text{elec}} = qE_0 + (p_0 \cdot \nabla)E|_0
\]

where \( p_0 \) is the dipole moment of the distribution about \( r_0 \). Here, \( E_0 = E(r_0) \).

P3.19. Show that the dipole moment of an arbitrary charge distribution is invariant to translation of the coordinate system if and only if the distribution has zero net charge.

P3.20. Use Eq. (C.24) to show that \( \oint r \times dr = 2 \int dS \), thus verifying the equivalence of Eqs. (3.35) and (3.36).

P3.21. A magnetic dipole is located at the origin of a coordinate system with its dipole moment \( m \) oriented perpendicular to a constant magnetic induction field \( B \). Show that the work that must be done on the dipole to move it to an arbitrary point \( r \) in this field and rotate it so that \( m \) makes an angle \( \theta \) different from 90° with \( B \) is given by \(-m \cdot B \).

P3.22. (a) Show that the magnetic dipole moment \( m \) of a charge \( q \) moving in a circle and having mass \( \mu \) is related to the angular momentum \( L \) of the charge by \( m = (q/2\mu)L \). (b) According to the rotational analog of Newton’s second law, the time rate of change of the angular momentum of an object is equal to the external torque acting on the object. Show that, when the charge in part (a) is placed in a constant magnetic induction field \( B \), its magnetic moment satisfies the equation of motion

\[
\frac{dm}{dt} = \frac{q}{2\mu} m \times B
\]

and then, taking \( B = Br \) and \( m(0) \) to have magnitude \( m_0 \) and to lie in the \( xz \)-plane at an angle \( \theta \) to the \( z \)-axis, i.e., \( m(0) = m_0 \sin \theta \hat{i} + m_0 \cos \theta \hat{k} \), find \( m(t) \) and describe its behavior geometrically as a function of time. Hint: An available symbol manipulating program like MAXIMA, MAPLE or Mathematica may be helpful in solving the differential equations symbolically.

P3.23. The force on a current loop in an arbitrary magnetic induction \( B(r) \) is given by Eq. (3.25). Suppose the loop is centered at \( r_0 \) and is small enough so that the approximation

\[
B(r) = B_0 + (x - x_0) \frac{\partial B_0}{\partial x_0} + (y - y_0) \frac{\partial B_0}{\partial y_0} + (z - z_0) \frac{\partial B_0}{\partial z_0}
\]

where \( B_0 = B(r_0) \), can be accurately made at all points on the loop. Show that the force \( \mathbf{F} \) on the loop is given by \( \mathbf{F} = \nabla_0 (\mathbf{m} \cdot \mathbf{B}_0) \), where \( \nabla_0 \) involves derivatives with respect to \( x_0, y_0, \) and \( z_0 \) and \( \mathbf{m} \) is the magnetic dipole moment of the loop. This force on a dipole in

the \( xy \)-plane. Describe the trajectory for several different initial velocities—perhaps use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to plot the solutions—and in particular examine the dependence of the points at which the particle crosses the \( x \)-axis on the initial velocity.
an inhomogeneous magnetic induction is exploited in the Stern-Gerlach experiment\textsuperscript{15,16} to separate particles in a beam according to their magnetic moments. *Hints:* (1) Show first that
\[
\mathbf{F} = \oint I \, dl \times \left( x \frac{\partial \mathbf{B}_0}{\partial x_0} + y \frac{\partial \mathbf{B}_0}{\partial y_0} + z \frac{\partial \mathbf{B}_0}{\partial z_0} \right)
\]
and then use Eq. (C.24) to show, for example, that
\[
\oint I \, dl \times \left( x \frac{\partial \mathbf{B}_0}{\partial x_0} \right) = \frac{\partial}{\partial x_0} \oint I \, dl \times (x \mathbf{B}_0) = -m \frac{\partial B_{0x}}{\partial x_0} + i \frac{\partial}{\partial x_0} (\mathbf{m} \cdot \mathbf{B}_0)
\]
[See Eq. (3.35).] (2) Anticipating a result to be demonstrated in Chapter 5, note that every magnetic induction field \( \mathbf{B}_0(\mathbf{r}_0) \) satisfies \( \nabla \cdot \mathbf{B}_0 = 0 \).

**P3.24.** Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to examine the motion of a charged particle in an inverse square electric field for which the (dimensionless) equations of motion are
\[
\frac{d^2 x}{dt^2} = \pm \frac{x}{[x^2 + y^2]^{3/2}} ; \quad \frac{d^2 y}{dt^2} = \pm \frac{y}{[x^2 + y^2]^{3/2}}
\]
Here the + sign refers to a repulsive force and the − sign to an attractive force. Explore both scattering and bound orbits. As a start, take the minus sign and try the initial values (i) \([X_0 V_0 Y_0 W_0] = [4 \ 0 \ 0 \ .5]\) and (ii) \([X_0 V_0 Y_0 W_0] = [-10 \ 1 \ 5 \ 0]\). Then try other values.

**P3.25.** Consider a particle of mass \( m \) and charge \( q \) moving in the \( xy \)-plane under the influence of a constant magnetic field \( \mathbf{B} = B \mathbf{k} \) and a constant electric field \( \mathbf{E} = E \mathbf{j} \). (a) Find the equations of motion. (b) Choosing a unit of length \( a \) and recognizing the frequency \( \omega = qB/m \), cast your equations in the dimensionless form
\[
\frac{d^2 X}{dT^2} = \frac{dY}{dT} ; \quad \frac{d^2 Y}{dT^2} = -\frac{dX}{dT} + s
\]
where \( X = x/a \), \( Y = y/a \), \( T = \omega t \), and \( s = mE/aqB^2 \). (c) Use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to obtain an analytic solution of this set of equations for \( X(t) \) and \( Y(t) \) subject to the initial conditions \( X(0) = 0 \), \( Y(0) = 0 \), \( dX(0)/dT = U_0 \), and \( dY(0)/dT = V_0 \). (d) Treating the parameter \( s \) as a single quantity to be supplied as a parameter, create a suitable file to communicate these equations to the routines in an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON and then examine the motion for several different values of \( s \) and a variety of different initial values. *Suggestion:* Start with the initial values \([X_0 V_0 Y_0 W_0] = [-10 \ 0 \ 0 \ 2]\), but then try other values.

**P3.26.** A particle having mass \( m \) and carrying charge \( q \) moves in the \( xy \)-plane while experiencing an electric field given by \( \mathbf{E}(x, y) = -\alpha y \mathbf{j} \), where \( \alpha \) is a constant. Assume that \( \alpha \) and \( q \) are both positive. (a) Show that the equations of motion for this particle are
\[
\frac{d^2 x}{dt^2} = 0 ; \quad \frac{d^2 y}{dt^2} = -\frac{\alpha q}{m} y = -by
\]
(b) With \( b \) a parameter, create a suitable file defining these equations for an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON. (c) Use that program to obtain graphs of the trajectories in the \( xy \)-plane of several particles projected from the origin with different angles and different speeds. (d) Speculate on a use for this field.

\textsuperscript{15}German physicist Otto Stern, b. 17 February 1888 in Sohrau, Kingdom of Prussia (now Zory, Poland); d. 17 August 1969 in Berkeley, California.

\textsuperscript{16}German physicist Walter Gerlach, b. 1 August 1889 in Biebrich, German Empire; d. 10 August 1979 in Munich, West Germany.
Chapter 4

The Electric Field of Static Charges

In this and the next two chapters, we shall be concerned principally with the question, “How is the electromagnetic field determined from its sources?”. We shall, however, also examine a number of consequences of the principal relationships between the fields and the sources. The three chapters treat static electric fields, static magnetic induction fields, and time-dependent electromagnetic fields in sequence. Throughout these chapters, we shall focus on determining the fields produced by given sources and on determining what sources must be set up to establish a required field. The forces that these fields may subsequently exert on other distributions of charge play no role in our present considerations.

4.1 Finding the Electrostatic Field from its Sources

The starting point for relating the static electric field to its sources is Coulomb’s Law for the force of interaction between two point charges, which we shall now write in a general vector form. Although Coulomb’s Law has the simplest form in a coordinate system whose origin coincides with one of the two point charges, we are here interested in obtaining a more general expression so that the choice of a coordinate origin can be dictated by its convenience to later problems. We therefore select now a coordinate system that may seem unnecessarily cumbersome for the shorter-range objectives of this paragraph. Let point charges $q$ and $q'$ be located at $\mathbf{r}$ and $\mathbf{r}'$, as shown in Fig. 4.1. Further, let $\mathbf{R}$ be the vector from $\mathbf{r}'$ to $\mathbf{r}$. The force $\mathbf{F}_q(\mathbf{r})$ on $q$ due to the presence of $q'$ (as given by the experimental properties outlined in Section 1.2) may then be expressed in mks units (which we now explicitly select) by

$$\mathbf{F}_q(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{qq'}{R^2} \hat{\mathbf{R}} = \frac{qq'}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

(4.1)

the second form following because $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = |\mathbf{r} - \mathbf{r}'|$, and $\hat{\mathbf{R}} = \mathbf{R}/R$ (P0.11). All of the observed features are incorporated in this law, including in particular the repulsive and attractive nature of the force. If $q$ and $q'$ have the same sign, the coefficient of $\hat{\mathbf{R}}$ is overall positive and the force on $q$ has the direction of $\hat{\mathbf{R}}$, i.e., is repulsive (directed away from $q'$). If $q$ and $q'$ have opposite signs, the coefficient of $\hat{\mathbf{R}}$ is negative and the force—again on $q$—is attractive (directed towards $q'$).
Before we can use Coulomb’s Law to relate a general electrostatic field to its sources, we need one additional *experimental* observation: The force between two given point charges is unaffected by the presence of still other charges in their vicinity. Thus, the force exerted on a charge \( q \) at \( r \) by an assembly of point charges \( q_i \) residing at points \( r_i \) is given (in mks units) by

\[
F_q(r) = \frac{q}{4\pi\epsilon_0} \sum_i q_i \frac{r - r_i}{|r - r_i|^3}
\] (4.2)

which expresses the *principle of superposition*. Its content is more far-reaching than the limited superposition built into the definition of charge—earlier we took two point charges at the same place to superpose. In essence, there are two independent experimental observations underlying electrostatics: Coulomb’s Law and the principle of superposition.

The charge densities introduced in Section 2.1 now facilitate a reexpression of Eq. (4.2) that is more appropriate for finding the force exerted on a point charge \( q \) by a space-filling distribution of charge. The volume, surface, or line making up the source distribution can be divided into elements that are small enough to be regarded as point charges. If the \( i \)-th element carries charge \( \Delta q_i \), then the total force \( F_q(r) \) on \( q \) at \( r \) is simply a sum of contributions from each element, i.e.,

\[
F_q(r) = \frac{q}{4\pi\epsilon_0} \sum_i \Delta q_i \frac{r - r_i}{|r - r_i|^3}
\] (4.3)

(See Fig. 4.2.) If all of the elements now become arbitrarily small, Eq. (4.3) is replaced by the integral

\[
F_q(r) = \frac{q}{4\pi\epsilon_0} \int \frac{r - r'}{|r - r'|^3} dq'
\] (4.4)

in which the limits are chosen so that the integration extends over the source charge distribution and \( dq' = \rho(r') du' \), \( \sigma(r') dS' \), or \( \lambda(r') d\ell' \) as appropriate to the distribution. If, in fact, the distribution contains several portions, the integral in Eq. (4.4) includes a sum of integrals, each related to a single portion of the distribution, and the charge element may be differently expressed for different portions of the distribution. Further, interpreting the integral in Eq. (4.4) to include a sum over any point charges present, we can view Eq. (4.2) as a special case of Eq. (4.4).

The final step in obtaining an expression relating the field to its sources is to apply the definition in Eq. (3.3) to Eqs. (4.1), (4.2), and (4.4). We find from Eq. (4.1), for example, that the electric field \( E(r) \) established at a point \( r \) by a point charge \( q' \) located at \( r' \) is given by

\[
E(r) = \frac{q'}{4\pi\epsilon_0} \frac{r - r'}{|r - r'|^3}
\] (4.5)

Similarly, Eq. (4.2) leads to

\[
E(r) = \frac{1}{4\pi\epsilon_0} \sum_i q_i \frac{r - r_i}{|r - r_i|^3}
\] (4.6)

for the field established at \( r \) by an array of point charges, and Eq. (4.4) leads to

\[
E(r) = \frac{1}{4\pi\epsilon_0} \int \frac{r - r'}{|r - r'|^3} dq'
\] (4.7)
4.1. FINDING THE ELECTROSTATIC FIELD FROM ITS SOURCES

Figure 4.1: Two point charges in an arbitrary coordinate system.

Figure 4.2: An arbitrary charge distribution.

for the field established at \( \mathbf{r} \) by a more general distribution of charge. Equations (4.5)–(4.7) not only provide the means explicitly to determine the field established by a known charge distribution, but they also express the relationship between the field and its sources even if the charge distribution itself is not known. In conventional terminology, the point \( \mathbf{r} \), at which the field is evaluated, is called the field point and the point \( \mathbf{r}' \), at which an element of the source is located, is called the source point. In more advanced texts, the convergence of the integral in Eq. (4.7) is explored in some detail, particularly when the point \( \mathbf{r} \) lies within the source distribution.

4.1.1 The Electric Monopole (Point Charge)

We shall now present several examples of specific electric fields, beginning with the field of a single point charge, sometimes called a monopole. Let the charge have strength \( Q \) and be
Figure 4.3: The vector representation of the field of (a) a positive and (b) a negative point charge. The field vectors are directed everywhere away from a positive charge and toward a negative charge. The magnitude of the vectors is inversely proportional to the square of the distances from the charge.

Figure 4.4: The field line representation of the field of (a) a positive and (b) a negative point charge. The field lines are directed radially outward from a positive charge and radially inward toward a negative charge, and the magnitude of the field is conveyed by the spacing of the field lines.

Located at the origin. Then Eq. (4.5) gives

$$E(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}$$

(4.8)

for the resulting field. This field is at every point directed away from the source charge when \( Q > 0 \) and toward the source when \( Q < 0 \), and its magnitude varies as \( 1/r^2 \). The field given by Eq. (4.8) for \( Q > 0 \) and \( Q < 0 \) is shown in Figs. 4.3 and 4.4. Figure 4.4, with field lines emerging from positive charges and terminating on negative charges, illustrates graphically the statement that charge is the source \( (Q > 0) \) or sink \( (Q < 0) \) of the electric field, but this observation can be taken seriously only after it has been proved that conventions relating the density of field lines to the strength of the field can be adhered to without starting or stopping field lines at points where there are no charges. (See P0.18.)
4.1. FINDING THE ELECTROSTATIC FIELD FROM ITS SOURCES

Figure 4.5: Field lines of a dipole. (The light lines indicate equipotential surfaces, which will be discussed in Section 4.6; those equipotential surfaces lying very close to the point charges are not shown.)

Figure 4.6: Coordinate system for expressing the field of a dipole.

4.1.2 The Electric Dipole

Another important charge distribution called the electric dipole is composed of two charges of the same magnitude but of opposite sign. The field lines of this dipole are shown graphically in Fig. 4.5. Near the positive charge, the field is directed radially away from the charge; near the negative charge, the field is directed radially toward the charge; at points on the perpendicular bisector of the line joining the charges, the field is parallel to the line joining the charges. (Why?) Analytically the field of this dipole is given by Eq. (4.6) or, in terms of the symbols introduced in Fig. 4.6, by
\[
E(r) = q \frac{4 \pi \varepsilon_0}{r} \left( \frac{r - \frac{1}{2} a}{\sqrt{r^2 - \frac{1}{2} a^2}} - \frac{r + \frac{1}{2} a}{\sqrt{r^2 + \frac{1}{2} a^2}} \right) \tag{4.9}
\]

In this result, the charges \( q \) and \(-q\) are taken to be at the points \((0, 0, \frac{1}{2} a)\) and \((0, 0, -\frac{1}{2} a)\) and the vector \( a \) is directed from the negative charge to the positive charge. Now, in most cases when the dipole is a reasonable approximation to a more complicated physically important distribution (e.g., an asymmetric molecule), the distance from the field point to the dipole is large compared to the separation of the charges, i.e., \( r \gg a \). Using the binomial theorem (Appendix B) and keeping only the first nonzero term in an expansion in powers of \( a/r \), we find—see P4.2—that the field given by Eq. (4.9) can be approximated when \( r \gg a \) by the simpler expression

\[
E(r) = 4 \pi \varepsilon_0 \frac{1}{r^3} \left[ 3(p \cdot \hat{r})\hat{r} - p \right] \tag{4.10}
\]

\[
= \frac{p}{4 \pi \varepsilon_0 r^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] \tag{4.11}
\]

where

\[
p = qa \tag{4.12}
\]

is the (electric) dipole moment of the source distribution. In the present context, however, \( p \) pertains to the source distribution and determines the resulting field; in Section 3.3, \( p \) pertained to the test distribution and determined (among other things) the torque exerted on the distribution by a uniform electric field. It is a convenient coincidence that this single characteristic of a distribution plays these two very different roles. Although Eqs. (4.10) and (4.11) strictly apply only when \( r \gg a \), it is sometimes useful to visualize an idealization in which \( r \gg a \) for every point in space. Then Eqs. (4.10) and (4.11) apply for all \( r \), and the corresponding ideal charge distribution—called a point dipole—is characterized only by a dipole moment; one does not ask about the individual charges composing a point dipole. The field of a dipole varies inversely as the cube of \( r \). Further, it has no azimuthal components and its radial and polar components are independent of the azimuthal angle \( \phi \), properties which reflect the invariance of the charge distribution to arbitrary rotation about the axis of the dipole.

### 4.1.3 Electric Multipoles

One way to visualize the formation of a dipole is to think first of a monopole. Now place near this monopole a second monopole of opposite sign. The result is a dipole. Using this same sort of visualization, we can construct higher-order “poles”. Location of one dipole adjacent to a second of opposite dipole moment, for example, gives rise to a quadrupole, of which there are several sorts, since the direction of the displacement of the two dipoles has a more profound effect on the overall distribution than does the direction of the displacement of one charge in a dipole from the other. Two different quadrupoles are shown in Fig. 4.7.

Octupoles are made by placing two quadrupoles side by side, in one of which the charges have opposite signs to the corresponding charges in the other. And so it goes to higher moments. That a charge distribution of a particular character can be produced by combining two slightly displaced distributions of the next lower order suggests the fascinating thought that the field of the higher-order “pole” might be obtained by differentiating the field of the lower-order “pole”, and such a relationship indeed exists (P4.49).
4.1. FINDING THE ELECTROSTATIC FIELD FROM ITS SOURCES

Figure 4.7: Two quadrupoles: (a) the linear quadrupole and (b) a more general quadrupole.

\[ +q \bullet -q \]
\[ +q \]
\[ -2q \]
\[ +q \]
\[ -q \]
\[ +q \]

(a) (b)

4.1.4 Other Distributions of Point Charges

The various multipole distributions described in the previous paragraphs require very specific relationships among the strengths of the component charges. The field lines for some representative distributions that do not conform exactly to any multipole are shown in Fig. 4.8.

4.1.5 An Infinite Sheet

To illustrate the use of Eq. (4.7), let the source distribution be an infinite two-dimensional plane sheet of charge characterized by a constant surface density \( \sigma \) and lying in the \( xy \)-plane (Fig. 4.9). Since the source looks the same to an observer at \((0,0,z)\) as it looks to an observer at \((x,y,z)\), the resulting field cannot depend on \(x\) or \(y\), i.e., \( \mathbf{E}(x,y,z) = \mathbf{E}(0,0,z) \).

To find \( \mathbf{E}(0,0,z) \) using Eq. (4.7), we select cylindrical coordinates. An element of the charged sheet is then the small area shown in Fig. 4.9 and we have that

\[ \mathbf{r} = \text{position vector of field point} = \hat{z} \]
\[ \mathbf{r}' = \text{position vector of source point} = \mathbf{r}' \cos \phi' \hat{i} + \mathbf{r}' \sin \phi' \hat{j} \]
\[ dq' = \sigma \mathbf{r}' \, d\mathbf{r}' \, d\phi' \]

Hence,

\[ \mathbf{r} - \mathbf{r}' = z \hat{k} - r' \cos \phi' \hat{i} - r' \sin \phi' \hat{j} \]
\[ |\mathbf{r} - \mathbf{r}'| = \left[ (r')^2 + z^2 \right]^{1/2} \]

and we find on substitution of these quantities into Eq. (4.7) that

\[ \mathbf{E}(\mathbf{r}) = \frac{\sigma}{4\pi \varepsilon_0} \int_0^\infty \int_0^{2\pi} \frac{z \hat{k} - r' \cos \phi' \hat{i} - r' \sin \phi' \hat{j}}{[(r')^2 + z^2]^{3/2}} r' \, d\phi' \, dr' = \pm \frac{\sigma}{2\varepsilon_0} \hat{k} \] (4.13)

where the upper sign applies when \( z > 0 \) and the lower sign when \( z < 0 \). The \( z \)-component of the field is shown in Fig. 4.10; it exhibits a discontinuity at the sheet \((z = 0)\) but has everywhere the same magnitude. Thus, this (infinite) sheet establishes a constant field directed away from the sheet when \( \sigma > 0 \) and toward the sheet when \( \sigma < 0 \). Note that

\[ ^1 \text{In filling in the details of this integration, note that } \sqrt{z^2} = |z|, \text{ not } z. \text{ The ambiguous sign arises from this subtlety in the mathematics.} \]
Figure 4.8: Field lines for several point charge distributions: (a) charge of +2 units on the left and −1 unit on the right; (b) charge of +2 units on the left and +1 unit on the right; (c) three equal charges at the corners of an *isosceles* triangle with vertices at \((x, y) = (1, 0), (0, 1),\) and \((-1, 0)\); and (d) charge of +2 units on the left and two charges, each of −1 unit, in the upper and lower right corners. [The light lines indicate equipotential surfaces, which will be discussed in Section 4.6. Note particularly that the equipotential surfaces are perpendicular to the field lines and that in part (a) the marked equipotential is a true sphere in three dimensions. Because they are very close together in the immediate vicinity of point charges, the equipotential surfaces in those regions are not shown.]

Eq. (4.13) was obtained by following a very systematic procedure that involved identifying the pertinent variables, substituting into the general expression of Eq. (4.7), and evaluating an integral. The reader is urged to adopt this approach whenever the problem at hand involves setting up Eq. (4.7) for a charge distribution.
4.2 Mapping Electric Fields

The basis of the computer-based algorithm used to produce all of the two-dimensional field maps shown to this point in this chapter is quite simple. We must have available equations or integrals from which the $x$- and $y$-components of a particular electric field $\mathbf{E}$ in the $xy$-plane can be calculated. We can then trace the field lines as follows.\footnote{This method has been discussed, for example, by J. R. Merrill, Am. J. Phys. 39, 791 (1971).} Starting at some point $(x_1, y_1)$, we move a distance $d$ in the direction of the (calculable) field at $(x_1, y_1)$. If $d$ is not too large, the point $(x_2, y_2)$ that we reach lies approximately on the field line through $(x_1, y_1)$. But, as shown in Fig. 4.11, the coordinates of the two points are related by

$$
x_2 = x_1 + d \cos \theta = x_1 + d \frac{E_x(x_1, y_1)}{|\mathbf{E}(x_1, y_1)|} \tag{4.14}
$$

$$
y_2 = y_1 + d \sin \theta = y_1 + d \frac{E_y(x_1, y_1)}{|\mathbf{E}(x_1, y_1)|} \tag{4.15}
$$

This method has been discussed, for example, by J. R. Merrill, Am. J. Phys. 39, 791 (1971).
Figure 4.11: Geometry for tracing an electric field line.

Figure 4.12: A simple algorithm for tracing an electric field line. We imagine the program to be controlling a pen plotter. For drawing on a computer screen, the statements should stipulate the positioning of the “beam” and indicate whether the beam is on or off during the motion.

```
PROGRAM FIELDLINE
   INPUT parameters
   INPUT X1, Y1 ! Starting point for field line
   INPUT D ! Step size
   MOVE (raised) pen to X1, Y1 ! Initialize position of pen
   LOWER pen
   UNTIL time to stop
      E1 <-- EX(X1,Y1) ! Calculate x-component of field
      E2 <-- EY(X1,Y1) ! Calculate y-component of field
      E <-- SQRT( E1^2 + E2^2 ) ! Calculate magnitude of field
      X2 <-- X1 + D*(E1/E) ! X-, Y-coordinates of next point
      Y2 <-- Y1 + D*(E2/E) ! on field line
      X1 <-- X2 ! Next point becomes current point
      Y1 <-- Y2
      MOVE (lowered) pen to X1, Y1 ! Draw next segment of field line
   END UNTIL
   RAISE pen
END PROGRAM
```

where $\theta$ is the angle between the field at $(x_1, y_1)$ and the $x$-axis. Once the first step has been made, we replace the starting point with the new point and repeat the calculation. The algorithm is summarized in Fig. 4.12; its output is a graph of the field line through the starting point. Modifications are necessary to achieve accuracy with reasonable step sizes, but the basic idea of the systematic tracing of an electric field line is exposed in this simple algorithm.

The program `electfield`, a listing of which is included in Appendix F, and instructions for the use of which are included in Appendix G, is based on a predictor-corrector embellishment\(^3\) of the above-described algorithm. Alternative routines for drawing fields are

\[^3\]See Fig. 3.2 and the associated text.
Figure 4.13: Geometry for Calculating the Field of a Semicircular Arc.

typically built in to symbolic and numeric programs like MAXIMA, MAPLE, Mathematica, IDL, MATLAB, OCTAVE, and PYTHON.

4.3 Numerical Evaluation of Fields

When the integral expressing the field produced by a prescribed source cannot be evaluated analytically, one must turn to numerical means. As an example, suppose we are interested in the electric field established at points on the $x$ axis by a uniformly charged semicircular arc located as shown in Fig. 4.13. Suppose the arc has radius $a$ and carries total charge $Q$. Then, in preparation for setting up Eq. (4.7), we identify

\[ r = \text{position vector of field point} = x \hat{i} \]
\[ r' = \text{position vector of source point} = a \cos \phi' \hat{i} + a \sin \phi' \hat{j} \]
\[ dq' = \frac{Q}{\pi a} (a d\phi') = \frac{Q}{\pi} d\phi' \]

Hence,

\[ r - r' = (x - a \cos \phi') \hat{i} - a \sin \phi' \hat{j} \]
\[ |r - r'| = [x^2 + a^2 - 2ax \cos \phi']^{1/2} \]

and we find on substitution of these quantities into Eq. (4.7) that

\[ E(x, 0, 0) = \frac{Q}{4\pi \varepsilon_0} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{(x - a \cos \phi') \hat{i} - a \sin \phi' \hat{j}}{[x^2 - 2ax \cos \phi' + a^2]^{3/2}} d\phi' \] (4.16)

We can simplify this expression in two ways. First, notice that the integrand giving the $x$ component of this field is an even function of $\phi'$ while the integrand associated with the $y$ component is an odd function of $\phi'$. Thus, because the integral extends over symmetric limits, the $y$ component will integrate to zero while the $x$ component can be expressed as twice the integral over the positive half of the original interval of integration. We find that

\[ E_x(x, 0, 0) = \frac{Q}{4\pi \varepsilon_0} \frac{2}{\pi} \int_{0}^{\pi/2} \frac{(x - a \cos \phi')}{[x^2 - 2ax \cos \phi' + a^2]^{3/2}} d\phi' \] (4.17)
Second, we can choose to express both $x$ and $E_x$ as multiples of more natural units, thereby absorbing all of the dimensional parameters into a rescaling of variables and greatly simplifying the numerical analysis of this field. Begin by factoring an $a$ out of the numerator of the integrand and an $a^2$ out of the quantity in square brackets in the denominator. If we then substitute $X$ for $x/a$ and divide the entire equation by the quantity $Q/4\pi\varepsilon_0a^2$, we find the much simpler expression

$$
\frac{E_x(X,0,0)}{Q/4\pi\varepsilon_0a^2} = \frac{2}{\pi} \int_0^{\pi/2} \frac{(X - \cos \phi')}{[X^2 - 2X \cos \phi' + 1]^{3/2}} \, d\phi'
$$

We seek information about this integral as a function of $X$, where $X = 0$ at the origin and $X = 1$ at the point where the arc intersects the $x$ axis. By introducing dimensionless measures of $x$ and $E_x$, we have removed all dimensional parameters from the results. Said another way, the differences between the fields produced by arcs of different sizes or arcs carrying different charges are purely differences of scale. There is no essential physics lurking in the size of the arc or the magnitude of its charge.

This integral can be simply evaluated only in special situations. For example, at $X = 0$, we find

$$
\frac{E_x(0,0,0)}{Q/4\pi\varepsilon_0a^2} = \frac{2}{\pi} \int_0^{\pi/2} - \cos \phi' \, d\phi' = -\frac{2}{\pi} \Rightarrow \mathbf{E}(0,0,0) = -\frac{Q}{\pi^2\varepsilon_0a^2} \hat{i}
$$

for the field at the origin. When $X \gg 1$ ($x \gg a$), we can approximate the integrand by ignoring terms guaranteed to be small compared to $X$, and we find

$$
\frac{E_x(X,0,0)}{Q/4\pi\varepsilon_0a^2} = \pm \frac{2}{\pi X^2} \int_0^{\pi/2} \, d\phi' = \pm \frac{1}{X^2} \Rightarrow \mathbf{E}(X,0,0) = \pm \frac{Q}{4\pi\varepsilon_0X^2} \hat{i}
$$

for the field at points on the $x$ axis far to the right (upper sign) or left (lower sign) of the arc. Reassuringly, the formalism yields the field of a point charge $Q$ at the origin, which agrees with the appearance of the source when viewed from a remote vantage point.

At other points along the $x$-axis, we must resort to a numerical evaluation to obtain information about this integral. To use IDL, we begin by creating the function arcfield.pro containing the IDL statements

```idl
function arcfield, phi
    ; ARCFIELD defines the integrand for computing the electric field produced at a point on the x axis by a uniformly charged semicircular arc.
    ;
    ; common param, X
    tmp1 = cos(phi)
```

---

4Note the subtlety that $X^{2.3/2} = |X|^3$, not $X^3$.
5IDL’s `common` storage is used to pass the parameter in the integrand from the main sequence of commands through the routine `qsimp` called by the main sequence to the routine defining the integrand and called by `qsimp`.
6All Lawrence-generated IDL programs described in this chapter can be downloaded from `$HEADEM/idl`

The translation of `$HEADEM` for your site is identified in your `Local Guide`.

4.3. NUMERICAL EVALUATION OF FIELDS

Figure 4.14: The electric field on the $x$ axis for the semicircular source in Fig. 4.13. This graph was produced with IDL.

```
tmp2 = X^2 - 2.0*X*tmp1 + 1.0
return, (X-tmp1) / tmp2^1.5
end
to define the integrand for IDL’s integration routine. Basically, we wish to evaluate $E_x$ for a succession of values of $X$ ranging from, say, $X = -5.0$ to $X = +5.0$ and then plot a graph of $E_x$ versus $X$ over that range. Only one complication occurs: the integrand is ill-defined at $X = 1$ when $\phi' = 0$. To pay suitable respect to the numerical nastiness of that one point, we elect to generate the points for the graph in two pieces. Using the built-in IDL routine qsimp.pro for numerical integration, the commands

```idl
IDL> common param, X ; Create common area
IDL> uplim = !Pi/2.0 ; Set upper limit for integration

; ***** Evaluate field at values of X less than 1 *****
IDL> X1 = -5.0 + findgen(60)/10.0 ; Vector of values from -5.0 to 0.9
IDL> q = size(X1) & N = q[1]-1 ; Set N to highest index in X1
IDL> EX1 = fltarr(N+1) ; Prepare EX1 for field values
IDL> for i = 0, N do $ ; Evaluate and store integral
IDL> begin X=X1[i] & $ ; for each value of X
IDL> EX1[i] = qsimp( "arcfield", 0.0, uplim ) & $
IDL> endfor

; ***** Evaluate field at values of X greater than 1 *****
IDL> X2 = 1.1 + findgen(40)/10.0 ; Vector of values from 1.1 to 5.0
IDL> q = size(X2) & N = q[1]-1 ; Set N to highest index in X2
IDL> EX2 = fltarr(N+1) ; Prepare EX2 for field values
IDL> for i = 0, N do $ ; Evaluate and store integral
IDL> begin X=X2[i] & $ ; for each value of X
IDL> EX2[i] = qsimp( "arcfield", 0.0, uplim ) & $
IDL> endfor

; ***** Plot field *****
IDL> plot, X1, EX1, xrange = [-5.0,5.0], yrange = [-10.0,10.0],
IDL> ticklen = 1.0, xtitle = '!3x/a',
IDL> ytitle = '!3E!Dx!N(x,0,0)/(Q/4!4pe!3!B0!N a!A2!N)', thick=4
IDL> oplot, X2, EX2, thick=4

to IDL will accomplish the desired graph. The output of this run is shown in Fig. 4.14. Note that the default (fractional) tolerance of $10^{-6}$ has been accepted in the numerical integrations.

PROBLEMS

P4.1. Three point charges, each of strength $10^{-9}$ C, are placed one at each of three corners of a square 10 cm on a side. (a) Calculate the electric field (magnitude and direction) at the fourth corner of the square. (b) Using FORTRAN, C, or another language with which you are familiar, write a short computer program that (1) accepts the coordinates and strength of, say, three point charges as input, (2) accepts the coordinates of a selected field point as input, (3) calculates and prints out the components of the electric field at the specified field point, and (4) returns to step (2). Test your program using the charge distribution in this problem and several different field points.

P4.2. (a) Apply the binomial theorem (Appendix B) to expand Eq. (4.9) in powers of $a/r$ to derive Eq. (4.10). Hint: First convince yourself that

$$
\frac{1}{|r \pm \frac{1}{2}a|} = \left( r \pm \frac{1}{2}a \right) \cdot \left( r \pm \frac{1}{2}a \right)^{-3/2} = \left( r^2 \pm a \cdot r + \frac{1}{3}a^2 \right)^{-3/2}
$$

$$
= \frac{1}{r^3} \left( 1 \pm \frac{a}{r} \cdot \hat{r} + \frac{a^2}{4r^2} \right)^{-3/2} \approx \frac{1}{r^3} \left( 1 \pm \frac{a}{r} \cdot \hat{r} \right)^{-3/2} + O \left( \frac{a^2}{r^2} \right)
$$

$$
= \frac{1}{r^3} \left( 1 \pm \frac{3a}{2r} \cdot \hat{r} \right) + O \left( \frac{a^2}{r^2} \right)
$$

which implies that

$$
\frac{r \pm \frac{1}{2}a}{|r \pm \frac{1}{2}a|^3} = \frac{r(\hat{r} \pm \frac{1}{2}(a/r))}{|r \pm \frac{1}{2}a|^3} \approx \frac{1}{r^2} \left( \hat{r} \pm \frac{1}{2} \frac{a}{r} \pm \frac{3}{2} \left( \frac{a}{r} \cdot \hat{r} \right) \hat{r} \right) + O \left( \frac{a^2}{r^2} \right)
$$
4.3. NUMERICAL EVALUATION OF FIELDS

Figure 4.15: Figure for Problem P4.7.

Here, one takes either the upper sign or the lower sign throughout. (b) Show that Eq. (4.11) follows from Eq. (4.10). Hint: See Table 0.1. Optional: Use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to express Eq. (4.9) in spherical coordinates and then apply appropriate commands to obtain Eq. (4.11) through Taylor expansion.

P4.3. Let a point charge \( Q \) be located at the origin and a dipole with dipole moment \( \mathbf{p} = p(\cos \alpha \hat{r} + \sin \alpha \hat{\theta}) \) be located at the point \((r, \theta, \phi)\) in spherical coordinates. Determine the force experienced by this dipole and draw sketches of the three spherical components of the force as functions of \( \alpha \). Explain physically the origin of the \( \hat{\theta} \)-component.

P4.4. Sketch the field lines of the linear quadrupole shown in Fig. 4.7(a). Then, use electfield or an equivalent routine in an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica or an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to check the figure that you sketch on your own. In using any of these commercial programs, you will have to invent a means to avoid problems from the divergences in the field at the locations of the charges. Optional: In similar fashion, explore the fields of other distributions of your choice. Make sure the patterns you observe make sense to you.

P4.5. (a) Draw a linear octupole consisting of four equally spaced charges, showing particularly the relative strengths of each charge. Note the appearance of the binomial coefficients. (b) Sketch representative field lines for this linear octupole. The procedure electfield or appropriate routines from an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica or an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON may be helpful, though in the latter two cases, you will have to invent a means to avoid problems from the divergences in the field at the locations of the charges.

P4.6. By direct integration, show that the electric field at a point a distance \( r \) from an infinitely long line charge carrying a constant (linear) charge density \( \lambda \) is given by \( \mathbf{E}(r) = \left(\lambda/2\pi\epsilon_0 r\right)\hat{z} \), where the line defines the \( z \)-axis and cylindrical coordinates are used. Don’t be afraid of integral tables or use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica.

P4.7. A total charge \( Q \) is distributed on a plane circular ring of radius \( a \) with a linear charge density \( \lambda(\phi) = \lambda_0(1 + \sin \phi) \), with \( \phi \) defined as in Fig. 4.15. By direct integration, determine the electric field at the point \((0, 0, b)\) on the \( z \)-axis. Express your answer in terms of \( Q \) and examine the field in the limit \( b \gg a \).

P4.8. A total charge \( Q \) is uniformly distributed over the surface of the region in the \( xy \)-plane bounded by the circle \( x^2 + y^2 = a^2 \). (a) Obtain an integral giving the electric field at the
point \((0, y, z)\) in the \(yz\)-plane. (b) Evaluate the integral when the field point is on the \(z\)-axis, i.e., for \(y = 0\). (Remember that an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica can evaluate integrals.) (c) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to obtain a careful graph of \(E_z(0, 0, z)\) as a function of \(z/a\) over the range \(-\infty < z/a < \infty\). (d) Examine the result of part (b) in the limit \(z \gg a\). (Remember that an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica can generate Taylor expansions.)

**P4.9.** A sphere of radius \(a\) carries a total charge \(Q\) uniformly distributed throughout its volume. By direct integration, determine the electric field at a point on a diameter (or extension thereof) at a distance \(b\) from the center of the sphere. Consider both \(b < a\) and \(b > a\). Express the results in terms of \(Q\) and show that the field at the specified point is the same as the field produced by a point charge located at the center with strength equal to the net charge lying inside a sphere of radius \(b\). Sketch a graph of \(|E|\) versus \(b\).

**P4.10.** Use Eqs. (4.5)–(4.7) and electfield, an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica, or an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to examine the field produced by one or more charge distributions of your own invention.

**P4.11.** Consider a circular ring of radius \(a\) carrying charge \(Q\) uniformly distributed over its circumference. Let this ring lie in the \(xy\)-plane with its center at the origin. (a) Show—perhaps using an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica for some of the manipulations—that the electric field produced by this ring at the observation point \((x, 0, 0)\) on the \(x\)-axis is given by

\[
E(x, 0, 0) = \frac{Qi}{4\pi\epsilon_0 a^2} \frac{1}{\pi} \int_0^\pi \frac{(s - \cos \phi')}{(s^2 - 2s\cos \phi' + 1)^{3/2}} d\phi'
\]

where \(s = x/a\). (b) Use the numerical integration routine from an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to evaluate the \(x\)-component of the field as a function of \(s\) and then use that same tool to plot a graph of \(E_x(x, 0, 0)/(Q/4\pi\epsilon_0 a^2)\) versus \(s\) for \(0 < s < 4\). Watch out for the point \(s = 1\). (c) Show that, in the limit as \(s \to \infty\), \(E(x, 0, 0) \to Qi/4\pi\epsilon_0 x^2\), i.e., the field approaches that of a point charge!

### 4.4 Gauss’s Law

The basic information about electrostatics contained in Coulomb’s Law and in the principle of superposition can be reexpressed in a variety of ways. In this section we shall develop the first of two laws that are direct consequences of the information already presented. First, however, we must define the concept of the flux of the electric field across a specified surface \(\Sigma\) placed in the field. This *electric flux*, denoted by \(\Phi_e\), is defined by the surface integral

\[
\Phi_e = \int_\Sigma \mathbf{E}(r) \cdot d\mathbf{S}
\]  

and in effect counts the net number of lines of \(\mathbf{E}\) piercing \(\Sigma\). (See Section 2.3.) In particular, if \(\mathbf{E}\) is constant and \(\Sigma\) is a plane surface of area \(S\), \(\Phi_e = \mathbf{E} \cdot \mathbf{S}\) and is the product of the surface area and the component of \(\mathbf{E}\) perpendicular to (the plane of) the surface. The electric flux may be either positive or negative and, in (rationalized) mks units, is expressed in \(\text{N}\cdot\text{m}^2/\text{C}\). If \(\Sigma\) happens to be a *closed* surface (Fig. 4.16), we take \(d\mathbf{S}\) to represent an
Figure 4.16: Arbitrary closed surface in an electric field. The contribution to the net electric flux is negative along that portion of the surface where the field enters the enclosed volume and positive along that portion where the field emerges from the volume.

outward normal and the flux $\Phi_e$, now defined by $\oint_\Sigma \mathbf{E} \cdot d\mathbf{S}$, represents the net outward flux of $\mathbf{E}$ over the surface. Regions in which the field lines enter the surface make negative contributions to the flux while regions in which the field lines leave the surface make positive contributions.

Now, the flux of a static electric field across a closed surface has a simple evaluation. Consider first the flux of the electric field of a point charge $q_0$ located at $r_0$ across an arbitrary closed surface $\Sigma$. The field is given by Eq. (4.5) with $r' = r_0$ and $q' = q_0$, and the required flux is then given by

$$\Phi_e = \frac{q_0}{4\pi\epsilon_0} \oint_\Sigma \frac{r - r_0}{|r - r_0|^3} \cdot d\mathbf{S} \quad (4.22)$$

We have, however, already evaluated the integral in Eq. (4.22). Substituting from Eq. (2.37), we find the very simple value

$$\Phi_e = \begin{cases} \frac{q_0}{\epsilon_0} & \text{if } q_0 \text{ is inside } \Sigma \\ 0 & \text{if } q_0 \text{ is outside } \Sigma \end{cases} \quad (4.23)$$

regardless of the shape of $\Sigma$!\footnote{Note the disappearance of the factor $4\pi$ in Eq. (4.23)—a consequence of our earlier selection of rationalized mks units.}

If the field is produced by a distribution of charge, the principle of superposition enables us to write immediately that

$$\Phi_e = \oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{(\text{net charge within } \Sigma)}{\epsilon_0} \quad (4.24)$$
When the charge distribution is described by a volume charge density $\rho$, for example, Eq. (4.24) assumes the more specific form

$$\oint_{\Sigma} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dv$$  \hspace{1cm} (4.25)$$

where the volume integral extends over the volume $V$ bounded by the closed surface $\Sigma$. Note particularly that $\Sigma$ is arbitrary; Eq. (4.25) applies to any (closed) surface whatever, even if that surface is only an imaginary surface introduced for the purpose of exploiting this relationship. Equations (4.23)–(4.25) are forms of Gauss’s Law, one of the basic laws of electricity. When the fields are static, Gauss’s Law is a direct consequence of Coulomb’s Law and merely expresses some of the information in Coulomb’s Law in a new form. In fact, Gauss’s Law carries over to non-static fields without change of form. That extension, however, can be made only on the basis of experimental observations beyond those supporting Coulomb’s Law, and the “generalized” Gauss’s Law is therefore more than merely a restatement of a portion of Coulomb’s Law. We shall examine non-static fields in more detail in later chapters.

Gauss’s Law in the form of Eq. (4.25) is particularly useful in at least two contexts. We shall consider first its use to calculate electric fields when the source distribution manifests sufficient symmetry that some properties of the field can be inferred without complete knowledge of the field. The essential premise of a symmetry argument is that the field established by some source distribution must exhibit whatever symmetries the distribution itself exhibits. If, for example, the source is invariant to rotation about some axis, the resulting field must be invariant to the same transformation. Thus, any symmetry of the source constrains the field. If these symmetries are extensive enough, the field is sufficiently constrained that Gauss’s Law is adequate to determine those properties of the field not completely fixed by the symmetries.

We shall illustrate the approach by applying symmetry and Gauss’s Law to determine the field established by an infinitely long line charge with constant linear charge density $\lambda$. For simplicity let this line define the $z$-axis of a cylindrical coordinate system. The most general field then has the form

$$\mathbf{E}(\mathbf{r}) = E_r(r,\phi,z) \hat{\mathbf{r}} + E_\phi(r,\phi,z) \hat{\phi} + E_z(r,\phi,z) \hat{\mathbf{k}}$$  \hspace{1cm} (4.26)$$

where $\hat{\mathbf{r}}$, $\hat{\phi}$, and $\hat{\mathbf{k}}$ are unit vectors in the radial, azimuthal, and axial directions, respectively. The source distribution, however, is invariant to rotation about the $z$-axis, and the field exhibits this invariance only if $E_r$, $E_\phi$, and $E_z$ are independent of $\phi$. In addition, the source distribution is invariant to translation along the $z$-axis, and the field exhibits this invariance only if $E_r$, $E_\phi$, and $E_z$ are independent of $z$. The most general field consistent with these two symmetries then is

$$\mathbf{E}(\mathbf{r}) = E_r(z) \hat{\mathbf{r}} + E_\phi(z) \hat{\phi} + E_z(z) \hat{\mathbf{k}}$$  \hspace{1cm} (4.27)$$

But, in addition, the source distribution is—and hence the field must be—invariant to reflection in any plane perpendicular to the $z$-axis. Since a non-zero $z$ component for this field at any point is incompatible with invariance of the field to reflection in the plane through that point and perpendicular to the $z$ axis, we conclude that $E_z(z) = 0$. Finally, the source distribution is—and hence the field must be—invariant to reflection in any plane containing...
4.4. GAUSS’S LAW

Figure 4.17: A Gaussian pillbox for determining the field of a line charge.

the z axis. Since a non-zero \( \phi \) component for this field at any point is incompatible with invariance of the field to reflection in the plane defined by the z axis and that point, we conclude that \( E_\phi(z) = 0 \). Thus, the symmetries of the selected source distribution constrain the field to be no more complicated than\(^9\)

\[
E = E_i(z) \hat{r}
\]

We now use Gauss’s Law to determine \( E_i(z) \). Although the law applies to any closed surface, the most useful surfaces to choose are those that take advantage of at least some of the symmetries of the source. The appropriate surface for this problem might be described as a cylindrical fruit juice can of radius \( r \) and length \( L \) with its axis coincident with the line of charge (Fig. 4.17). The flux of the electric field out of this surface (sometimes called a Gaussian surface or, especially when small, a Gaussian pillbox) possesses potentially three contributions:

\[
\Phi_e = \oint E \cdot dS = \int_{\text{top}} E \cdot dS + \int_{\text{bottom}} E \cdot dS + \int_{\text{cylinder}} E \cdot dS
\]

With \( E \) given by Eq. (4.28), however, \( E \cdot dS = 0 \) on the top and bottom surfaces (\( dS \parallel \hat{k} \)). Only the integral over the cylindrical surface makes a contribution. On that surface, \( dS = \epsilon d\phi dz \hat{r} \) and

\[
\Phi_e = \int_{z_0}^{z_0+L} dz \int_0^{2\pi} d\phi E_i(z) \hat{r} \cdot \hat{r} = 2\pi L \epsilon_i E_i(z)
\]

since \( \epsilon \) is constant (albeit arbitrary) insofar as the integral is concerned. Now, the charge enclosed within the selected surface is \( \lambda L \). Since Gauss’s Law requires that \( \Phi_e = \lambda L/\epsilon_0 \),

\(^9\)As can be seen, the statement “because of symmetry, the field must be radially directed with a radial component dependent only on \( z \)”, while true, nonetheless hides the fuller symmetry argument that supports it. The reader is urged to quote symmetry in support of various conclusions only after supplying the full argument.
CHAPTER 4. THE ELECTRIC FIELD OF STATIC CHARGES

Figure 4.18: A Gaussian pillbox for determining the field at the surface of a conductor.

\[ \mathbf{E} = E_n \hat{n} \]

Eq. (4.30) gives

\[ 2\pi L \xi E_\xi(\tau) = \frac{\lambda L}{\epsilon_0} \]

Thus \( E_\xi(\tau) = \frac{\lambda}{2\pi \epsilon_0 \xi} \) and from Eq. (4.28) we find that

\[ \mathbf{E} = \frac{\lambda}{2\pi \epsilon_0 \xi} \hat{\xi} \]

in agreement with the result obtained by direct integration in P4.6.

Gauss’s Law can also be fruitfully applied to some situations involving conductors, which are by definition pieces of matter whose intrinsic composition includes microscopic charges that are free to move in response to any forces applied to them. Thus, if a conductor is placed in a region of space initially containing a static electric field, these free charges will move in response to the field, thereby being redistributed within the conductor and in turn add their own contribution to the field within (and without) the conductor. This adjustment of the charge distribution within the conductor will continue as long as there remains any field within the conductor to apply forces to the free charges. Once the fields are again static, the field within the conductor must be zero; any other value contradicts the requirement that the fields be static. Thus, when the fields outside a conductor are static, the region occupied by the conductor itself is free of fields. Further, the electric field at the surface of such a conductor must be normal to the surface, for any tangential component would cause a motion of free charges along the surface, again contradicting the requirement that the fields be static. These properties of the fields in and around conductors facilitate the use of Gauss’s Law, and we shall illustrate by determining the field just outside a conducting surface at a point \( P \) where the (surface) charge density is \( \sigma \). A suitable Gaussian pillbox is shown in Fig. 4.18; its sides consist of a cylindrical surface with its axis normal to the conducting surface at \( P \) and two plane surfaces, one on each side of the conducting surface. Further, the pillbox is chosen small enough to enclose an element of the conducting surface that is approximately plane and over which \( \sigma \) and \( \mathbf{E} \) are both approximately constant. Let the cross-sectional area of the pillbox be \( \Delta S \) and let \( \hat{n} \) be a unit vector in the direction of the outward normal to the conducting surface at \( P \). Then the charge within the pillbox is \( \sigma \Delta S \), and the flux out of the pillbox is \( E_n \Delta S \), the latter following because only the plane surface of the pillbox outside the conductor contributes to the flux. (The field is zero inside the conductor and does not pierce the cylindrical walls
of the pillbox outside the conductor.) Applying Gauss’s Law as expressed in Eq. (4.24), we find that \( E_n \Delta S = \sigma \Delta S / \varepsilon_0 \) or \( E_n = \sigma / \varepsilon_0 \) or finally that the field \( E \) just outside a conducting surface at a point \( P \) where the charge density is \( \sigma \) is given by

\[
E = \frac{\sigma}{\varepsilon_0} \hat{n}
\]

(4.33)

where \( \hat{n} \) is a unit outward normal at the point \( P \).

For many purposes, a differential form of Gauss’s Law is more convenient than the integral form, Eq. (4.25). Let \( \Sigma \) and \( V \) become arbitrarily small. In that limit, we can use the identity in Eq. (2.25) to write

\[
\oint_{\Delta \Sigma} E \cdot dS \approx \nabla \cdot E \Delta V
\]

(4.34)

(We now denote the small surface and volume elements by \( \Delta \Sigma \) and \( \Delta V \).) Further, if \( \Delta V \) is small enough, \( \rho \) can be regarded as a constant throughout \( \Delta V \), and

\[
\frac{1}{\varepsilon_0} \int_{\Delta V} \rho \, dv \approx \frac{1}{\varepsilon_0} \rho \Delta V
\]

(4.35)

Finally, substituting Eqs. (4.34) and (4.35) into Eq. (4.25), dividing by \( \Delta V \), and passing to the limit \( \Delta V \to 0 \) [in which limit the approximations in Eqs. (4.34) and (4.35) become exact], we find the differential relationship

\[
\nabla \cdot E = \frac{\rho}{\varepsilon_0}
\]

(4.36)

which is equivalent to the integral expression in Eq. (4.25). Although we shall occasionally need this result in the rest of this chapter, we shall postpone detailed consideration of the differential form of the basic laws until Chapter 6.

**PROBLEMS**

**P4.12.** An infinitely long straight rod having a circular cross section of radius \( b \) is charged to a uniform volume charge density \( \rho \). Using symmetry and Gauss’s Law, determine the electric field at a distance \( \tau \) from the axis of the rod. Consider both \( \tau < b \) and \( \tau > b \). Sketch a graph of \( E \tau \) versus \( \tau \).

**P4.13.** Use symmetry and Gauss’s Law to obtain the field established by an infinite plane sheet of charge characterized by a constant (surface) charge density \( \sigma \). Compare your result with Eq. (4.13).

**P4.14.** Use Gauss’s Law to show that a region containing an electrostatic field but devoid of charge includes no point at which a test charge would be in stable equilibrium. **Hint:** What must the field lines look like near a point of stable equilibrium?

**P4.15.** Let all of space be uniformly charged to a volume charge density \( \rho \). This distribution is invariant to arbitrary translations and to arbitrary rotations about any arbitrary axis. Symmetry therefore requires the field everywhere to be zero. Hence, any arbitrary closed surface lying in this distribution has no electric flux across it. But that surface certainly contains some charge, and we have an apparent contradiction of Gauss’s Law. Where is the error? Why is there no contradiction in applying symmetry arguments and Gauss’s Law to infinite sheets and infinite line charges?
P4.16. An uncharged conducting object has a hollow cavity in its interior. If a point charge \( q \) is placed in the cavity, prove that a charge \( -q \) is induced on the surface of the cavity and a charge \( q \) is induced on the outer surface of the conductor.

P4.17. One of two plane parallel conducting plates of nonzero thickness carries a charge \( Q \) and the other carries a charge \( -Q \). Each charge assumes a static distribution on the surfaces of its plate. The situation is shown in Fig. 4.19. (a) Neglect fringing at the ends (i.e., treat the plates as infinite) and use symmetry to argue that \( \sigma_1, \sigma_2, \sigma_3, \) and \( \sigma_4 \) must be constant, that \( \sigma_2 = -\sigma_3 \), that \( \sigma_1 = -\sigma_4 \), and that the electric field everywhere is parallel to \( \hat{n} \). (b) Use Gauss’s Law to show that the field in the region between the plates is uniform and given by \( E = (\sigma_2/\epsilon_0)\hat{n} \). (c) Using the result in Eq. (4.13), sketch graphs of the \( \hat{n} \)-component of the electric field established by each sheet of charge in Fig. 4.19 and then sketch a graph of the net field established by all four sheets. Show that the field in the interior of each plate will be zero only if \( \sigma_1 = -\sigma_4 = 0 \), thus showing that all of the charge placed on the plates accumulates on the facing surfaces. (d) Show that the field in the region outside that bounded by the plates is identically zero. The arrangement in this problem is called a parallel plate capacitor.

4.5 The Restricted Faraday Law

The second law of electrostatics that follows directly from Coulomb’s Law and the principle of superposition is a restricted form of what we shall later call the Faraday Law; in essence it states that the force field \( \mathbf{F}_q(r) = q\mathbf{E}(r) \) corresponding to the electrostatic field \( \mathbf{E}(r) \) is conservative. This law is expressed analytically in the integral form

\[
\oint \mathbf{F}_q(r) \cdot d\mathbf{r} = 0 \quad \Rightarrow \quad \oint \mathbf{E}(r) \cdot d\mathbf{r} = 0
\]

(4.37)

where the (closed) path of integration is arbitrary [see Eq. (0.36)], or in the differential form

\[
\nabla \times \mathbf{F}_q = 0 \quad \Rightarrow \quad \nabla \times \mathbf{E} = 0
\]

(4.38)
4.6. THE ELECTROSTATIC POTENTIAL

[See Eq. (0.54).] Since the validity of Eq. (4.37) [and hence of Eq. (4.38)] for an arbitrary static field follows by superposition from its validity for the static field of a point charge, we need only establish Eq. (4.37) for a point charge. Let the charge have strength $q'$ and be located at $r'$. Then its field is given by Eq. (4.5) and we find that

$$\oint E \cdot dr = \frac{q'}{4\pi\epsilon_0} \oint \frac{r - r'}{|r - r'|^3} \cdot dr = \frac{q'}{4\pi\epsilon_0} \oint \frac{(x - x') \, dx + (y - y') \, dy + (z - z') \, dz}{\left(\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{3/2}\right)}$$

$$= -\frac{q'}{4\pi\epsilon_0} \oint \left(\frac{dx}{dx} \frac{\partial}{\partial x} + \frac{dy}{dy} \frac{\partial}{\partial y} + \frac{dz}{dz} \frac{\partial}{\partial z}\right) \frac{1}{\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2}}$$

$$= -\frac{q'}{4\pi\epsilon_0} \oint d\left(\frac{1}{|r - r'|}\right) = -\frac{q'}{4\pi\epsilon_0} \frac{1}{|r - r'|}\bigg|_{\text{starting } r}^{\text{finishing } r} = 0!! \quad (4.39)$$

the final form following because the path is closed and the starting and finishing points are the same point. Equations (4.37) and (4.38) are therefore established for the general electrostatic field. In contrast to Gauss’s Law, this restricted form of Faraday’s Law is valid only for static electric fields. In Chapter 6, we shall look to experiment to determine the appropriate modifications when the electric field ceases to be static. The resulting generalization, of course, is then more than merely a consequence of Coulomb’s Law.

4.6 The Electrostatic Potential

As discussed in Section 0.3, every conservative force field $F_q(r)$ has associated with it a scalar potential energy field $U(r)$ given by Eq. (0.41), which assumes the form

$$U(r) = U(r_0) - \int_{r_0}^{r} qE(r) \cdot dr \quad (4.40)$$

for the electrostatic force field $F_q = qE$. Here, $r_0$ is an arbitrary reference point and $U(r_0)$ may be assigned an arbitrary value; both are commonly chosen to simplify the form of $U(r)$. Further, by Eq. (0.49), the force field can be recovered from the potential energy field by evaluating the negative gradient,

$$F_q(r) = qE(r) = -\nabla U(r) \quad (4.41)$$

and by Eq. (0.40) the potential energy field appears in the expression

$$\frac{1}{2}mv^2 + U(r) = \text{constant} \quad (4.42)$$

for the conservation of the mechanical energy of a particle of mass $m$ moving in the given force field with velocity $v$. It is customary to eliminate the test charge $q$ by dividing Eqs. (4.40) and (4.41) by $q$ and introducing the electrostatic potential field $V(r)$ as the potential energy per unit charge, viz.,

$$V(r) = \frac{U(r)}{q} \quad (4.43)$$

Equation (4.40) then becomes

$$V(r) = V(r_0) - \int_{r_0}^{r} E(r) \cdot dr \quad (4.44)$$
and permits calculation of $V(r)$ if $E(r)$ is known. Determination of $E(r)$ from knowledge of $V(r)$ involves Eq. (4.41), which assumes the form

$$E(r) = -\nabla V(r) \quad (4.45)$$

Finally, the conservation law, Eq. (4.42), becomes

$$\frac{1}{2}mv^2 + qV(r) = \text{constant} \quad (4.46)$$

From Eq. (4.43), we infer that the mks unit of potential is the Joule/Coulomb (J/C), a combination given the name volt (V). We then conclude from Eq. (4.45) that the mks unit of electric field, previously identified as the N/C, must also be expressible as the V/m; the second name is the more common.

We have now shown the existence of a potential function for an electrostatic field and we have obtained expressions for finding the potential from the field [Eq. (4.44)] and for finding the field from the potential [Eq. (4.45)]. We next obtain an expression for determining the potential directly from the source distribution. Let the source first be a point charge of strength $q$ at the origin. If the reference point $r_0$ is taken at infinity and $V(r_0)$ is set equal to zero, then combining Eq. (4.5) (with $q' \rightarrow q$, $r' = 0$) with Eqs. (4.44) and Eq. (0.31) gives

$$V(r) = -\int_{r_0}^{r} \frac{q}{4\pi\epsilon_0 r^2} \bullet (\hat{r} dr + \hat{\theta} r d\theta + \hat{\phi} r \sin \theta d\phi) = -\int_{r_0}^{r} \frac{q}{4\pi\epsilon_0 r^2} dr = \frac{q}{4\pi\epsilon_0 r} \quad (4.47)$$

Translation of the coordinate system so that the point charge is at $r_0$ and the field point is at $r$ merely replaces $r$ in Eq. (4.47) with $|r - r_0|$. Thus, the potential established by a point charge $q$ located at $r_0$ is given by

$$V(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{|r - r_0|} \quad (4.48)$$

Finally, by superposition we obtain the expression

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum q_i \frac{1}{|r - r_i|} \quad (4.49)$$

for the potential established by an array of point charges $q_i$ at $r_i$. Thus, for example, the potential established by the dipole shown in Fig. 4.6 is given by

$$V(r) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{|r - \frac{1}{2}a|} - \frac{1}{|r + \frac{1}{2}a|} \right] \quad (4.50)$$

which reduces to the simpler form

$$V(r) = \frac{\mathbf{p} \bullet \mathbf{r}}{4\pi\epsilon_0 r^2} = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}, \quad \mathbf{p} = qa \quad (4.51)$$
in the limit \( r \gg a \) (P4.24) and varies as \( r^{-2} \), contrasting with the variation of the potential of a monopole as \( r^{-1} \).

We could relate the potential established by a distributed source to the source itself by applying to Eq. (4.49) an argument such as that leading to Eq. (4.4). To set the stage for the corresponding magnetic development, however, we shall here adopt a different approach. Note first the mathematical identity

\[
\frac{r - r'}{|r - r'|^3} = -\nabla \left( \frac{1}{|r - r'|} \right)
\]

where the gradient involves derivatives with respect to the components of \( r \) (see P0.14). Substitution of this identity into Eq. (4.4) leads ultimately to

\[
E(r) = -\nabla \left( \frac{1}{4\pi \epsilon_0} \int \frac{dq'}{|r - r'|} \right)
\]

in which the gradient has been written in front of the integral because it does not involve differentiations with respect to any of the variables of integration. Comparison with Eq. (4.45) then results in the identification

\[
V(r) = \frac{1}{4\pi \epsilon_0} \int \frac{dq'}{|r - r'|} + \chi(r)
\]

where \( \chi(r) \) is arbitrary except that its gradient must vanish if the negative gradient of Eq. (4.54) is to give the correct field, i.e.,

\[
\nabla \chi = \frac{\partial \chi}{\partial x} i + \frac{\partial \chi}{\partial y} j + \frac{\partial \chi}{\partial z} k = 0
\]

Equation (4.55) is a vector equation and each component must be separately zero, i.e., \( \partial \chi / \partial x = 0 \), etc., i.e. \( \chi \) cannot depend on \( x \), \( y \), or \( z \); at worst, it can be a constant and we have finally that

\[
V(r) = \frac{1}{4\pi \epsilon_0} \int \frac{dq'}{|r - r'|} + \text{constant}
\]

where the integral extends over the charge distribution and \( dq' \) is expressed as appropriate to the distribution, e.g.,

\[
V(r) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{|r - r'|} dv' + \text{constant}
\]

for a volume distribution. Equations (4.56) and (4.57) not only provide for a calculation of \( V \) directly from known sources but also express the relationship between (a static) \( V \) and its sources even when the sources are not explicitly known.\(^\text{13}\)

An extremely important equation for the electrostatic potential is obtained by substituting the equation \( \mathbf{E} = -\nabla V \) into Gauss’s Law, \( \nabla \cdot \mathbf{E} = \rho / \epsilon_0 \), to obtain

\[
\nabla \cdot (\nabla V) = \nabla^2 V = -\frac{\rho}{\epsilon_0}
\]

\(^{13}\)The existence of \( V \) might also have been inferred from the equation \( \nabla \times \mathbf{E} = 0 \) by applying the theorem in item (3) of Section 2.5. This route to Eq. (4.45), however, does not provide an expression for calculating \( V \) directly from its sources.
Table 4.1: Interrelationships Among Static $\rho$, $\mathbf{E}$, and $V$.

<table>
<thead>
<tr>
<th></th>
<th>$\rho$</th>
<th>$\mathbf{E}$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$- \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{</td>
<td>\mathbf{r} - \mathbf{r}'</td>
<td>^3} \mathbf{r}' , dv'$</td>
</tr>
<tr>
<td>$\mathbf{E}$</td>
<td>$\epsilon_0 \nabla \cdot \mathbf{E}$</td>
<td>$-\nabla V$</td>
<td>$-\int \mathbf{E} \cdot d\mathbf{r}$</td>
</tr>
<tr>
<td>$V$</td>
<td>$-\epsilon_0 \nabla^2 V$</td>
<td>$-\nabla V$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

where the Laplacian operator $\nabla^2$ is defined in Section 2.5. This equation can be used to determine the source of a field if the potential is known; it is called Poisson’s equation\textsuperscript{14} if $\rho \neq 0$ and Laplace’s equation\textsuperscript{15} if $\rho = 0$. One solution to Eq. (4.58) is, of course, expressed by Eq. (4.57), but this solution is not always useful. A detailed examination of this equation is postponed to Chapter 8.

The interrelationships we have now developed among the charge density, the electrostatic field, and the electrostatic potential are summarized in Table 4.1.

### 4.7 Numerical Evaluation of Potentials

When the integral giving the potential produced by a prescribed source cannot be evaluated analytically, one must turn to numerical means. We illustrate with the uniformly charged semicircular arc shown in Fig. 4.13. The quantities we need for insertion into Eq. (4.56) are the same as before, namely

$$\begin{align*}
\mathbf{r} & = \text{position vector of field point} = x \mathbf{i} \\
\mathbf{r}' & = \text{position vector of source point} = a \cos \phi' \mathbf{i} + a \sin \phi' \mathbf{j} \\
dq' & = \frac{Q}{\pi a} (a \, d\phi') = \frac{Q}{\pi} \, d\phi'
\end{align*}$$

Hence,

$$\begin{align*}
\mathbf{r} - \mathbf{r}' & = (x - a \cos \phi') \mathbf{i} - a \sin \phi' \mathbf{j} \\
|\mathbf{r} - \mathbf{r}'| & = \sqrt{x^2 + a^2 - 2ax \cos \phi'}^{1/2}
\end{align*}$$

and we find on substitution of these quantities into Eq. (4.56) that

$$V(x, 0, 0) = \frac{Q}{4\pi \epsilon_0} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{d\phi'}{[x^2 - 2ax \cos \phi' + a^2]^{1/2}} \quad (4.59)$$

\textsuperscript{14} French mathematician and physicist Siméon Denis Poisson, b. 21 June 1781, Pithiviers, Orléanais, France (now Loiret, France); d. 25 April 1840 in Sceaux, Hauts-de-Seine, France.

\textsuperscript{15} French mathematician, physicist, and astronomer Pierre-Simon Laplace, b. 23 March 1749 in Beaumont-en-Auge, Normandy, France; d. 5 March 1827 in Paris, France.
We can simplify this expression in two ways. First, notice that the integrand is an even function of $\phi'$. Because the integral extends over symmetric limits, we can reexpress it as twice the integral over the positive half of the original interval of integration. We find that

$$V(x, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{2}{\pi} \int_0^{\pi/2} \frac{d\phi'}{[x^2 - 2ax\cos\phi' + a^2]^{1/2}}$$

(4.60)

Second, we can choose to express both $x$ and $V$ as multiples of more natural units, thereby absorbing all of the dimensional parameters into a rescaling of variables and greatly simplifying the numerical analysis. Begin by factoring an $a^2$ out of the quantity in square brackets in the denominator. If we then substitute $X$ for $x/a$ and divide the entire equation by $Q/4\pi\varepsilon_0 a$, we find the much simpler expression

$$V(x, 0, 0) = \frac{Q}{4\pi\varepsilon_0 a} \frac{2}{\pi} \int_0^{\pi/2} \frac{d\phi'}{|X^2 - 2X\cos\phi' + 1|^{1/2}}$$

(4.61)

We seek information about this integral as a function of $X$, where $X = 0$ at the origin and $X = 1$ at the point where the arc intersects the $x$ axis. As with the earlier example, introducing dimensionless measures of $x$ and $V$ has removed all dimensional parameters from the results. As with the fields, the differences between the potentials produced by arcs of different sizes or arcs carrying different charges are purely differences of scale. There is no essential physics lurking in the size of the arc or the magnitude of its charge.

This integral can be simply evaluated only in special situations. For example, at $X = 0$, we find

$$V(0, 0, 0) = \frac{Q}{4\pi\varepsilon_0 a} \frac{2}{\pi} \int_0^{\pi/2} d\phi' = 1 \implies V(0, 0, 0) = \frac{Q}{4\pi\varepsilon_0 a}$$

(4.62)

for the potential at the origin. When $X \gg 1$ ($x \gg a$), we can approximate the integrand by ignoring terms guaranteed to be small compared to $X$, and we find

$$V(X, 0, 0) = \frac{Q}{4\pi\varepsilon_0 a} \frac{2}{\pi|X|} \int_0^{\pi/2} d\phi' = \frac{1}{|X|} \implies V(X, 0, 0) = \frac{Q|X|}{4\pi\varepsilon_0 a} = \frac{Q}{4\pi\varepsilon_0 |x|}$$

(4.63)

for the potential at points on the $x$ axis. Reassuringly, the formalism yields the potential of a point charge $Q$ at the origin, which agrees with the appearance of the source when viewed from a remote vantage point.

At other points along the axis, we must resort to a numerical evaluation to obtain information about this integral. To use IDL again, we begin by creating the function `arcpot.pro` containing the IDL statements

```idl
function arcpot, phi

; ARCPOT defines the integrand for computing the electrostatic
; potential produced at a point on the x axis by a uniformly
; charged semicircular arc.

; common param, X
tmp1 = cos(phi)
pot = 1.0/sqrt(X^2 - 2.0*X*tmp1 + 1.0)
return, pot
end
```
to define the integrand for IDL’s integration routine. Basically, we wish to evaluate $V$ for a succession of values of $X$ ranging from, say, $X = -5.0$ to $X = +5.0$ and then plot a graph of $V$ versus $X$ over that range, we proceed as before, again using qsimp.pro for the numerical integration and paying attention to the singularity at $X = 1, \phi' = 0$, we invoke the statements

IDL> common param, X ; Create common area
IDL> uplim = !Pi/2.0 ; Set upper limit for integration

; ***** Evaluate potential at values of X less than 1 *****
IDL> X1 = -5.0 + findgen(60)/10.0 ; Vector of values from -5.0 to 0.9
IDL> q = size(X1) & N = q[1]-1 ; Set N to highest index in X1
IDL> V1 = fltarr(N+1) ; Prepare V1 for field values
IDL> for i = 0, N do $
IDL> \quad$ begin X=X1[i] & $\\$
IDL> \quad$ V1[i] = qsimp( "arcpot", 0.0, uplim ) & $\\$
IDL> \quad$ endfor

; ***** Evaluate potential at values of X greater than 1 *****
IDL> X2 = 1.1 + findgen(40)/10.0 ; Vector of values from 1.1 to 5.0
IDL> q = size(X2) & N = q[1]-1 ; Set N to highest index in X2
IDL> V2 = fltarr(N+1) ; Prepare V2 for field values
IDL> for i = 0, N do $
IDL> \quad$ begin X=X2[i] & $\\$
IDL> \quad$ V2[i] = qsimp( "arcpot", 0.0, uplim ) & $\\$
IDL> \quad$ endfor

; ***** Plot potential *****
IDL> plot, X1, V1, xrange = [-5.0,5.0], yrange = [0.0,5.0], $\\$
IDL> ticklen = 1.0, xtitle = '!3x/a', $\\$
IDL> ytitle = '!3V(x,0,0)/(Q/4!4pe!3!B0!Na)', thick=4
IDL> oplot, X2, V2, thick=4

The output of this run is shown in Fig. 4.20. Note that the default (fractional) tolerance of $10^{-6}$ has been accepted in the numerical integrations.

### 4.8 Mapping Electrostatic Potentials

In addition to representing an electric field graphically by displaying its field lines, we can also represent the field by showing its **equipotential contours**, which are curves or surfaces at every point of which the electrostatic potential has the *same* value. The equipotential contours for several selected fields are shown by the light curves in Figs. 4.5 and 4.8. Technically, equipotentials should be drawn at equal increments in the potential so that their separation gives a hint as to the strength of the electric field at each point.\(^{16}\) So as to give a

\(^{16}\)See Eq. (4.45), which confirms that the field is the rate of change of the potential with respect to distance.
Figure 4.20: The electrostatic potential on the $x$ axis for the semicircular source in Fig. 4.13. This graph was produced with IDL.

A clearer idea of the structure of the potential, the diagrams here identified do not respect that convention. These diagrams do reveal, however, that the field is at every point perpendicular to the equipotential contours, a property that we prove by evaluating the potential difference $V(r_a + d\mathbf{r}) - V(r_a)$ between two infinitesimally separated points $r_a + d\mathbf{r}$ and $r_a$ lying in the same equipotential contour. By hypothesis, this potential difference is zero, but it is also calculable directly from Eq. (4.44); we find that

$$0 = -\int_{r_a}^{r_a + d\mathbf{r}} \mathbf{E} \cdot d\mathbf{r} = -\mathbf{E}(r_a) \cdot d\mathbf{r}$$

Thus, for the particular displacement assumed, the electric field is perpendicular to $d\mathbf{r}$. But $d\mathbf{r}$ was any displacement lying in an equipotential contour and the electric field is therefore perpendicular to the equipotential contour. Q.E.D.\textsuperscript{17} This property is clearly evident in Figs. 4.5 and 4.8.

Algorithms for exploiting knowledge of the field to trace equipotential curves are based on the property that equipotential curves are at every point perpendicular to the field line through that point. Thus, simple modification of Eqs. (4.14) and (4.15) to

$$x_2 = x_1 - d\sin \theta = x_1 - d \frac{E_y(x_1, y_1)}{|\mathbf{E}(x_1, y_1)|}$$

\textsuperscript{17}These contours can be drawn by the program \texttt{electfield} introduced earlier or by routines in an available symbolic manipulating program like MAXIMA, MAPLE, or \texttt{Mathematica} or an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON.
\[ y_2 = y_1 + d \cos \theta = y_1 + d \frac{E_x(x_1, y_1)}{|E(x_1, y_1)|} \]

(4.66)

with no other changes in the overall procedure will shift the trace from field to equipotential.

4.9 Conductors as Equipotential Regions

That the electric field at the surface of a conductor is perpendicular to that surface now leads us to suspect that the surface of a conductor must be an equipotential. Indeed, since there is no field anywhere within the body of a conductor in a static field, the potential difference between any two points in such a conductor—not just between two points on the surface—is zero and the entire volume of space occupied by a conductor in a static field is therefore an equipotential region. Thus, we can speak unambiguously of the potential of a conductor.

4.10 Capacitance

Now, suppose that two conductors, one carrying charge \( Q \) and the other carrying charge \( -Q \), are placed near one another in a region previously free of fields. Once the static fields produced by the charges on the conductors have been established, the charge on each conductor is distributed so that each conductor is at a definite potential. Any device of the sort described is called a capacitor, and the ratio of the charge \( Q \) to the difference \( \Delta V \) in potential between the two conductors is called the capacitance \( C \) of the arrangement,

\[ C = \frac{Q}{\Delta V} \]

(4.67)

In most cases, \( \Delta V \) is proportional to \( Q \), and \( C \) is therefore independent of \( Q \); \( C \) is essentially a geometric quantity, determined largely by the size, shape, and separation of the two conductors. The mks unit of capacitance is the coulomb per volt (C/V), called the farad (F), which turns out to be inconveniently large; the microfarad (\( \mu \)F; \( 10^{-6} \) F) and even the picofarad (pF; \( 10^{-12} \) F) occur more commonly.\(^{18}\)

PROBLEMS

P4.18. (a) Show that the electrostatic potential function associated with a constant electric field \( E \) is given by \( -E \cdot r \), where the reference point of zero potential is taken at the coordinate origin. (b) Show that the potential energy of a dipole of moment \( p \) placed in a constant electric field \( E \) is given by \(-p \cdot E\). Hint: The potential energy of a pair of point charges is, of course, the sum of the potential energies of the individual charges.

P4.19. (a) Substituting the field obtained in Eq. (4.32) into Eq. (4.44), show that the electrostatic potential at a distance \( r \) from the axis of an infinite line charge is given by

\[ V(r) = -\frac{\lambda}{2\pi \varepsilon_0} \ln \frac{r}{a} \]

if the potential is taken to be zero at \( r = a \). Why is the reference point not taken either at \( \infty \) or at the axis? (b) Recover the field from the potential by applying Eq. (4.45).

\(^{18}\)Michael Faraday, see footnote 5 in Chapter 6, page 158.
4.10. CAPACITANCE

P4.20. Starting with Eq. (4.33), show that the charge density $\sigma$ at a point on the surface of a conductor in a static field described by the potential $V$ is given by $\sigma = -\epsilon_0 \nabla V \cdot \hat{n}$, where $\hat{n}$ is a unit outward normal to the conductor and the gradient is evaluated at the point where the charge density is desired.

P4.21. A nonrelativistic alpha particle of mass $m$ and charge $2q$ with initial speed $v$ experiences a head-on collision with a heavy nucleus of atomic number $Z$, and hence of charge $Zq$. Use conservation of energy to determine the distance of closest approach to the nucleus.

P4.22. Two spherical conducting raindrops of radius $a$ are both charged to the same potential $V$ relative to a reference point at infinity. Determine the potential of the single drop that results when the two drops are allowed to merge.

P4.23. What is the highest potential to which a spherical conductor of radius 30 cm can be charged if air becomes conducting when a field exceeding $3 \times 10^6$ V/m is produced?

P4.24. Derive Eq. (4.51) for the potential of a dipole by expanding Eq. (4.50) in powers of $a/r$ and then calculate the field of the dipole. Compare your result with Eq. (4.11). Hint: As a prototype, note the expansion in P4.2. In calculating the field, use the gradient in spherical coordinates, Eq. (0.67).

P4.25. Determine the electrostatic potential of a linear quadrupole [Fig. 4.7(a)] having its axis along the $z$-axis when the quadrupole is observed from a distance large compared to its dimensions, and then find the field. Let the separation of adjacent charges be $a$. The coordinates in Fig. 4.6 with the center of the quadrupole at the origin are also convenient here.

P4.26. Consider a uniformly charged plane circular disc of radius $a$ lying in the $xy$-plane with its center at the origin. (a) Write an integral (multiple if necessary) giving the electrostatic potential at the point $r$. (b) Determine the potential $V(z)$ at the point $(0, 0, z)$ and sketch a graph of $V(z)$ versus $z$. (c) Obtain approximate expressions for the potential $V(z)$ in the two regions $z \gg a$ and $z \ll a$ and interpret the two limiting expressions physically. (d) Calculate the $z$-component of the electric field at the point $(0, 0, z)$ and sketch a graph of this component as a function of $z$. (e) Can you calculate the $x$- and $y$-components of the electric field at $(0, 0, z)$ from the potential obtained in part (b)? If so, do it; if not, explain why not. Optional: By applying the integral form of Gauss’s Law to a small cylindrical pillbox centered on the axis of the disc and a distance $z$ above its plane, show that the radial component of the electric field at points close to the axis is given by

$$E_r(r, z) = -\frac{r}{2} \frac{\partial E_z}{\partial z} \bigg|_{r=0}$$

where $r$ is the distance of the point from the axis. Can you obtain this same result by manipulating with the differential form of Gauss’s Law?

P4.27. Let a portion of the $z$-axis in the region $-a \leq z \leq a$ carry a charge $Q$ uniformly distributed along its length with linear density $\lambda$. (a) Set up an integral giving the electrostatic potential at any point in space. (b) Evaluate the potential at points on the positive $z$-axis for which $z > a$. (c) Show that the result obtained in part (b) approaches $Q/4\pi \epsilon_0 z$ in the limit $z \gg a$. Note the power series expansion $\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$. (d) Explain how you could have predicted the limiting value in part (c) without actually evaluating the integral in part (b).

P4.28. Determine the electric field and the source distribution corresponding to the Yukawa potential, $V(r) = Q e^{-\alpha r}/4\pi \epsilon_0 r$, where $\alpha$ is a constant.

P4.29. An infinite conducting plate in the plane $z = 0$ is maintained at potential zero and a similar plate in the plane $z = d$ is maintained at potential $V_0$. Assume that there is no charge in the region between the plates. Because the plates are invariant to arbitrary
translation parallel to the \( xy \)-plane, the potential can be a function only of \( z \), \( V = V(z) \), and Laplace’s equation reduces to \( d^2V/dz^2 = 0 \). By solving Laplace’s equation and imposing the given values of \( V \) at the two plates, find \( V(z) \) for \( 0 < z < d \). Then find the field between the plates and the charge density on each plate. \textit{Hint:} See Eq. (4.33).

P4.30. Describe the equipotential contours corresponding to the three-dimensional field of a point charge.

P4.31. Obtain a carefully drawn sketch of the equipotentials established in the \( yz \)-plane by a point dipole having dipole moment \( \mathbf{p} = p\mathbf{k} \). (Compare P0.18.)

P4.32. The equipotential curves established by a particular two-dimensional charge distribution are shown in Fig. 4.21. Determine both the magnitude and the direction of the electric field at point \( A \).

P4.33. A potential difference is applied between the two conducting plates shown in Fig. 4.22. Sketch the electric field lines and the equipotential lines in the region between the plates.

P4.34. Accepting that the field between two equally but oppositely charged parallel conducting plates is given by \( \left( \sigma/\epsilon_0 \right) \hat{n} \), where \( \sigma \) is the charge density on the plates and \( \hat{n} \) is a unit vector perpendicular to the plates (P4.17), show that the capacitance of this parallel plate capacitor is given by \( C = \epsilon_0 A/d \), where \( A \) is the surface area of the plates and \( d \) their separation. Ignore fringing at the edges of the plates.

P4.35. Show that the capacitance of a capacitor consisting of two coaxial cylindrical shells of length \( L \) and radii \( a \) and \( b \), \( a < b \), is given by \( C = 2\pi\epsilon_0 L/\ln(b/a) \). Ignore fringing at the ends of the capacitor.
P4.36. (a) Show that the capacitance of a capacitor consisting of two concentric spherical shells of radii \(a\) and \(b\), \(a < b\), is given by \(C = 4\pi\epsilon_0 ab/(b-a)\). (b) The capacitance of an isolated sphere is defined by allowing \(b \to \infty\). What (numerically) is the radius of an isolated sphere having a capacitance of 1 F? Compare your result with the radius of the earth, \(R_e = 6.37 \times 10^6\) m. (The distance from the earth to the moon is about 60\(R_e\).)

4.11 Potential Energy in the Electrostatic Field

Because it takes work to assemble a distribution of charges from the state in which all elementary charges composing the distribution are infinitely remote from one another, the distribution (once assembled) possesses the capacity to do useful work, i.e., has a potential energy. Further, since the electrostatic forces against which the agent assembling the distribution does work are conservative, the amount of recoverable energy stored in the distribution (i.e., the potential energy) is exactly equal to the amount of work needed to assemble the distribution in the first place. Consider, for example, the buildup of an assembly of point charges \(q_i\) located at positions \(r_i\) by bringing the charges in one at a time to their final positions. The first charge can be brought from infinity to its final position \(r_1\) with no work, since there are as yet no charges in the distribution. To bring in the second charge, work must be done against the forces exerted by the first charge on the second. This work is equal to the potential energy of the second charge when it finally reaches the position \(r_2\).

Accepting the electrostatic potential given by Eq. (4.48), we find that, once the first two charges have been positioned, the potential energy of the system is

\[
\Delta W_2 = q_2 \left[ \frac{q_1}{4\pi\epsilon_0} \frac{1}{|r_1 - r_2|} \right] = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|r_1 - r_2|} \tag{4.68}
\]

To bring up the third charge, work must be done against both of the charges that are already in position. The work expended to bring in the third charge then is

\[
\Delta W_3 = q_3 \left[ \frac{q_1}{4\pi\epsilon_0} \frac{1}{|r_1 - r_3|} + \frac{q_2}{4\pi\epsilon_0} \frac{1}{|r_2 - r_3|} \right] \tag{4.69}
\]

and so on; the amount of work needed to bring in the \(j\)-th charge is given by

\[
\Delta W_j = q_j \sum_{i=1}^{j-1} \frac{q_i}{4\pi\epsilon_0} \frac{1}{|r_i - r_j|} \tag{4.70}
\]

and the total potential energy of the charge distribution then is

\[
W = \sum_{j=2}^{N} \sum_{i=1}^{j-1} \frac{q_i q_j}{4\pi\epsilon_0} \frac{1}{|r_i - r_j|} \tag{4.71}
\]

The double sum appearing in Eq. (4.71) can now be rewritten in a more useful form. If the two summation indices are regarded as locating points in a two-dimensional plane, the sum as it stands can be interpreted as a sum over the points marked with a circle in Fig. 4.23. Now, for every circled point with coordinates \((i,j)\) there is a “squared” point with coordinates \((j,i)\). Since the summand in Eq. (4.71) has the same value of \((i,j)\) as it
has for \((j, i)\), the sum might alternatively be viewed as extending over the “squared” points instead of over the circled points. Or—what is more useful—the sum may be regarded as extending over all the marked points provided a factor of \(\frac{1}{2}\) is inserted to correct for the resulting double counting. We thus find that Eq. (4.71) can be rewritten in the form

\[
W = \frac{1}{2} \sum_{i,j=1; i \neq j}^{N} \frac{q_i q_j}{4 \pi \varepsilon_0 |\mathbf{r}_i - \mathbf{r}_j|}
\]

(4.72)

If we now recognize the sum over \(j\) in Eq. (4.72) as the potential established at point \(i\) by all of the other charges, we can rewrite Eq. (4.72) in the form

\[
W = \frac{1}{2} \sum_{i=1}^{N} q_i V_i
\]

(4.73)

Finally, if \(q_i\) in Eq. (4.73) is interpreted as an element of a broad charge distribution, it might more appropriately be denoted \(\Delta q_i\). Then the sum over these elements becomes an integral as the size of the elements is diminished to zero. Thus, for a broad distribution, we find that the stored potential energy is given by

\[
W = \frac{1}{2} \int V(\mathbf{r}) \, dq
\]

(4.74)

where \(dq\) would be written more explicitly in terms of a volume, surface, or linear charge density as appropriate to the distribution.

One is tempted to ask, “Where is this energy stored?” The most fruitful answer can be inferred from the following rewriting of Eq. (4.74). Assume a volume density of charge. Then, Eq. (4.74) becomes

\[
W = \frac{1}{2} \int V \rho \, dv
\]

(4.75)

Now, note that

\[
\nabla \cdot (V \mathbf{E}) = \nabla V \cdot \mathbf{E} + V \nabla \cdot \mathbf{E} = -\mathbf{E} \cdot \mathbf{E} + \frac{1}{\varepsilon_0} V \rho
\]

(4.76)
where Gauss’s Law, Eq. (4.36), has been invoked. Solving for $V\rho$ and substituting the result into Eq. (4.75), we find that

$$W = \frac{1}{2} \varepsilon_0 \int \mathbf{E} \cdot \mathbf{E} \, dv + \frac{1}{2} \varepsilon_0 \oint \mathbf{V} \mathbf{E} \cdot d\mathbf{S}$$  \hspace{1cm} (4.77)

where the divergence theorem has been invoked in the second term on the right. Now, the volume integral extends over all space and the surface integral therefore extends over a very large surface at infinity. If the sources of the field are confined to some finite region of space, $V$ and $\mathbf{E}$ fall off with increasing $r$ at least as rapidly as $r^{-1}$ and $r^{-2}$, respectively, but $d\mathbf{S}$ increases only as $r^2$. Thus, as $r$ becomes large, $V\mathbf{E} \cdot d\mathbf{S}$ falls off as $r^{-1}$, and the second integral in Eq. (4.77) becomes negligible in the limit as the surface over which it extends becomes indefinitely large. We find, therefore, that

$$W = \int \left( \frac{1}{2} \varepsilon_0 \mathbf{E} \cdot \mathbf{E} \right) \, dv$$  \hspace{1cm} (4.78)

the integral extending over all space. Because of its role in Eq. (4.78), it is appropriate to introduce an electric energy density $u_\mathbf{E}$ given by

$$u_\mathbf{E} = \frac{1}{2} \varepsilon_0 \mathbf{E} \cdot \mathbf{E}$$  \hspace{1cm} (4.79)

and to regard the energy stored in a charge distribution as residing in the electric field itself. From one point of view, this association merely enables easy bookkeeping regarding the energy required to assemble a charge distribution. From another point of view, however, we have endowed the field with another “real” property and have thereby attached further reality to the field itself.

PROBLEMS

P4.37. Two positive and two negative charges, all of the same magnitude $q$, are placed at the vertices of a square of side $a$. If the positive charges are on adjacent vertices, determine the work required to assemble the distribution.

P4.38. (a) Suppose two initially uncharged conductors are charged to $Q$ and $-Q$, respectively, by moving small amounts of charge $dq$ slowly from one to the other. Let the potential difference between the conductors when the charges have reached the values $q$ and $-q$ be $V(q)$. Show that the work done to produce the final distribution is given by $W = \int_0^Q V(q) \, dq$. (b) Suppose the capacitance of the arrangement is independent of the charge. Show that $W = Q^2/2C$. (c) Accepting the capacitance and the field given in P4.34, express $W$ for a parallel plate capacitor in terms of the field within it and show that $W = \text{(energy density)} \times \text{(volume between plates)}$.

P4.39. Starting with Eq. (4.74), show that the energy stored in a system of conductors, the $i$-th one being at potential $V_i$ and carrying charge $q_i$, is given by $W = \frac{1}{2} \sum q_i V_i$.

P4.40. (a) By integrating the energy density of the electric field over all space, determine the energy required to distribute a total charge $Q$ uniformly over the surface of a spherical shell of radius $a$. (b) Assuming that the charge on an electron is uniformly distributed over a spherical shell and, further, assuming that the relativistic rest energy of the electron has an electromagnetic origin, estimate (numerically) the radius of the electron.

P4.41. (a) Show that the energy of an arbitrary charge distribution described by a charge density $\rho(r)$ is given by $\int \rho V \, dv$ when the charge distribution is placed in an external potential $V$. (b) Show that, when $V$ corresponds to a uniform field $\mathbf{E}$, this energy reduces to $-\mathbf{p} \cdot \mathbf{E}$, where $\mathbf{p}$ is the dipole moment of the distribution. (See P4.18.)
4.12 The Multipole Expansion of the Electrostatic Potential

Exact evaluation of Eq. (4.57) is possible only for a very few charge distributions. Fortunately, we are very often interested in the potential only in regions remote from its source. In those regions, the potential is given approximately (but quite accurately) by the first contributing term in a binomial expansion, as we saw, for example, in our treatment of the electric dipole in Section 4.1.2. If the subsequent terms in the binomial expansion are retained, a series expressing the potential as an infinite sum is obtained. As the observation point moves closer to the source distribution, more and more of these terms must be preserved in order to obtain an accurate approximation for the potential. This infinite sum for the potential is called the multipole expansion of the potential and the individual terms in the sum are the monopole, dipole, quadrupole, etc., contributions. The different terms are distinguished from one another in part by their dependence on the distance $r$ from the source to the observation point; each term in the series has one more power of $r$ in the denominator than its predecessor. Successive terms also have progressively more involved dependences on the angles specifying the direction of the observation point relative to the source.

A more detailed development of the multipole expansion is based on a binomial expansion of the quantity $1/|\mathbf{r} - \mathbf{r'}|$. Using the binomial theorem, we find that

$$
\frac{1}{|\mathbf{r} - \mathbf{r'}|} = \frac{1}{\sqrt{r^2 - 2\mathbf{r} \cdot \mathbf{r'} + (\mathbf{r'})^2}} = \frac{1}{r} \left[ 1 - 2\mathbf{r} \cdot \mathbf{r'} \left( \frac{\mathbf{r'}}{r} \right)^2 \right]^{-1/2} = \frac{1}{r} \left[ 1 + \mathbf{r} \cdot \mathbf{r'} + \frac{3(\mathbf{r'} \cdot \mathbf{r})^2 - (\mathbf{r'})^2}{2r^2} + \cdots \right]
$$

(4.80)

A more convenient alternative expression of this expansion is obtained if the components of $\mathbf{r}$ and $\mathbf{r'}$ are denoted by $x_1, x_2, x_3$ and $x'_1, x'_2, x'_3$, rather than by $x, y, z$ and $x', y', z'$. We then find from Eq. (4.80) that

$$
\frac{1}{|\mathbf{r} - \mathbf{r'}|} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r'}}{r^2} + \frac{3(\sum_i x'_i x_i)^2 - (\mathbf{r'})^2 \sum_j x_j^2}{2r^5} + \cdots
$$

(4.81)

The object now is to separate the primed and unprimed coordinates. Let the numerator in the third term be written as follows:

$$
3 \left( \sum_i x'_i x_i \right)^2 - (\mathbf{r'})^2 \sum_j x_j^2 = 3 \sum_{i,j} x'_i x_i x'_j x_j - (\mathbf{r'})^2 \sum_{i,j} x_i \delta_{ij} x_j = \sum_{i,j} x_i x_j [3x'_i x'_j - \delta_{ij} (\mathbf{r'})^2]
$$

(4.82)

where the Kronecker delta $\delta_{ij}$ is unity when $i = j$ and zero when $i \neq j$. We thus find that

$$
\frac{1}{|\mathbf{r} - \mathbf{r'}|} = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r'}}{r^3} + \frac{1}{2r^5} \sum_{i,j} x_i x_j [3x'_i x'_j - \delta_{ij} (\mathbf{r'})^2] + \cdots
$$

(4.83)

$^{19}$German mathematician Leopold Kronecker, b. 7 December 1823 in Liegnitz, Province of Silesia, Prussia; d. 29 December 1891 in Berlin, Germany.
4.12. THE MULTIPole EXPANSION OF THE ELECTROSTATIC POTENTIAL

and in each term the source coordinates are explicitly separated from the observation coordinates.

The multipole expansion for the potential is now obtained by substituting Eq. (4.83) into Eq. (4.57); writing each term separately, we find that

\[
V(\mathbf{r}) = \frac{\int dq'}{4\pi\varepsilon_0 r} + \frac{\mathbf{r} \cdot \int r' dq'}{4\pi\varepsilon_0 r^3} + \frac{\sum x_i x_j \int (3x'_i x'_j - \delta_{ij}(r')^2) \, dq'}{8\pi\varepsilon_0 r^3} + \cdots \quad (4.84)
\]

All integrals extend over the source distribution. For a given source, these integrals are constants characterizing the distribution; they do not depend on the observation point.

Each of the integrals in Eq. (4.84) has a physical interpretation. The first, for example, is given by

\[
\int dq' = Q \quad (4.85)
\]

where \( Q \) is the net charge in the distribution and the first term in Eq. (4.84) is the potential of a point charge. It is the most significant contribution to \( V(\mathbf{r}) \) when \( r \) is very large. Comparison of the second term with Eq. (4.51) leads to the identification of a dipole moment

\[
\int r' \, dq' = \mathbf{p} \quad (4.86)
\]

classifying an arbitrary charge distribution. If the total charge \( Q \) happens to be zero, the most significant contribution to the potential at large distances is the dipole contribution. The third term is the quadrupole contribution to the electrostatic potential and the nine numbers

\[
Q_{ij}^{(e)} = \int [3x'_i x'_j - \delta_{ij}(r')^2] \, dq' \quad (4.87)
\]

are the elements of the (electric) quadrupole moment tensor, which is clearly symmetric, since \( Q_{ij}^{(e)} \) necessarily equals \( Q_{ji}^{(e)} \). (Why?) For specific charge distributions, some elements may be zero and furthermore some components other than those that are necessarily equal may happen accidentally to be equal. Although \( Q, \, p, \, Q_{ij}^{(e)} \), and higher moments do not depend in any way on the observation point, they are moments of the charge distribution relative to a particular coordinate system and are not all necessarily invariant to changes in that coordinate system. (See P3.19.) Usually, however, a most natural coordinate system is apparent from the charge distribution and the selection of that specific coordinate system is left understood.

In terms of the monopole, dipole, and quadrupole moments, the first three terms of the multipole expansion of the potential are

\[
V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\mathbf{r} \cdot \mathbf{p}}{4\pi\varepsilon_0 r^3} + \frac{\sum x_i x_j Q_{ij}^{(e)}}{8\pi\varepsilon_0 r^3} + \cdots \quad (4.88)
\]

Every subsequent term in this expansion can also be written in terms of the coordinates of the observation point and a set of numbers determined solely by the charge distribution. We shall not carry the expansion beyond this point.

A corresponding expansion for the electric field of an arbitrary charge distribution is most easily obtained by computing the gradient of Eq. (4.88).
Problem 4.42. Consider a linear quadrupole consisting of a point charge of strength \(2q\) at the origin and point charges of strength \(-q\) at \((0, 0, a)\) and \((0, 0, -a)\). Show that the monopole and dipole moments of this distribution are zero, calculate the elements of the quadrupole moment tensor, and write out the electrostatic potential as a function of the spherical polar coordinates of the observation point. Finally, calculate the field of this quadrupole by evaluating the gradient of the potential. **Hint:** Let the integrals become appropriate sums over the point charges in the quadrupole.

Problem 4.43. Use `electfield` to explore the electric field and equipotential lines for the electric quadrupole described in P4.42 and then explore the equipotential patterns for other distributions of your choice.

Problem 4.44. (a) Calculate the monopole, dipole, and quadrupole moments of the ring in Fig. 4.15, which carries a linear charge density given by \(\lambda(\phi) = \lambda_0 (1 + \sin \phi)\). (b) Write out the first three terms in the multipole expansion of the electrostatic potential of this ring in terms of the spherical coordinates of the observation point.

Problem 4.45. A spherically symmetric charge distribution is described by a charge density that is a function only of the spherical radial coordinate. Show that the dipole and quadrupole moments of such a distribution are zero.

Problem 4.46. Consider a charge distribution that is invariant to rotation about the z-axis. Such a distribution is described by a charge density \(\rho \) that depends only on the spherical coordinates \(r, \theta\): \(\rho = \rho(r, \theta)\). (a) Obtain a two-dimensional integral giving the total charge in the distribution. (b) Show that the dipole moment of this distribution has only a z-component and obtain a two-dimensional integral for this component. (c) Show that the components of the electric quadrupole moment satisfy \(Q_{12} = Q_{21} = Q_{32} = Q_{31} = Q_{13} = 0\) and \(Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}\), and obtain an integral giving \(Q_{33}\). (d) Write out the electrostatic potential (through the quadrupole term), express the result in terms of the spherical coordinates of the observation point, and note the appearance of the Legendre polynomials. (See Table 8-1 with \(x = \cos \theta\).) (e) The electron cloud in one of the excited states of the hydrogen atom is described by the charge density

\[
\rho(r) = -\frac{q}{64\pi a_0^6} r^2 e^{-r/a_0} \sin^2 \theta
\]

where \(q\) is the magnitude of the electronic charge and \(a_0\) is the Bohr radius. Calculate the monopole, dipole, and quadrupole moments for this state. Don’t be afraid of integral tables or of an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica, and don’t overlook the contributions of the nucleus.

Problem 4.47. Consider a line distribution of charge lying along the \(z\)-axis and carrying linear charge density \(\lambda(z)\). Further, suppose \(\lambda(z) = 0\) except in some finite range of \(z\) near \(z = 0\). Write an integral for the potential at an arbitrary point in space and then expand the integrand to show that

\[
V(r) = \frac{1}{4\pi \epsilon_0} \sum_{n=0}^{\infty} \frac{P_n(\cos \theta)}{r^{n+1}} \int (z')^n \lambda(z') \, dz'
\]

where the integral in each term can be interpreted as a multipole moment. Use spherical coordinates for the field point and note the generating function for the Legendre polynomials \(P_n(t)\) in PB.5 in Appendix B.

Problem 4.48. As shown in P4.41(a), the energy \(W\) of a charge distribution \(\rho\) placed in an external potential \(V\) is given by \(W = \int \rho V \, dv\). Assuming \(V\) is established entirely by charges
outside the region of integration, expand \( V \) in a (three-dimensional) Taylor series about the origin to show that

\[
W = QV(0) - p \cdot E(0) - \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial E_j}{\partial x_i} \bigg|_0 + \cdots
\]

where \( Q, p, \) and \( Q_{ij} \) are the monopole, dipole, and quadrupole moments of the distribution. **Hint:** For the external field, \( \nabla \cdot E = 0 \) throughout the volume of integration.


**SUPPLEMENTARY PROBLEMS**

P4.49. Let the electric field established at \( \mathbf{r} \) by a point charge \( q \) at \( \mathbf{r}' \) be written as \( q \mathbf{E}(\mathbf{r}, \mathbf{r}') \). (a) Write an analytic expression for \( \mathbf{E} \) and interpret \( \mathbf{E} \) physically. (b) A charge \( q \) is located at \( \mathbf{r}_0 + \frac{1}{2} \mathbf{a} \) and a charge \( -q \) is located at \( \mathbf{r}_0 - \frac{1}{2} \mathbf{a} \). Write a formal expression for the field of the resulting dipole in terms of \( \mathbf{E} \) and, using a Taylor expansion about \( \mathbf{r}_0 \), show that

\[
\mathbf{E}_{\text{dipole}}(\mathbf{r}) = (\mathbf{p} \cdot \nabla_0) \mathbf{E}(\mathbf{r}, \mathbf{r}_0)
\]

where \( \mathbf{p} = qa \) and \( \nabla_0 \) acts on the components of \( \mathbf{r}_0 \). In effect the field of the dipole can be obtained as the derivative of the field of a monopole. (c) Substitute the explicit analytic expression for \( \mathbf{E} \) into the result of part (b) and derive an expression for the field of a dipole in terms of \( \mathbf{p}, \mathbf{r}, \) and \( \mathbf{r}_0 \). **Note:** A discussion of this property of the dipole field may be found, for example, in R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley Publishing Company, Inc., Reading, Mass, 1964), Volume II, Section 6-4.

P4.50. The hydrogen atom in its ground state consists of a positively charged (point) nucleus carrying charge \( q \) surrounded by a (spherically symmetric) electron cloud described by the volume charge density

\[
\rho(\mathbf{r}) = -\frac{q}{\pi a_0^3} e^{-2r/a_0}
\]

where the coordinate origin is taken at the nucleus and \( a_0 \) is the Bohr radius. Use symmetry and Gauss’s Law to find the electric field as a function of \( \mathbf{r} \). Don’t be afraid of integral tables or of an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica.

P4.51. A total charge \( Q \) is uniformly distributed throughout a sphere of radius \( a \) with its center at the origin. Let the (constant) charge density be \( \rho \). (a) Use symmetry and Gauss’s Law to determine the electric field at all points in space. Make sure the arguments supporting each step of the calculation are clearly presented. Do not overlook the region \( r < a \). (b) Sketch a graph of the radial component of the field as a function of \( r \). Sketch also the magnitude of the field for a point charge of strength \( Q \) and compare the two fields. (c) Taking the reference point at infinity, calculate and sketch a graph of the electrostatic potential as a function of \( r \). (d) Suppose a point charge of strength \( -Q \) and mass \( m \) is released from rest at a point inside the distribution described in this problem. Ignoring collisions of this charge with the charges composing the distribution, describe its subsequent motion and determine its frequency of oscillation. Evaluate this frequency explicitly for \( Q = 1.6 \times 10^{-19} \) C, \( a_0 = 0.5 \times 10^{-10} \) m, and \( m = 9.1 \times 10^{-31} \) kg and compare the result with typical frequencies of light. The system described by these numerical values is a simplified version of the now-discarded “plum-pudding” model of the hydrogen atom.

P4.52. Consider a uniformly charged infinite plane sheet lying in the \( xy \)-plane. Set up Eq. (4.56) for the potential of this distribution and explain physically why this approach does not
yield a useful evaluation of the potential. Obtain and sketch a graph of the potential as a function of \( z \) by some alternative means.

**P4.53.** Consider two infinite plane sheets, one lying in the plane \( z = 0 \) and carrying a constant surface charge density \( \sigma \) and the other lying in the plane \( z = d \) with surface charge density \( -\sigma \). Using the results in Eq. (4.13), sketch graphs of the \( z \)-component of the electric field established by each plane separately and by the combination. Then, taking the reference point at the point \( z = 0 \), determine the electrostatic potential and sketch a graph. Finally, show that, if \( d \) becomes vanishingly small and \( \sigma \to \infty \) so that \( \sigma d \) remains constant, a macroscopic observer walking from one side of the arrangement to the other experiences a discontinuity in the potential but no discontinuity in the field.

**P4.54.** Consider two dipoles, one of moment \( p \) at the origin and the other of moment \( p' \) at \( \mathbf{R} \).

(a) Show that the potential energy \( U \) of the dipole at \( \mathbf{R} \) in the field of the dipole at the origin is given by

\[
U = \frac{\mathbf{p} \cdot \mathbf{p}' - 3(\mathbf{R} \cdot \mathbf{p})(\mathbf{R} \cdot \mathbf{p}')}{4\pi\varepsilon_0 R^3}
\]

*Hint:* Treat the second dipole as two closely spaced charges and use Eq. (4.51) to obtain the potential energy of each charge in the field of the dipole at the origin. (b) Examine the characteristics of this dipole-dipole interaction. For example, how does \( U \) depend on the relative orientation of the two dipoles? Are there any configurations of static equilibrium? If so, are they stable or unstable? What forces and torques exist on each dipole? Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to help in making any graphs you might want.

**P4.55.** By integrating the energy density in the field of two point charges, derive an expression for the potential energy of one charge in the field of the other. *Hints:* (1) Let the fields of each charge separately be \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \). The energy density is then related to

\[
(\mathbf{E}_1 + \mathbf{E}_2) \cdot (\mathbf{E}_1 + \mathbf{E}_2) = \mathbf{E}_1 \cdot \mathbf{E}_1 + 2\mathbf{E}_1 \cdot \mathbf{E}_2 + \mathbf{E}_2 \cdot \mathbf{E}_2
\]

Interpret the divergent integrals arising from the first and third terms as electron self-energies and look more closely at the second term, which gives rise to a convergent integral. (2) Let one charge be located at the origin and the other on the \( z \)-axis, and use spherical coordinates.

**P4.56.** Determine the force of attraction between the plates of a parallel plate capacitor (P4.17 and P4.34) in terms of the area \( A \) and the separation \( d \) of the plates and the applied potential \( \Delta V \). *Hint:* How much work is required to effect a small virtual displacement of one of the plates? Assume first that the charge on the plates remains constant during the virtual displacement but then show that the same force is found if the potential difference between the plates is assumed constant instead. *Note:* In the Kelvin absolute electrometer, a measurement of the force between the plates of a parallel plate capacitor is used to determine the potential difference between the plates.

**P4.57.** A charged ring is oriented as in Fig. 4.15 but carries a charge \( Q \) uniformly distributed over its circumference. (a) Show that the potential \( V(\tau, \phi, z) \) established by this ring at the point \( (\tau, \phi, z) \) in cylindrical coordinates is given by

\[
V(\tau, z) = \frac{V(\tau, \phi, z)}{Q/4\pi\varepsilon_0 a} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi'}{|1 - 2(\tau/a) \cos \phi' + (\tau^2/a^2) + (z^2/a^2)|^{1/2}}
\]

Then, using an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON, plot a careful graph of \( V(0, z) \) versus \( z/a \). (b) Use numerical integration to evaluate this potential for various values of \( \tau/a \) and \( z/a \) and then obtain appropriate graphs to examine the behavior of \( V(\tau, z) \) as a function of \( \tau/a \) for representative, fixed values of \( z/a \) (including \( z/a = 0 \)).
4.12. THE MULTIPOLE EXPANSION OF THE ELECTROSTATIC POTENTIAL

P4.58. (a) Write a computer program to implement the algorithm shown in Fig. 4.12 on an available computer. If you don’t have graphics routines available, you will have to change the output statements to print coordinates of points rather than to plot lines. (b) Test your program for the field produced by an array of a few point charges. You might, for example, generate points on several field lines and reproduce one or more of the plots in Figs. 4.4, 4.5, 4.8. Can you invent a procedure for assuring that the density of field lines conforms to the established convention? Optional: (1) Modify your program to plot (or print) only every $N$-th point so that small steps can be taken without generating reams of output. (2) Develop a three-dimensional version. (3) Improve the algorithm by regarding each new point as a prediction of the correct value, using that prediction to obtain a better estimate of the average field in the region of the step, and finally using that better field to obtain improved values for the coordinates of the next point on the field line. (4) Modify your program so that it traces equipotentials rather than field lines. Hint: Since equipotentials are perpendicular to field lines, a new point on an equipotential is reached by “stepping” at right angles to the field line.

P4.59. Consider a circular ring of radius $a$ carrying charge $Q$ uniformly distributed over its circumference. Let this ring lie in the $xy$ plane with its center at the origin. (a) Show—perhaps using an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica for some of the manipulations—that the electric field produced by this ring at the observation point $(x, 0, z)$ in the $xz$ plane is given by

$$
\mathbf{E}(x, 0, z) = \frac{Q}{4\pi \epsilon_0 a^2} \frac{1}{\pi} \int_0^\pi \frac{(X - \cos \phi') \hat{i} + Z \hat{k}}{(X^2 + Z^2 - 2X \cos \phi' + 1)^{3/2}} \, d\phi'
$$

where $X = x/a$ and $Z = z/a$. (b) Use a numerical integration from an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to evaluate the $x$ and $z$ components of this field as functions of $X$ and $Z$ and then obtain graphs—your choice of format—showing these components versus $X$ for various values of $Z$ (or in some other arrangement that seems appropriate to you). Watch out for the point $X = 1, Z = 0$, at which the integrand will have a zero denominator.

P4.60. Deduce (perhaps using an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica) the expression

$$
V(x, y, z) \frac{Q}{4\pi \epsilon_0 a} = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{d\phi'}{\sqrt{X^2 + Y^2 + Z^2 - 2X \cos \phi' - 2Y \sin \phi' + 1}}
$$

for the electrostatic potential established at the general point $r = x \hat{i} + y \hat{j} + z \hat{k}$ by a ring of radius $a$ lying in the $xy$-plane with its center at the origin and carrying charge $Q$ uniformly distributed around its perimeter. Then use a numerical integration from an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to evaluate this potential and provide data for generating appropriate graphs as you explore this potential in several planes of your choice. In this equation, $X = x/a, Y = y/a$, $Z = z/a$, and $\phi'$ is the angle locating a representative point on the ring relative to the $x$-axis.

P4.61. Consider a linear quadrupole consisting of a charge $2q$ at the origin and charges $-q$ at $r = \mathbf{a} = a \hat{k}$ and $r = -\mathbf{a}$. (a) Write an expression analogous to Eq. (4.43) for the potential established by this source. (b) Use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to evaluate the Taylor series of this potential, finding the first significant contribution in the region $a/r << 1$. (c) Using the gradient operator in spherical coordinates, find the distant electric field produced by a quadrupole.

P4.62. Use computational tools to explore the equipotential lines for the two electric quadrupoles shown in Fig. 4.7 and then explore the equipotential patterns for other distributions of your choice. (See page 91.)
Chapter 5

The Magnetic Induction Field of Steady Currents

Two apparently distinct sources of magnetic induction fields are known: permanent magnets (e.g., bar magnets) and electric currents. A direct treatment of the field produced by permanent magnets involves introducing the concept of a point magnetic pole, realized physically by one end of a long slender bar magnet whose second end has a negligible effect in the region near the first end. Two types of poles—plus and minus—can be identified; the interactions between these poles can be expressed by a law similar to Coulomb’s Law; the magnetic induction field is defined as the force on a unit positive test pole; the principle of superposition can be applied to obtain expressions for the magnetic induction field established by a distribution of magnetic poles in space; and a magnetic scalar potential, whose gradient gives the negative of the magnetic induction field, can be introduced. Approached this way, the magnetic induction field of permanent magnets has a description very similar to that of the electric field of fixed charges.

This approach to permanent magnetism, however, has at least three disadvantages. First, there are experimental difficulties in isolating individual magnetic poles. Even if the second end of the slender bar magnet is far away, the two ends are still intimately related, for the remote end cannot simply be broken off and discarded. Experimentally, a magnet broken in two produces two magnets, each with the usual positive pole at one end and negative pole at the other end. In effect, magnetic dipoles exist in nature, and consequently all higher magnetic multipoles can be found, but the magnetic monopole seems not to exist. Second, the pole formalism is simple only for determining the fields in the region outside of permanent magnets. Finally, the pole formalism cannot be easily extended to include the magnetic induction fields produced by electric currents. That electric currents give rise to magnetic induction fields is a comparatively recent discovery (Oersted, c. 1820). Since Oersted’s time, it has been found that all magnetic induction fields, including those of permanent magnets, can be traced to currents, and we therefore adopt the more general approach in which currents are viewed as the primary source of magnetic induction fields. In this chapter, we shall explore the relationships between steady (i.e., time-independent) currents and the resulting static magnetic induction fields, paralleling the development as

---

1Danish physicist and chemist Hans Christian Ørsted, b. 14 August 1777 in Rudkøbing, Denmark; d. 9 March 1851 in Copenhagen, Denmark.
closely as possible after that of the static electric field in Chapter 4. The fields produced by time-dependent currents will be treated in Chapter 6.

5.1 The Law of Biot-Savart

The starting point for relating the static magnetic induction field to its sources is the law for the force per unit length between two parallel current-carrying wires (Section 1.4). In essence, we seek to reformulate this law so that the result gives the force of interaction between two short segments or elements of the wires. Unfortunately, two such elements cannot be isolated physically without destroying the current in the circuits. Thus, the forces of interaction between two isolated current elements cannot be examined experimentally. The best that we can do is postulate a law for that elementary interaction, choosing a form that we hope will give experimentally verifiable results when it is integrated over all the elements making up a physically constructible circuit. Let us therefore suppose that there exists between two small parallel elements—one in each of two wires—a force given in magnitude by

\[ dF = K \frac{(I \, d\ell)(I' \, d\ell')}{(r^{''})^2} \sin \theta \]  

(5.1)

where \( K \) is a constant and the other symbols are defined in Fig. 5.1. To show that this assumed elementary force gives the correct value for the force per unit length between two parallel wires, we integrate Eq. (5.1) over the “primed” wire, obtaining

\[ F_{II'} = K I I' \int_{-\infty}^{\infty} \frac{\sin \theta}{(r^{''})^2} \, d\ell' \]  

(5.2)
for the force per unit length on the “unprimed” wire. From the geometry of Fig. 5.1, we find that \( r'' = \sqrt{s^2 + (\ell - \ell')^2} \) and that \( \sin \theta = s/r'' \), and Eq. (5.2) then becomes

\[
F_{II'} = KII's \int_{-\infty}^{\infty} \frac{d\ell'}{s^2 + (\ell - \ell')^2} = 2K \frac{II'}{s}
\]

(5.3)

which is in agreement with Eq. (1.7) provided we set \( K = \frac{1}{2}k_2 \). In particular, in mks units (which we now explicitly select), we set \( K = \mu_0/4\pi \) and Eq. (5.1) for the assumed interaction between current elements becomes

\[
dF = \frac{\mu_0}{4\pi} \frac{(I'd\ell')(I'd\ell') \times (r''/r''^3)}{\sin \theta}
\]

(5.4)

We now press our assumed reformulation of Eq. (1.7) still further by writing Eq. (5.4) in a vector form, thereby including the direction of the force of interaction. The presence of the factor \( \sin \theta \) in Eq. (5.4) suggests introducing the cross product

\[
I'd\ell' \times r'' = I'd\ell' \times r'' \sin \theta \hat{n}
\]

(5.5)

where \( \hat{n} \) is a unit vector directed into the page in Fig. 5.1. In addition, the force between the two elements in Fig. 5.1 is an attractive force if both currents are positive, and the cross product \( I'd\ell \times \hat{n} \) then has the proper direction for the force \( dF \) on the element \( I'd\ell \). We are thus led to assume the vector form

\[
dF = \frac{\mu_0}{4\pi} \frac{I'd\ell' \times (I'd\ell' \times r'')}{r''^3}
\]

(5.6)

for Eq. (5.4). Because \( \hat{n} \) and \( d\ell \) are perpendicular, the right-hand side of Eq. (5.6) has the correct magnitude; by its construction, it also has the correct direction.

At this point, we are confident only that Eq. (5.6) gives the correct force between two infinite parallel wires. We cannot even be confident that Eq. (5.6) is the only vector expression that reduces correctly in that special case. Further, because of the experimental impossibility of isolating individual current elements, we cannot subject Eq. (5.6) for the force between current elements to the sort of direct experimental test that can be given to Coulomb's Law for the force between charge elements. Nonetheless, we now adopt Eq. (5.6) for the force of interaction between two current elements arbitrarily positioned and arbitrarily oriented in space, fully realizing (1) that this law has been postulated and (2) that its experimental support is found by examining how satisfactorily its integral predicts the forces between physical circuits.

An expression for determining the magnetic induction field from knowledge of its sources emerges if we integrate Eq. (5.6) over a portion \( \Gamma \) of the unprimed circuit in Fig. 5.2 and over the entire primed circuit. We then find that the force \( \mathbf{F} \) on \( \Gamma \) is given by

\[
\mathbf{F} = \int_{\Gamma} I'd\ell' \times \left[ \frac{\mu_0}{4\pi} \oint (I'd\ell' \times (r - r')) \frac{1}{|r - r'|^3} \right]
\]

(5.7)

where \( r'' \) has been replaced by its equivalent \( r - r' \) (Fig. 5.2). Comparison of Eq. (5.7) with Eq. (3.25) now leads us to interpret the primed circuit as the source of a magnetic induction field given by

\[
\mathbf{B}(r) = \frac{\mu_0}{4\pi} \oint (I'd\ell' \times (r - r')) \frac{1}{|r - r'|^3}
\]

(5.8)
Although it is very much an abstraction, it is convenient at times to suppose that the
differential element \( I' \, dl' \) at \( r' \) makes a differential contribution

\[
\frac{dB(r)}{4\pi} = \frac{\mu_0}{4\pi} \frac{I' \, dl' \times (r - r')}{|r - r'|^3}
\]

(5.9)
to the field at \( r \). Equivalently, when the currents in a distribution not guided by wires
are steady (i.e., when \( \nabla \cdot \mathbf{J} = 0 \) [Eq. (2.47)], which implies that the lines of \( \mathbf{J} \) close on
themselves [Section 2.3], which in turn implies that the current distribution can be regarded
as a superposition of current loops), then Eqs. (5.8) and (5.9) can be written in terms of
current densities by making the replacement \( I' \, dl' \rightarrow J(r') \, dv' \) (P2.7). Thus, we obtain
from Eq. (5.8), for example, that the magnetic induction established by a steady current
described by the current density \( J(r) \) is given by

\[
B(r) = \frac{\mu_0}{4\pi} \int J(r') \times \frac{(r - r')}{|r - r'|^3} \, dv'
\]

(5.10)
where, as usual, \( r \) and \( r' \) represent the field and source points, respectively. Equations (5.8)–
(5.10) are all forms of the Biot-Savart Law,\(^2\) which not only determines the magnetic induction
field of given steady sources but also relates the field to the sources even when neither
is known explicitly. Equation (5.10) should be compared with the expression for the electric
field obtained by setting \( dq' = \rho(r') \, dv' \) in Eq. (4.7). Equation (5.10) is more complicated
because the sources of the magnetic induction field (currents) are vectors, while the sources
of the electric field (charges) are scalars.

\(^2\)French physicist Jean-Baptiste Biot, b. 21 April 1774 in Paris, France; d. 3 February 1862 in Paris,
France. French physicist Félix Savart, b. 30 June 1791 in Charleville-Mezieres, France, d. 16 March 1841 in
Paris, France.
5.2 Applications of the Law of Biot-Savart

5.2.1 The Infinitely Long Straight Wire

As a first example using the Biot-Savart Law, we calculate the magnetic induction of a long straight wire carrying a current $I'$ along the $z$-axis toward $z = +\infty$ [Fig. 5.3(a)]. In cylindrical coordinates, we have that

\[ r = \hat{r} + z \hat{k} \quad r' = z' \hat{k} \quad dl' = dz' \hat{k} \]

\[ r - r' = \hat{r} + (z - z') \hat{k} \]

\[ |r - r'| = [r^2 + (z - z')^2]^{1/2} \]
Figure 5.4: A Circular Current Loop.

\[ I' d' \times (r - r') = I' \, dz' \hat{\phi} \]

Substitution of these quantities into Eq. (5.8) then gives

\[ B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{I' \, dz'}{r^2 + (z - z')^2} \hat{\phi} = \frac{\mu_0 I'}{2\pi \hat{\phi}} \]

Thus, the magnetic induction of a long wire is directed everywhere in the \( \hat{\phi} \) direction, i.e., in circles centered on the wire, and varies in magnitude inversely as the distance from the wire to the observation point. The field lines for this field are shown in Fig. 5.3(b). Equation (5.11), of course, includes the right-hand rule: If the right hand grasps the wire with the thumb pointing in the direction of the current, the fingers curl around the wire in the direction of the field. Further, Fig. 5.3(c) shows a graph of \( B_\phi / (\mu_0 I / 2\pi a) \) versus \( \tau / a \), where \( a \) is an arbitrarily selected length and the scaling of \( B_\phi \) and \( \tau \) is done so as to produce a graph that has general applicability; expressing the plotted variables in dimensionless form permits drawing of a graph that is not specific to any particular current \( I' \) or for that matter to any particular choice of basic units.

5.2.2 Circular Loops

As a second example of the use of the Biot-Savart Law, consider a circular current loop. Let the loop lie in the \( xy \) plane with its center at the origin, have radius \( a \), and carry current \( I' \) flowing counterclockwise as viewed from a point on the positive \( z \) axis above the loop. Suppose we seek the magnetic induction field at a point in the \( xz \) plane. The geometry is shown in Fig. 5.4. Using Cartesian unit vectors but cylindrical coordinates for the source.
Substitution of these quantities into Eq. (5.8) yields the expression
\[ r = x \hat{i} + z \hat{k} = a (X \hat{i} + Z \hat{k}) \]
\[ r' = a (\cos \phi' \hat{i} + \sin \phi' \hat{j}) \]
\[ r - r' = a [(X - \cos \phi') \hat{i} - \sin \phi' \hat{j} + Z \hat{k}] \]
\[ |r - r'| = [X^2 + Z^2 - 2X \cos \phi' + 1]^{1/2} \]
\[ dl' = a \sin \phi' \hat{\phi} = a \sin \phi' (-\sin \phi' \hat{i} + \cos \phi' \hat{j}) \]
where \( X = x/a \), \( Z = z/a \), and the unit vector \( \hat{\phi} \) has been recast in terms of its Cartesian components. (See Table 0.1) Finally, we find that
\[
I' dl' \times (r - r') = I' a^2 \sin \phi' \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \phi' & \cos \phi' & 0 \\ X - \cos \phi' & -\sin \phi' & Z \end{vmatrix}
\]
\[ = I' a^2 \sin \phi' [Z \cos \phi' \hat{i} + Z \sin \phi' \hat{j} + (1 - X \cos \phi') \hat{k}] \]
Substitution of these quantities into Eq. (5.8) yields the expression
\[ B(x, 0, z) = \frac{\mu_0 I'}{4\pi a} \int_{-\pi}^{\pi} \left[ Z \cos \phi' \hat{i} + Z \sin \phi' \hat{j} + (1 - X \cos \phi') \hat{k} \right] \left[ X^2 + Z^2 - 2X \cos \phi' + 1 \right]^{3/2} d\phi' \quad (5.12) \]
for the field at the selected observation point. This integral can be simplified by recognizing (1) that the \( y \) component involves the integral of an odd function of \( \phi' \) over symmetric limits and hence has the value zero and (2) that the \( x \) and \( z \) components involve integrals of even functions of \( \phi' \) and hence can be expressed as twice the integral over the positive half of the interval. Further, because the source is invariant to rotation by any amount about the \( \phi \) axis, we can obtain an expression for the field at any point \((x, \phi, z)\) by replacing \( \hat{i} \) with \( \hat{\phi} \) and \( x \) with \( \hat{r} \) (or \( X \) with \( \mathcal{R} = \hat{r}/a \)). We find then, that the field at any point is given by
\[ B(\hat{r}, \phi, z) = \frac{\mu_0 I'}{2\pi a} \left[ \hat{\phi} \int_{0}^{\pi} \frac{Z \cos \phi'}{[\mathcal{R}^2 + Z^2 - 2\mathcal{R} \cos \phi' + 1]^{3/2}} d\phi' \right.
\[ + \hat{k} \int_{0}^{\pi} \frac{(1 - \mathcal{R} \cos \phi')} {[\mathcal{R}^2 + Z^2 - 2\mathcal{R} \cos \phi' + 1]^{3/2}} d\phi \] (5.13)
The integrals in Eq. (5.13) are, of course, analytically intractable for arbitrary \( \mathcal{R} \) and \( Z \). Indeed, they are simple only if we restrict our attention to an observation point on the \( Z \) axis, where \( \mathcal{R} = 0 \), and Eq. (5.13) reduces to the much simpler expression
\[ B(0, \phi, z) = \frac{\mu_0 I'}{2\pi a} \hat{k} \int_{0}^{\pi} \frac{d\phi'}{[Z^2 + 1]^{3/2}} = \frac{\mu_0 I'}{2a} \frac{1}{[Z^2 + 1]^{3/2}} \hat{k} \quad (5.14) \]
which we choose to write in the form
\[ \frac{B(0, 0, z)}{\mu_0 I'/2a} = \frac{1}{[Z^2 + 1]^{3/2}} \hat{k} \quad (5.15) \]
by measuring the field in units of the field at the center of the loop. Clearly, the field at a point on the axis of the loop—or, better, the \( z \) component of that field—is an even function.
Figure 5.5: Z Component of On-Axis Magnetic Field of a Circular Loop. In this graph, \( B_0 = \mu_0 I'/2a \). This graph was produced with IDL.

of the \( z \) coordinate, has a maximum value at \( Z = 0 \) (the center of the loop), and falls off symmetrically as one proceeds further away from the center to points either above or below the loop.

We can invoke IDL to generate a quick graph of the \( z \)-component of the magnetic field of a current loop at points on the axis of the loop. Simply invoking the statements

\[
\text{IDL> } Z = -4.0 + \text{findgen(81)}/10.0 ; \text{ Create } Z \text{ with values from } \\
\text{ } -4.0 \text{ to } 4.0 \text{ in steps of } 0.1 \\
\text{IDL> } BZ = 1.0 /\left(Z^2+1\right)^{\frac{1}{2}} ; \text{ Calculate corresponding } BZ \\
\text{IDL> plot, } Z, BZ, \text{ ytitle=' } B/B0, \text{ xtitle=' } z/a; \text{ Plot graph}
\]

to IDL will produce the graph shown in Fig. 5.5.

The combination of two identical loops, one located a distance \( c \) above the \( xy \) plane and the other located a distance \( c \) below the \( xy \) plane is particularly interesting. The contribution of each member of this pair is readily obtained by interpreting the quantity \( Z \) in Eq. (5.15) as the coordinate of the observation point relative to the center of the corresponding loop. Then, with \( C = c/a \), we find that

\[
\frac{B_{\text{pair}}(0,0,z)}{\mu_0 I'/2a} = \left(\frac{1}{[(Z-C)^2 + 1]^{3/2}} + \frac{1}{[(Z+C)^2 + 1]^{3/2}}\right) \hat{k} \tag{5.16}
\]

To use IDL to create a surface plot of the field in Eq. (5.16), we could invoke the statements

\[
\text{IDL> lugen_grid, } z, c, \text{ xrange=[-2.0,2.0], nx = 40, yrange=[0.0,1.0], ny=20 }
\]

\[\text{3See Section F.1.1 in Appendix F for a description of the Lawrence-generated IDL routine lugen_grid. The source file lugen_grid.pro and other Lawrence-generated IDL programs described in this chapter can be downloaded from $HEADEM/idl. The translation of $HEADEM for your site is identified in your Local Guide.}\]
Figure 5.6: The On-Axis Magnetic Induction Field of a Pair of Coils. Z locates the point on the axis; C measures the separation of the coils. Part (a) displays the field in a single unit of measure; part (b) has been scaled so that the field at each separation is 1 at \( Z = 0 \). The latter graph distorts the relative magnitude of the fields for various separations but shows more clearly the special significance of the Helmholtz separation. This graph was produced with IDL.

IDL> \( bz = \frac{1.0}{(z-c)^2 + 1.0}^{1.5} + \frac{1.0}{(z+c)^2 + 1.0}^{1.5} \)

IDL> surface, bz, charsize=3.0, xtitle='z', ytitle='c', ztitle='scaled B', $  
    xtickname=['-2','-1', '0', '1', '2'], $  
    ytickname=['0.00', '0.25', '0.50', '0.75', '1.00']

To produce Fig. 5.6(a), which shows this function over the \( ZC \) plane. The on-axis field for a particular separation of the loops, for example, is shown by following this surface along a path parallel to the z axis. Unfortunately, some of the features of this surface are hidden behind higher parts of the surface in the foreground. In an alternative presentation, obtained by inserting the commands

IDL> \( b0 = \frac{2.0}{(c^2+1.0)^{1.5}} \)

IDL> \( bz = \frac{bz}{b0} \)

before the surface command in the above script, the magnetic field is scaled differently for each separation to make the field at the center (\( Z = 0 \)) unity (Fig. 5.6(b)) for all separations.

The scaling of the second graph distorts the comparison between different separations but shows particularly clearly the special flatness of the curve \( \frac{B_z}{B_0} \) versus \( z/a \) at the special separation \( C = \frac{1}{2} \) (\( c = \frac{1}{2}a \))—loop separation equal to the common radius. At this particular separation, the field along the axis is nearly constant over a particularly large portion of the axis centered on the center of the pair of loops. Pairs of loops made with this special separation are especially useful in laboratory contexts and are called Helmholtz coils.
CHAPTER 5. THE MAGNETIC INDUCTION FIELD OF STEADY CURRENTS

5.3 Mapping Magnetic Fields

The programs magnetfield and magnetloop, listing of which are included in Appendix F, and instructions for the use of which are included in Appendix G, are based on a predictor-corrector embellishment of the algorithm described in Fig. 4.12. Alternative routines for drawing fields are typically built in to symbolic and numeric programs like MAXIMA, MAPLE, Mathematica, IDL, MATLAB, OCTAVE, and PYTHON. Figure 5.7 shows several field patterns produced by routines of this sort.

5.4 Numerical Evaluation of Fields

The use of numerical integration to study fields has already been illustrated in Chapter 4. Those techniques can, of course, also be used to determine magnetic fields. We illustrate here by examining the off-axis field of a single current loop as given by Eq. (5.13). In a plane containing the axis of the loop, that field has only two components, a radial component ($\hat{r}$) given by

$$\frac{B_r(r, \phi, z)}{\mu_0 I/2a} = \frac{1}{\pi} \int_0^\pi \frac{Z \cos \phi'}{[R^2 + Z^2 - 2R \cos \phi' + 1]^{3/2}} d\phi'$$

(5.17)

and an axial ($\hat{k}$) component given by

$$\frac{B_z(r, \phi, z)}{\mu_0 I/2a} = \int_0^\pi \frac{(1 - R \cos \phi')}{[R^2 + Z^2 - 2R \cos \phi' + 1]^{3/2}} d\phi'$$

(5.18)

We can, of course, generate graphs of these components by invoking an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON. Using IDL, for example, we would define the integrands for the integration routines with the file looprad.pro

```idl
function looprad, phi
; LOOPRAD defines the integrand for computing the radial component of the magnetic field produced at a point in the RZ plane by a circular current loop.
; common param, R, Z
tmp1 = cos(phi)
tmp2 = R^2 + Z^2 - 2*R*tmp1 +1
tmp3 = Z*tmp1/tmp2^1.5
return, tmp3/!Pi
end
```

for the radial component of the field and the file loopax.pro

```idl
; LOOPRAD defines the integrand for computing the axial component of the magnetic field produced at a point in the RZ plane by a circular current loop.
; common param, R, Z
```

---

4See Fig. 3.2 and the associated text.
5All Lawrence-generated IDL programs described in this chapter can be downloaded from $\textbf{$HEADEM/idl}$. The translation of $\textbf{HEADEM}$ for your site is identified in the Local Guide.
Figure 5.7: The magnetic induction field produced by several selected current distributions. Each distribution consists of one or more circular current loops, all carrying currents of the same strength. The current in each loop goes into the page at the point marked \(\times\) and comes out of the page at the point marked \(\bullet\). The plane of each loop is perpendicular to the page and intersects the page in the horizontal line connecting the associated \(\times\) and \(\bullet\).

Part (a) shows the field of a single loop (magnetic dipole) and should be compared with the field of an electric dipole shown in Fig. 4.5. In regions remote from the dipole the two fields are similar; marked differences are apparent in regions near to and “inside of” the dipoles. Part (b) shows the field of a Helmholtz coil, in which the separation of the two constituent loops is equal to the radius of the loops (P5.5). Part (c) shows the field of a magnetic quadrupole (two parallel loops with opposite currents) and part (d) shows the field of two loops with parallel axes but lying in the same plane. The field lines drawn are chosen to reveal the shape of the field and their spacing does not convey the relative strength of the field at various points.
CHAPTER 5. THE MAGNETIC INDUCTION FIELD OF STEADY CURRENTS

function loopax, phi
    ; LOOPAX defines the integrand for computing the axial component of the magnetic field produced at a point in the RZ plane by a circular current loop.
    ;
    ; common param, R, Z
    ;
    tmp1 = cos(phi)
    tmp2 = R^2 + Z^2 - 2*R*tmp1 +1
    tmp3 = ( 1 - R*tmp1 )/tmp2^1.5
    return, tmp3/%pi
end

for the axial component of the field. As a simple illustrating of the use of these files, we calculate the two components of the field along a radial line (varying $R$) at the fixed value $Z = 0.2$: 

```
IDL> common param, RR, ZZ ; Create common area for looprad and loopax
IDL> Z = 0.2 ; Set value for Z
IDL> R = findgen(81)/40.0 ; Vector of values from 0.0 to 2.0 in steps of 0.025
IDL> BR = fltarr(81) ; Prepare BR for radial components
IDL> BZ = fltarr(81) ; Prepare BZ for axial components

; Evaluate radial and axial components
IDL> for i=0, 80 do begin $
IDL>     RR=R[i] & ZZ= Z & $ ; Set coordinates of next point on plot
IDL>     BR[i] = qsimp( 'looprad', 0.0, %pi ) & $ ; Evaluate BR
IDL>     BZ[i] = qsimp( 'loopax', 0.0, %pi ) & $ ; Evaluate BZ
IDL> endfor

; Plot axial component
IDL> plot, R, BZ, thick=4, xtitle='r/a', ytitle='B!Dz!N/B!D0!N', charsize=1.5

; Plot radial component
IDL> plot, R, BR, thick=4, xtitle='r/a', ytitle='B!Dr!N/B!D0!N', charsize=1.5
```

Figure 5.8 shows these graphs and also the corresponding graphs obtained by executing the same script with, however, $Z$ set to the value 0.4. Further exploration of this field is relegated to the problems.

PROBLEMS

P5.1. Describe a charge distribution having zero charge density but nonzero current density, thereby showing how it is possible for magnetic induction fields to exist in the absence of electric fields.

The routine qsimp may produce an error message that, in essence, warns the user that convergence may not have been achieved at one or more integrations, but that warning doesn’t prevent the routine from returning the results obtained for those integrations.
5.4. NUMERICAL EVALUATION OF FIELDS

Figure 5.8: The axial and radial components of the magnetic field produced by a circular loop in the plane $Z = 0.2$ (solid line) and $Z = 0.4$ (dashed line). These graphs were produced with IDL.

![Graph of magnetic field components](image1)

Figure 5.9: Figure for Problem P5.3.

![Diagram of rectangular loop near a wire](image2)

P5.2. A rigid object within which the charge density is nonzero moves with velocity $\mathbf{v}$ but does not rotate. Show that the magnetic induction $\mathbf{B}$ and the electric field $\mathbf{E}$ established by this distribution are related by $\mathbf{B} = \mu_0 \varepsilon_0 \mathbf{v} \times \mathbf{E}$. Hint: Use Eq. (5.10) and note that $\mathbf{J} = \rho \mathbf{v}$. (Strictly, this result is valid only if $v \ll$ speed of light.)

P5.3. A rectangular loop carrying current $I'$ is placed near a long straight wire lying in the plane of the loop and carrying current $I$ (Fig. 5.9). Find the net force on the loop. Is the loop attracted to or repelled from the straight wire?

P5.4. (a) Using Eqs. (5.15) and (5.16) and a suitable computer tool, generate a graph of $B_z/(\mu_0 I'/2a)$ for the single loop as a function of $z/a$ and then generate graphs of $B_{\text{pair}}/(\mu_0 I'/2a)$ for the pair of identical loops as a function of $z/a$ for various values of $c/a$ ranging from $c/a = 0$ (loops on top of one another) to $c/a = 1$ (loops separated by twice their common radius). (b) From your graphs, explain why the configuration known as the Helmholtz coil, for which $c/a = \frac{1}{2}$, is especially useful. (c) For the two cases $c/a = 0$ and $c/a = \frac{1}{2}$, determine how far from the origin one can go along the $z$-axis before the field has fallen to 90% of its value at the origin. (d) Using an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica, find a Taylor expansion
CHAPTER 5. THE MAGNETIC INDUCTION FIELD OF STEADY CURRENTS

in \( Z \) of Eq. (5.16) about the value of \( Z = 0 \), carrying the expansion to include terms as high as \( Z^5 \). Note that careful choice of \( C \) can eliminate the term involving \( Z^2 \), leaving the term involving \( Z^4 \) as the most significant correction from the zeroth order term. With this insight, explain again why the Helmholtz configuration is especially significant.

**P5.5.** A simple solenoid is made by winding a wire many times around a (nonmagnetic) cylindrical core, the resulting coil looking like a very tight coil spring. (a) Recast the result in Eq. (5.15) to the form

\[
B_z(0, 0, z) = \frac{\mu_0 I' a^2}{2} \frac{1}{(z^2 + a^2)^{3/2}}
\]

and then use this result to show that the field at \((0, 0, z)\) on the axis of a solenoid of radius \( a \) is given by

\[
B(0, 0, z) = \frac{\mu_0 n m}{2\pi} \int_{-L/2}^{L/2} \frac{dz'}{[a^2 + (z - z')^2]^{3/2}}
\]

where \( n \) is the number of turns per unit length, \( m \) is the magnetic moment of a single turn, \( L \) is the length of the solenoid, and the axis of the solenoid coincides with the \( z \)-axis. (b) Calculate the field \( B_\infty \) when \( L = \infty \) and reexpress the above equation in the dimensionless form \( B/B_\infty = \cdots \). (c) Using a numerical integration routine from an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON, evaluate this integral to study \( B_z(0, 0, z)/B_\infty \) as a function of \( z/a \) for several values of \( L/a \). For each case examined, determine how far from the center you can go before the field has changed by 1%. Under what circumstances can the field at the center be calculated to 1% by assuming the solenoid to be infinitely long?

**P5.6.** The circulation of an electron in an atomic orbit constitutes a current loop. If the orbit has radius \( a \) and the electron (charge \( -q \)) circulates with speed \( v \), determine the resulting current, find the magnetic dipole moment [Eq. (3.35)] of the orbiting electron, and find the magnetic induction field at the center of the orbit. Using values for the hydrogen atom, estimate the electron magnetic moment and the field at the nucleus numerically. **Hint:** Use the form of \( B_z \), as the most significant correction from the zeroth order term. With \( Z = 0 \), carrying the expansion to include terms as high as \( Z^5 \).

**P5.7.** A circular current loop of radius \( a \) lies in the \( xy \)-plane with its center at the origin and carries a current \( I' \) counterclockwise as viewed from a point on the positive \( z \)-axis. (a) Starting with the Biot-Savart Law, show that

\[
B(x, 0, z) = \frac{\mu_0 I' a}{2\pi} \int_0^\pi \frac{z \cos \phi' \hat{i} + (a - x \cos \phi') \hat{k}}{[x^2 + z^2 + a^2 - 2ax \cos \phi']^{3/2}} d\phi'
\]

where \( \phi' \) is the angle between the \( x \)-axis and a point on the loop. (b) Verify that this result reduces correctly to the result in Eq. (5.15) when \( x = 0 \). (c) Expand the integrand in powers of \( a/r \), where \( r^2 = x^2 + z^2 \), and show that, when the observation point is remote from the loop,

\[
B(\mathbf{r}) = \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{r}) \hat{r} - \mathbf{m}]
\]

where \( \mathbf{m} \) is the magnetic dipole moment [Eq. (3.35)] of the current loop and \( \hat{r} \) is a unit vector directed toward the point of observation from the center of the loop. **Hint:** Let \( \theta \) be the angle between the \( z \)-axis and the direction of observation. Then \( z = r \cos \theta \) and \( x = r \sin \theta \).

**P5.8.** Explore further characteristics of the field of two identical loops arranged as in the configuration to which Eqs. (5.17) and (5.18) apply.

**P5.9.** Study the magnetic induction field produced by two identical loops arranged as for the pair discussed in the text, but let one of the loops (say the upper one) carry a current...
5.5. **The Magnetic Flux Law**

Just as Coulomb’s Law gave rise to Gauss’s Law and the restricted Faraday Law for static electric fields, the Biot-Savart Law gives rise to the magnetic flux law and Ampere’s circuital law for static magnetic induction fields. Statement of the flux law, which is the topic of this section, involves the concept of the flux $\Phi_m$ of the magnetic induction across a surface $\Sigma$, where, by definition,

$$\Phi_m = \int_{\Sigma} \mathbf{B} \cdot d\mathbf{S} \quad (5.19)$$

Statements similar to those made following Eq. (4.21) apply also to Eq. (5.19). The mks unit of magnetic flux is the weber (Wb), and the older name—Wb/m$^2$—for the mks unit of $\mathbf{B}$ is thus consistent with Eq. (5.19).\(^7\)

Seeking a statement analogous to Gauss’s Law, we shall now examine the flux of a static magnetic induction out of an arbitrary closed surface. Using the divergence theorem, we can write this flux as a volume integral, i.e.

$$\oint_{\Sigma} \mathbf{B} \cdot d\mathbf{S} = \int_{V} \nabla \cdot \mathbf{B} \, dv \quad (5.20)$$

where $V$ is the volume bounded by $\Sigma$. With $\mathbf{B}$ given by Eq. (5.10), however, $\nabla \cdot \mathbf{B}$ can be evaluated; we find that

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \cdot \int J(r') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, dv' = -\frac{\mu_0}{4\pi} \int J(r') \cdot \left[ \nabla \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right] \, dv'$$

$$= \frac{\mu_0}{4\pi} \int J(r') \cdot \left[ \nabla \times \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] \, dv' \quad (5.21)$$

where the second equation is obtained from the first by (1) reversing the order of differentiation with respect to the components of $\mathbf{r}$ and integration on the components of $\mathbf{r}'$ and (2) applying Eq. (C.12), and the third equation is obtained from the second by using Eq. (4.52). Since, however, the curl of any gradient is zero [Eq. (C.17)], we find from Eq. (5.21) that

$$\nabla \cdot \mathbf{B} = 0 \quad (5.22)$$

and then from Eq. (5.20) that

$$\oint_{\Sigma} \mathbf{B} \cdot d\mathbf{S} = 0 \quad (5.23)$$

where $\Sigma$ is arbitrary. Equations (5.22) and (5.23) are equivalent (P5.11); they state the magnetic flux law in differential and integral form, respectively. When $\mathbf{B}$ is static, this law is a direct consequence of the Biot-Savart Law. In fact, the flux law carries over to non-static fields without change of form. That extension, however, can be made only on the basis of experimental observations beyond those supporting the Biot-Savart Law. In

\(^7\)Wilhelm Eduard Weber, see footnote 5 in Chapter 3, page 71.
that extension, therefore, the magnetic flux law becomes something more than a mere
restatement of a portion of the Biot-Savart Law. A fuller treatment of time-dependent
fields is postponed until Chapter 6.

Equation (5.23) can be given an illuminating physical interpretation. The electric
analog of Eq. (5.23) is Gauss’s Law,
\[ \oint_{\Sigma} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_{V} \rho \, dv \] (5.24)
That is, the net flux of \( \mathbf{E} \) out of a closed surface is proportional to the sum of all the point
electric charges inside the surface. To date, however, no one has succeeded in isolating a
corresponding magnetic point charge. If we were not aware of electric charge, we would put a
zero on the right-hand side of Eq. (5.24). Since in effect we are not aware of magnetic charge,
we put a zero on the right-hand side of the magnetic analog of Eq. (5.24), i.e., of Eq. (5.23).
Thus, Eq. (5.23) can be viewed as a mathematical statement that magnetic monopoles are
not needed to account for any known experimental results. Equivalently, Eq. (5.22) requires
the lines of the magnetic induction to close on themselves (Section 2.3), starting or stopping
of these lines occurring only on (the unneeded) magnetic charges. Should someone someday
successfully isolate a magnetic charge, Eqs. (5.22) and (5.23) would require modification to
incorporate these new results.

PROBLEMS

P5.10. Calculate the flux of the magnetic induction produced by the long straight wire across
the plane surface bounded by the rectangular loop in Fig. 5.9.
P5.11. Show that Eq. (5.22) can be derived from Eq. (5.23) and vice versa, thus verifying the
equivalence of these two expressions. Hint: To obtain Eq. (5.22) from Eq. (5.23), let \( \Sigma \)
become small and apply Eq. (2.25).

5.6 Ampere’s Circuital Law

The second basic magnetostatic law that follows as a consequence of the Biot-Savart Law
involves the line integral of the magnetic induction field about an arbitrary closed path \( \Gamma \).
Using Stokes’ theorem, we can write this line integral as a surface integral, i.e.,
\[ \oint_{\Gamma} \mathbf{B} \cdot d\mathbf{l} = \int_{\Sigma} \nabla \times \mathbf{B} \cdot d\mathbf{S} \] (5.25)
where \( \Gamma \) bounds the (open) surface \( \Sigma \). With \( \mathbf{B} \) given by Eq. (5.10), however, \( \nabla \times \mathbf{B} \) can be
evaluated; we find that
\[
\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \times \int \mathbf{J}(\mathbf{r'}) \times \frac{(\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3} \, dv' \\
+ \frac{\mu_0}{4\pi} \int [\mathbf{J}(\mathbf{r'}) \cdot \nabla'] \frac{(\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3} \, dv' \] (5.26)
where \( \nabla \) has been taken under the integral sign, the resulting triple product has been expanded using Eq. (C.16), and finally the operator \( \nabla \), which at this point acts only on
functions of the combination $\mathbf{r} - \mathbf{r}'$, has been replaced by the operator $-\nabla'$, where $\nabla'$ differentiates with respect to the components of $\mathbf{r}'$. Next, we use Eq. (C.26) to rewrite the second integral in Eq. (5.26), finding that

$$
\nabla \times \mathbf{B} = -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \left( \nabla' \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\mathbf{v}' - \frac{\mu_0}{4\pi} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \nabla' \cdot \mathbf{J}(\mathbf{r}') d\mathbf{v}' + \frac{\mu_0}{4\pi} \oint \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \cdot d\mathbf{S}'
$$

(5.27)

Here the third integral extends over the surface bounding the region in which $\mathbf{J}(\mathbf{r}')$ differs from zero. On this surface, however, $\mathbf{J}(\mathbf{r}') \cdot d\mathbf{S}'$—essentially the normal component of $\mathbf{J}(\mathbf{r}')$—is necessarily zero, for otherwise current would be flowing out of the region. Thus, the third integral in Eq. (5.27) has the value zero. But for the steady currents to which the Biot-Savart Law applies, $\nabla \cdot \mathbf{J}$ is also zero [Eq. (2.47)] and the second integral in Eq. (5.27) makes no contribution. Finally, differentiation shows that $\nabla' \cdot [(\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|^3] = 0$ except when $\mathbf{r}' = \mathbf{r}$. Thus, the remaining integral in Eq. (5.27) can be reduced to an integral over a small volume $\Delta V'$ surrounding the point $\mathbf{r}$, i.e.

$$
\nabla \times \mathbf{B} = -\frac{\mu_0}{4\pi} \int_{\Delta V'} \mathbf{J}(\mathbf{r}') \left( \nabla' \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\mathbf{v}'
$$

(5.28)

Now if we choose $\Delta V'$ sufficiently small that $\mathbf{J}(\mathbf{r}')$ varies insignificantly throughout $\Delta V'$, then $\mathbf{J}(\mathbf{r}')$ can be evaluated at the central point $\mathbf{r}$ and taken outside the integral. The remaining integral can then be rewritten as a surface integral by using the divergence theorem; we find that

$$
\nabla \times \mathbf{B} = +\frac{\mu_0 \mathbf{J}}{4\pi} \oint \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} \cdot d\mathbf{S}'
$$

(5.29)

where $\mathbf{J}(\mathbf{r})$ has been written simply $\mathbf{J}$ and the minus sign in front of Eq. (5.28) has been absorbed by reversing the terms in the numerator of the integrand. Now, the replacements $\mathbf{r}' \rightarrow \mathbf{r}$, $\mathbf{r} \rightarrow \mathbf{r}_0$, $d\mathbf{S}' \rightarrow d\mathbf{S}$ in Eq. (5.29) convert the integral into the integral evaluated at Eq. (2.37). Since $\mathbf{r}$ lies inside the surface of integration, the integral in Eq. (5.29) has the value $4\pi$ and we finally find that

$$
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}
$$

(5.30)

and then from Eq. (5.25) that

$$
\oint_{\Gamma} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\Sigma} \mathbf{J} \cdot d\mathbf{S}
$$

(5.31)

where the surface $\Sigma$ is any surface bounded by the curve $\Gamma$, $\Gamma$ is arbitrary, and the usual right-hand rule relates the direction in which $\Gamma$ is traversed to the direction assigned to $d\mathbf{S}$. Equations (5.30) and (5.31) are equivalent. (Why? Compare P5.11.) They state Ampere’s circuital law in differential and integral form, respectively. In words, Eq. (5.31) states that the line integral of $\mathbf{B}$ about a closed path is equal to $\mu_0$ times the current flowing across any surface bounded by that path. In contrast to the magnetic flux law, Ampere’s circuital law is valid only for static magnetic induction fields. We shall develop

---

8In fact, some current distributions extending to infinity can be permitted without changing this conclusion. In particular, currents such as those carried from minus infinity to plus infinity by infinitely long wires are allowed. See P5.12.
the necessary modifications in Chapter 6, noting now only that this later generalization of Ampere’s law rests on additional experimental observations and is therefore more than merely a consequence of the Biot-Savart Law.

As an example of the application of both the circuital law and the flux law, we shall evaluate the magnetic induction established by an infinitely long, cylindrical wire of radius $a$ having its axis coincident with the $z$-axis and carrying a total current $I$ uniformly distributed over its cross-section with current density $(I/\pi a^2) \hat{k}$ (Fig. 5.10). As with the line charge treated in Section 4.4, we here apply symmetry to simplify the most general field at the point $r$, viz.,

$$B(r) = B_r(r) \hat{r} + B_\phi(r, \phi, z) \hat{\phi} + B_z(r, \phi, z) \hat{k} \tag{5.32}$$

in cylindrical coordinates. The present distribution shares invariance to rotation about the $z$-axis and invariance to translation along the $z$-axis with the line charge, and the three components on Eq. (5.32) must therefore be independent of $\phi$ and $z$. But the current distribution is not invariant to reflection in the $xy$-plane, so the remaining conclusions concerning the symmetries of the field of a line charge do not apply to the current. Symmetry supports the reduction

$$B(r) = B_r(r) \hat{r} + B_\phi(r) \hat{\phi} + B_z(r) \hat{k} \tag{5.33}$$

which is applicable whether $r < a$ or $r > a$, but (as presented here) goes no further. Additional information about these components, however, can be obtained by applying the two basic laws to suitable surfaces or curves. Apply the flux law, for example, to the surface bounded by a cylinder of radius $r$ and height $h$ and two planes, as shown in Fig. 5.10. Only $B_z$ contributes to the flux across the upper and lower plane surfaces, but $B_z$ is independent
of $z$ and the flux entering at the lower surface is exactly equal to the flux leaving at the upper surface; the two surfaces together make no net contribution to $\int \mathbf{B} \cdot d\mathbf{S}$. The cylindrical portion of the surface, however, contributes an amount $2\pi r B_z(r)$, since the component of $\mathbf{B}$ normal to the cylindrical surface is everywhere $B_z$ and $B_z$ has the same value at every point on the surface. Thus,

$$\int \mathbf{B} \cdot d\mathbf{S} = 2\pi r B_z(r)$$

(5.34)

for this surface. Since the flux law requires $\int \mathbf{B} \cdot d\mathbf{S} = 0$, we conclude that $B_z = 0$ and hence that

$$\mathbf{B}(r) = B_\phi(r) \hat{\phi} + B_z(r) \hat{k}$$

(5.35)

Now, apply the circuital law to a path coming in along the $x$-axis to $x = r$, going up parallel to the $z$-axis to $x = r$, $z = d$, and then returning to infinity along the line $z = d$ in the $xz$-plane. Finally, regard the path to be closed at infinity. No current crosses the surface bounded by this path, even if the path extends to a point inside the wire. Thus, the circuital law requires that $\oint \mathbf{B} \cdot d\mathbf{l} = 0$. Assuming that $|\mathbf{B}| \to 0$ at large distances from the wire\(^9\) and taking $\mathbf{B}$ as in Eq. (5.35), we determine for this path that

$$\oint \mathbf{B} \cdot d\mathbf{l} = B_z(r) d$$

(5.36)

which will be zero only if $B_z = 0$. Thus, $\mathbf{B}$ is finally reduced to

$$\mathbf{B}(r) = B_\phi(r) \hat{\phi}$$

(5.37)

To determine $B_\phi$ more explicitly, we apply the circuital law again, this time to a circle of radius $r$ lying in the $xy$-plane with its center at the origin. On this circle $d\mathbf{l} = r \, d\phi \, \hat{\phi}$ and, with attention to the proper relationship between the direction of traversal of the path and the direction assigned the surface, $d\mathbf{S} = r' \, dr' \, d\phi' \, \hat{k}$. Thus, for this path

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_0^{2\pi} B_\phi(r) \, r' \, dr' \, d\phi' = 2\pi r B_\phi(r)$$

(5.38)

and

$$\int \mathbf{J} \cdot d\mathbf{S} = \int_0^r r' \, dr' \int_0^{2\pi} \frac{I}{\pi a^2} \, d\phi' = \begin{cases} \frac{I}{\pi a^2} \left( \frac{r}{a} \right)^2, & r < a \\ I, & r > a \end{cases}$$

(5.39)

and the circuital law gives

$$B_\phi(r) = \begin{cases} \frac{\mu_0 I}{2\pi a} \left( \frac{r}{a} \right), & r < a \\ \frac{\mu_0 I}{2\pi a} \left( \frac{a}{r} \right), & r > a \end{cases}$$

(5.40)

A graph of $B_\phi(r)$ versus $r$, plotted in the dimensionless form $B_\phi/\left(\mu_0 I/2\pi a\right)$ versus $r/a$, is shown in Fig. 5.11. Note that this result in the region outside the wire is identical with Eq. (5.11) for a wire of zero radius.

\(^9\)This statement follows from the Biot-Savart Law, but it also is part of our general expectation about the relationship between any field and its source, namely that the effect of a (localized) source must become less strong as the observation point becomes more remote. The difficulty in applying this general expectation to specific cases lies in deciding whether a specific source is sufficiently localized. For the infinite wire in the present context, this assumption proves to be realistic. It would not apply to the electric field of a two-dimensionally infinite sheet [Eq. (4.13)].
Figure 5.11: $B_\phi$ versus $r$ for the magnetic induction field of a long straight wire of non-zero radius.

PROBLEMS

P5.12. Let the source distribution for Eq. (5.27) be an infinitely long straight wire carrying current $I$ along the $z$-axis from $z = -\infty$ to $z = +\infty$. Regard the surface integral in Eq. (5.27) to extend over a cylindrical surface of finite length surrounding the wire and then let the ends of the cylinder recede to $\pm \infty$. Show that the surface integral approaches the value zero in the limit.

P5.13. A coaxial cable consists of a center wire of radius $a$ and a coaxial cylindrical shell of radius $b$, $b > a$. Suppose that the center wire of a long straight coaxial cable carries a current $I$ along the $z$-axis toward $z = +\infty$ and that the shell carries the same current in the opposite direction. Assuming that the current in the wire is uniformly distributed over the wire, find the magnetic induction field as a function of distance $r$ from the axis of the cable and draw a graph of the azimuthal component of the field versus $r$.

P5.14. A simple solenoid is made by winding a wire many times around a (nonmagnetic) cylindrical core, the resulting coil looking like a very tight coil spring. Consider an (idealized) infinitely long solenoid of circular cross section (radius $a$) wound with $n$ turns of wire per unit length, the wire carrying current $I$. Let the coil be wound tightly enough so that effects arising from the pitch of the windings can be ignored. (a) Use the flux law and symmetry to show that the resulting field has no (cylindrical) radial component. (b) Use the circuital law and symmetry to show that the resulting field has no azimuthal component. (c) Use the circuital law to show that the axial component of the resulting field is zero outside and $\mu_0 n I$ inside the solenoid and argue that the field is uniform across the cross section of the solenoid. (d) From the sign conventions built into the basic laws, verify the right-hand rule relating the direction of the current around the solenoid to the direction of the field. (Fingers around the solenoid in the direction of the current; thumb in the direction of the field.)

P5.15. A solenoid (P5.14) of finite length and radius $a$ is wound with a total of $N$ turns of a wire carrying current $I$ and is then bent so that its two ends meet, forming a toroid (doughnut) with central radius $b$, $b > a$. Let the axis of the toroid coincide with the $z$-axis and let its
median plane lie in the \textit{xy}-plane. (a) Show that the magnetic induction inside the toroid
is given by

$$B = \frac{\mu_0 NI}{2\pi r} \hat{\phi}$$

and show that the flux \(\Phi_m\) of this field across the plane
surface bounded by a single turn of the coil is given by

$$\Phi_m = \frac{\mu_0 NIa}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{\xi + (b/a)} d\xi$$

(b) Use an integration routine in an available numeric processing program like IDL,
MATLAB, OCTAVE, or PYTHON as a tool to explore this integral as a function of \(b/a\).

5.7 The Magnetic Vector Potential

An expression for a magnetic analog of the electrostatic potential can be derived by applying
an argument similar to that following Eq. (4.52) to the Biot-Savart Law. Substituting the
identity, Eq. (4.52), into Eq. (5.8), for example, we obtain

$$B(r) = \frac{\mu_0}{4\pi} \oint \left( \nabla \frac{1}{|r-r'|} \right) \times I' d'I'$$

(5.41)

Since \(I' d'I'\) does not depend on \(r\), however, it is constant in so far as the derivatives in \(\nabla\)
are concerned. Using Eq. (C.8) with \(\Phi = 1/|r-r'|\) and \(Q = I' d'I'\), we can therefore write
the integrand in Eq. (5.41) as \(\nabla \times (\Phi Q)\). Finally, since \(\nabla\) does not act on the variables of
integration, we can interchange the order of differentiation and integration to find that

$$B(r) = \nabla \times \left( \frac{\mu_0}{4\pi} \oint \frac{I' d'I'}{|r-r'|} \right)$$

(5.42)

A similar argument applied to Eq. (5.10) gives

$$B(r) = \nabla \times \left( \frac{\mu_0}{4\pi} \oint \frac{J(r')}{|r-r'|} dv' \right)$$

(5.43)

for a source distribution described by the current density \(J(r)\). From either form, we infer
that the magnetic induction field can be derived from the curl of another vector field,

$$B = \nabla \times A$$

(5.44)

where \(A\) is called the (magnetic) vector potential. From Eqs. (5.42) and (5.43), however,
we can conclude only that

$$A(r) = \frac{\mu_0}{4\pi} \oint \frac{I' d'I'}{|r-r'|} + W(r)$$

(5.45)

$$= \frac{\mu_0}{4\pi} \oint \frac{J(r')}{|r-r'|} dv' + W(r)$$

(5.46)

where \(W(r)\) is an arbitrary vector field satisfying \(\nabla \times W = 0\)—a restriction that is considerably
less constraining than the corresponding restriction on the function \(\chi\) in Eq. (4.54).
Since any field with zero curl can always be expressed as a gradient [item (3) in Section 2.5],
the condition \( \nabla \times \mathbf{W} = 0 \) will be satisfied if (and only if) \( \mathbf{W} = \nabla \lambda \) but imposes no constraint on the scalar field \( \lambda \). Thus Eqs. (5.42)–(5.44) together define \( \mathbf{A} \) no more explicitly than the equations

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint \frac{I' \, dl'}{|\mathbf{r} - \mathbf{r}'|} + \nabla \lambda(\mathbf{r}) \tag{5.47}
\]

\[
= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \, dv'}{|\mathbf{r} - \mathbf{r}'|} + \nabla \lambda(\mathbf{r}) \tag{5.48}
\]

where \( \lambda \) is arbitrary; whatever the value of \( \lambda \), \( \nabla \times \mathbf{A} \) will give the correct \( \mathbf{B} \)-field. Equations (5.47) and (5.48) not only provide for a calculation of \( \mathbf{A} \) directly from known sources but also express a relationship between (a static) \( \mathbf{A} \) and its sources even when the sources are not explicitly known. Equation (5.49) should be compared with the electrostatic analog, Eq. (4.57).

In consequence of Eqs. (5.47) and (5.48), the difference between two equivalent vector potentials \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \), distinguished by the choices \( \lambda_1 \) and \( \lambda_2 \) for \( \lambda \), is given by

\[
\mathbf{A}_2 - \mathbf{A}_1 = \nabla(\lambda_1 - \lambda_2) \implies \mathbf{A}_2 = \mathbf{A}_1 + \nabla \Lambda \tag{5.49}
\]

where \( \Lambda = \lambda_2 - \lambda_1 \) is arbitrary, since \( \lambda_2 \) and \( \lambda_1 \) are themselves independently arbitrary. This transformation from one vector potential to a second equivalent vector potential is called a gauge transformation, and the fact that addition of an arbitrary gradient to the vector potential does not change the corresponding \( \mathbf{B} \)-field is referred to as the gauge invariance of \( \mathbf{B} \); each vector potential by itself is said to be in a particular gauge. Now the vector potential can be chosen not only so that \( \mathbf{B} = \nabla \times \mathbf{A} \) but also so that \( \nabla \cdot \mathbf{A} \) has any chosen value whatever. We prove this property by noting first that Eq. (5.49) gives

\[
\nabla \cdot \mathbf{A}_2 = \nabla \cdot \mathbf{A}_1 + \nabla^2 \Lambda \tag{5.50}
\]

Thus if we have a vector potential \( \mathbf{A}_1 \) in hand and \( \nabla \cdot \mathbf{A}_1 \) does not have the desired value, we first find \( \Lambda \) by solving

\[
\nabla^2 \Lambda = -\nabla \cdot \mathbf{A}_1 + d \tag{5.51}
\]

where \( d = d(\mathbf{r}) \) is the desired divergence, and then find an equivalent vector potential \( \mathbf{A}_2 \) from Eq. (5.49). By its construction, \( \mathbf{A}_2 \) will then have the desired divergence. In summary, gauge invariance of \( \mathbf{B} \) corresponds to the freedom to select \( \nabla \cdot \mathbf{A} \) arbitrarily. When treating static fields, \( \nabla \cdot \mathbf{A} \) is commonly set equal to zero; we shall later see that other choices are more suitable for time-dependent fields.

A differential equation relating \( \mathbf{A} \) and \( \mathbf{J} \) when both are static can be obtained by substituting Eq. (5.44) into Eq. (5.30) and then using Eq. (C.19); we find that

\[
\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{5.52}
\]

Thus, choosing the so-called Coulomb gauge, in which \( \nabla \cdot \mathbf{A} = 0 \), we find finally that

\[
\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{5.53}
\]

---

10 The existence of \( \mathbf{A} \) might also have been inferred from the flux law, \( \nabla \cdot \mathbf{B} = 0 \), by applying the theorem in item (4) of Section 2.5. Since the flux law is valid for time-dependent fields, this route to Eq. (5.44) demonstrates the existence of \( \mathbf{A} \) more generally than the route adopted above. It does not, however, provide an expression for calculating \( \mathbf{A} \) directly from its sources.
5.7. THE MAGNETIC VECTOR POTENTIAL

Table 5.1: Interrelationships Among Static $J$, $B$, and $A$.

<table>
<thead>
<tr>
<th>Given ↓ / Find →</th>
<th>$J$</th>
<th>$B$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J$</td>
<td>$-\mu_0 \int \frac{J(r') \times (r - r')}{</td>
<td>r - r'</td>
<td>^3} dv'$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\mu_0 \nabla \times B$</td>
<td>$\nabla \times A$</td>
<td>$\nabla \times A$</td>
</tr>
<tr>
<td>$A$</td>
<td>$\frac{1}{\mu_0} \nabla^2 A^a$</td>
<td>$\nabla \times A$</td>
<td>$\nabla \times A$</td>
</tr>
</tbody>
</table>

$^a$ provided $A$ is in Coulomb gauge, for which $\nabla \cdot A = 0$.

which should be compared to the electrostatic analog, Eq. (4.58).

The interrelationships we have now developed among the current density, the magnetic induction, and the vector potential are summarized in Table 5.1. The entries in this table should be compared with those in Table 4.1.

To illustrate the application of some of the relationships developed in this section, we shall calculate the magnetic induction produced by an arbitrary current loop at a point remote from the loop. We first evaluate the vector potential, which we take to be given by Eq. (5.47) with $\lambda = 0$. When $|r| = r \gg |r'| = r'$ for all $r'$ within the region of integration (i.e., far from the loop), we can expand the denominator in powers of $r'/r$ and keep only the first two terms; we find that

$$A(r) = \frac{\mu_0 I'}{4\pi r} \oint dS' \times \nabla' (r \bullet r') = \int dS' \times [(r \bullet \nabla') r' + r \times (\nabla' \times r')]$$

where we have recognized that $dv'$ in Eq. (5.47) in fact stands for an increment in $r'$, which can equally well be denoted by $dr'$. Now, $\oint dS'$ is a vector from the start of the path to the end and is zero for a closed path. Thus, only the second term in Eq. (5.54) contributes. This term can be evaluated by applying Eq. (C.20) followed by Eq. (C.15) with all derivatives acting on the primed variables and $r$ being a constant; we find that

$$\oint (r \bullet r') dr' = \int dS' \times \nabla' (r \bullet r') = \int dS' \times [(r \bullet \nabla') r' + r \times (\nabla' \times r')] = \int dS' \times r$$

the last form following because $(r \bullet \nabla') r' = r$ and $\nabla' \times r' = 0$. Substituting Eq. (5.55) into the second term of Eq. (5.54) and omitting the (vanishing) first term, we have that

$$A(r) = \frac{\mu_0 I'}{4\pi r^3} \left( \int l' dS' \right) \times r = \frac{\mu_0}{4\pi} \frac{m \times r}{r^3}$$

where $m$ is the magnetic dipole moment of the current loop, which has already appeared in a different context in Eq. (3.35). Direct evaluation of $\nabla \times A$ then gives

$$B = \frac{\mu_0}{4\pi r^3} [3(m \bullet \hat{r}) \hat{r} - m]$$

$$= \frac{\mu_0 m}{4\pi r^3} [2 \cos \theta \hat{r} + \sin \theta \hat{\theta}]$$
the second form applying in spherical coordinates when \( \mathbf{m} = m \mathbf{k} \). The far field of a magnetic dipole thus has exactly the same form as the far field of an electric dipole [compare Eqs. (5.57) and (5.58) with Eqs. (4.10) and (4.11)], but the similarity does not extend to the near fields.

Although the vector potential \( \mathbf{A} \) is important to many theoretical developments, it is less useful in practical problems than the scalar potential \( V \). Part of the utility of \( V \), for example, arises because the electrostatic field \( \mathbf{E} \) is often easily determined by first evaluating \( V \) from a scalar integral [Eq. (4.56)] over known sources and then evaluating \( \mathbf{E} = -\nabla V \). The corresponding procedure for determining the magnetostatic field \( \mathbf{B} \)—namely evaluate \( \mathbf{A} \) from a vector integral Eq. (5.47) or (5.48) over known sources and then evaluate \( \mathbf{B} = \nabla \times \mathbf{A} \)—is only occasionally easier than determining \( \mathbf{B} \) directly from the Biot-Savart Law. Not only is the curl of a vector more difficult to evaluate than the gradient of a scalar, but the integrals for \( \mathbf{A} \) are often more difficult than those for \( V \), particularly because \( \mathbf{A} \) must be known over a broader region of space than \( V \). To be more specific, if we seek only \( E_z(0,0,z) \), then it is sufficient to know \( V(0,0,z) \) in order to obtain \( E_z(0,0,z) = -\partial V(0,0,z)/\partial z \). If, however, we seek \( B_z(0,0,z) \) we must know \( A_x(x,y,z) \) and \( A_y(x,y,z) \), for only then can we calculate \( (\nabla \times \mathbf{A})_z = \partial A_y/\partial x - \partial A_x/\partial y \) correctly for the point \((0,0,z)\). Said another way, symmetries constraining \( \mathbf{E} \) can be exploited to yield a useful \( V \), but symmetries constraining \( \mathbf{B} \) cannot be so easily exploited to yield a useful \( \mathbf{A} \).

A second part of the utility of \( V \) arises because its physical interpretation is very immediate and hence intuitive: \( qV \) gives the energy of a charge in the field \( V \); \( \frac{1}{2} \int \rho V \, dv \) gives the energy required to assemble the distribution \( \rho \) that is the source of \( V \). Although \( \mathbf{A} \) has physical interpretations—it plays a role in determining the energy required to establish a current distribution; \( \mathbf{A} \) and also \( V \) have effects on the phase of quantum wave functions—these interpretations are less immediate and less intuitive than those of \( V \). It is therefore natural to think of \( V \) as a “real” field—whatever that means—and to think of \( \mathbf{A} \) more as a mathematical tool, even though such a classification in fact misrepresents particularly the quantum significance of \( \mathbf{A} \).

**PROBLEMS**

**P5.16.** A portion of a circuit carrying current \( I \) lies along the \( z \)-axis in the region \( a \leq z \leq b \). Let \( I \) flow from \( z = a \) toward \( z = b \). Use Eq. (5.47) with \( \lambda = 0 \) to calculate the contribution of this portion of the circuit to the vector potential at an arbitrary point \( \mathbf{r} \). Don’t be afraid of integral tables. *Optional:* Let \( a = -b \) and use a computer to obtain a graph of \( A_z(\mu_0 I/4\pi) \) versus \( z/b \), where \( z \) is the distance from the wire to an observation point in the \( xy \)-plane.

**P5.17.** By setting up and evaluating Eq. (5.47) with \( \lambda = 0 \), find the vector potential established at a point on the axis and a distance \( z \) from the center of a circular loop of radius \( a \) carrying current \( I \). How much of the magnetic induction field can be correctly found from the curl of this potential? Explain your answer fully.

**P5.18.** Show that \( \mathbf{A} \) as given by Eq. (5.48) with \( \lambda = 0 \) has zero divergence. Assume that the currents are confined to a finite region of space. *Hint:* Remember that Eq. (5.48) applies only for steady currents, for which \( \nabla \cdot \mathbf{J} = 0 \).

---

5.8. ENERGY IN THE STATIC MAGNETIC INDUCTION FIELD

P5.19. Let \( \mathbf{B} = B_0 \hat{k} \) with \( B_0 \) constant. (a) Show that \( \mathbf{A}_1 = \frac{1}{2} \mathbf{B} \times \mathbf{r} \) and \( \mathbf{A}_2 = B_0 x \hat{j} \) are both acceptable vector potentials. (b) Find a function \( \Lambda \) such that \( \mathbf{A}_2 = \mathbf{A}_1 + \nabla \Lambda \). (c) Show that both \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) have zero divergence, thereby demonstrating that \( \nabla \times \mathbf{A} = \mathbf{B} \) and \( \nabla \cdot \mathbf{A} = 0 \) together do not necessarily define \( \mathbf{A} \) uniquely.

P5.20. Find a vector potential for the field of a long straight wire. Hints: (1) Assume \( \mathbf{A} = A_z(r) \hat{k} \) and require \( \nabla \times \mathbf{A} \) to give the field in Eq. (5.11). (2) See Eq. (0.73).

P5.21. (a) Use Stokes’ theorem to show that \( \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{S} \), where \( \mathbf{B} = \nabla \times \mathbf{A} \). (b) An infinitely long solenoid of radius \( a \) with \( n \) turns per unit length of a wire carrying current \( I \) produces the field \( \mathbf{B} = \mu_0 n I \hat{k} \) when \( r < a \) and \( \mathbf{B} = 0 \) when \( r > a \) (P5.14). Here, \( \hat{k} \) is a unit vector parallel to the axis of the solenoid. Assume a vector potential having only a \( \phi \) component, and use part (a) to find this vector potential. (c) Verify that \( \nabla \times \mathbf{A} = \mathbf{B} \) both inside and outside the solenoid. See Eq. (0.73). (d) Ponder what it means for there to exist a vector potential outside the solenoid where there is no field.

P5.22. Derive Eqs. (5.57) and (5.58) for the field of a magnetic dipole by evaluating the curl of the corresponding vector potential, Eq. (5.56).

5.8 Energy in the Static Magnetic Induction Field

Paralleling our development of electrostatics, we should at this point discuss energy storage in a static magnetic induction field. Unfortunately, we can develop that topic most naturally only after we are able to calculate the work required to establish a current, and that calculation in turn requires the (general) Faraday Law of electromagnetic induction, which we shall develop in Section 6.1. The best we can do at this point is obtain the correct result by a plausibility argument based on the analogous electrostatic expressions. Beginning with Eq. (4.75), we replace \( \rho \) with \( \mathbf{J} \), \( V \) with \( \mathbf{A} \), and the product \( \rho V \) with the dot product \( \mathbf{J} \cdot \mathbf{A} \).

The result for the work \( W \) required to establish the steady current distribution \( \mathbf{J} \) is

\[
W = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A} \, dv
\]  

(5.59)

where \( \mathbf{A} \) is the vector potential established by \( \mathbf{J} \) and the integral extends over all space. Using Eqs. (5.30), (C.12), and (5.44) and the divergence theorem, we find alternatively that

\[
W = \frac{1}{2\mu_0} \int (\nabla \times \mathbf{B}) \cdot \mathbf{A} \, dv = \frac{1}{2\mu_0} \int B^2 \, dv - \frac{1}{2\mu_0} \oint (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{S}
\]

(5.60)

where \( \Sigma \) is a large spherical surface that ultimately recedes to infinity, in which limit (if \( \mathbf{J} \) is localized) \( |\mathbf{A}| \propto r^{-2} \), \( |\mathbf{B}| \propto r^{-3} \)—see Eqs. (5.56) and (5.57)—and \( |d\mathbf{S}| \propto r^2 \). Thus, \( (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{S} \propto r^{-3} \) and the above integral over \( \Sigma \) approaches zero. From Eq. (5.59) we infer that a magnetic energy density

\[
u_B = \frac{B^2}{2\mu_0}
\]

(5.61)

should be assigned to a region of space where the magnetic induction is \( \mathbf{B} \). We stress, however, that Eq. (5.59), which is correct only for static fields, and Eq. (5.60), which turns out to be correct more generally, have not yet been rigorously derived.
PROBLEMS

P5.23. Let a toroidal coil be made by winding \( N \) turns on a core whose cross section is a rectangle of height \( h \) and width \( w \) and then bending the core into a “doughnut” with inside radius \( a \) and outside radius \( a + w \). If the coil carries current \( I \), show that the energy \( W \) stored in the coil is given by

\[
W = \frac{\mu_0}{4\pi} N^2 I^2 h \ln \left( \frac{a + w}{a} \right)
\]

P5.24. The energy stored in a static magnetic induction field as given by Eq. (5.60) is obviously gauge-invariant. Show that Eq. (5.59) is also gauge-invariant, i.e., that

\[
\frac{1}{2} \int \mathbf{J} \cdot \mathbf{A}_1 \, dv = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A}_2 \, dv,
\]

where \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) are related by Eq. (5.49), and state any assumptions in your proof. \textit{Hint}: Remember that \( \nabla \cdot \mathbf{J} = 0 \) for steady currents.

5.9 The Multipole Expansion of the Magnetic Vector Potential

A multipole expansion of the magnetic vector potential established by a current loop can be obtained by substituting Eq. (4.83) into Eq. (5.47) with \( \lambda = 0 \); we find that

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int I' \, dl' \left( \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \frac{1}{2r^5} \sum_{i,j} x_i x_j \left[ 3x_i' x_j' - \delta_{ij} (r')^2 \right] + \cdots \right)
\]

(5.62)

The first two terms are identical with those in Eq. (5.54); Eq. (5.56) expresses their value in terms of the magnetic dipole moment

\[
\mathbf{m} = \int I' \, dS' = \frac{1}{2} \int \mathbf{r}' \times I' \, dl'
\]

(5.63)

as defined in Eqs. 3.35 and 3.36. To express the third term more compactly, we introduce the nine-component quantity

\[
\mathbf{Q}^{(m)}_{ij} = \int [3x_i' x_j' - \delta_{ij} (r')^2] I' \, dl'
\]

(5.64)

each component of which is a vector. We then have from Eq. (5.62) that

\[
\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3} \mathbf{m} \times \mathbf{r} + \frac{\mu_0}{8\pi r^5} \sum_{i,j} x_i \mathbf{Q}^{(m)}_{ij} x_j + \cdots
\]

(5.65)

from which the corresponding field can be derived by evaluating \( \nabla \times \mathbf{A} \). We make three further observations: (1) If the source consists of several current loops, the corresponding values of \( \mathbf{m} \) and \( \mathbf{Q}^{(m)}_{ij} \) are merely sums of the values for each loop separately (Why?); (2) when the currents are steady (\( \nabla \cdot \mathbf{J} = 0 \)), a general current distribution can be thought of as a superposition of current loops and Eq. (5.65) applies with

\[
\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \, dv'
\]

(5.66)

\[
\mathbf{Q}^{(m)}_{ij} = \int [3x_i' x_j' - \delta_{ij} (r')^2] \mathbf{J}(\mathbf{r}') \, dv'
\]

(5.67)
and (3) the quantity $Q^{(m)}_{ij}$ introduced here is not the conventional magnetic quadrupole moment tensor, which is defined in terms of convenient (but fictitious) magnetic monopoles (see Section 11.4) and has nine scalar components.

**PROBLEMS**

**P5.25.** A current distribution consists of two parallel circular current loops of radius $a$, one located parallel to the $xy$-plane with its center at $(0, 0, -\frac{1}{2}b)$ and carrying current $I$ clockwise as viewed from a point on the positive $z$-axis and the other with its center at $(0, 0, \frac{1}{2}b)$ and carrying current $I$ counterclockwise. (a) Show that this distribution has zero magnetic dipole moment. (b) Calculate the nine vector components of $Q^{(m)}_{ij}$. (c) Write out the quadrupole term in the vector potential using spherical coordinates (Table 0.1) and calculate the corresponding magnetic induction. **Optional:** Evaluate this field by writing a superposition of two terms, each obtained from Eq. (5.57), and expanding in powers of $b/r$.

**P5.26.** Starting from Eq. (5.48), derive Eq. (5.65) in which $m$ and $Q^{(m)}_{ij}$ are defined by Eqs. (5.66) and (5.67).

**SUPPLEMENTARY PROBLEMS**

**P5.27.** When $\Gamma$ is a closed path, Eq. (5.7) gives the force between two complete circuits. Show that this force satisfies Newton’s third law. **Hint:** Use Eq. (C.1) and note that one of the resulting (closed) integrals involves an exact differential and hence has the value zero.

**P5.28.** In a region of space where $J$ is zero, $\nabla \times B$ is also zero and $B$ can be derived from a magnetic scalar potential $V^{(m)}$, defined so that $B = -\mu_0 \nabla V^{(m)}$. (a) Show that $V^{(m)}$ satisfies Laplace’s equation, $\nabla^2 V^{(m)} = 0$. (b) Given the field and potential of an electric dipole, Eqs. (4.10) and (4.51), and the field of a magnetic dipole, Eq. (5.57), infer the corresponding magnetic scalar potential. (c) Find a magnetic scalar potential from which the field outside an infinitely long wire, Eq. (5.11), can be derived and explain why your result is not a single-valued function of position. (d) Find a magnetic scalar potential from which the field on the axis of a circular current loop as given in Eq. (5.15) can be derived.

**P5.29.** A magnetic dipole with moment $m = m\hat{k}$, $m > 0$, is located at the origin and produces a field given by Eq. (5.57). A circular loop of radius $a$ and carrying current $I'$ is placed with its center at $(0, 0, b)$ and its plane parallel to the $xy$ plane. (a) Calculate the force between the dipole and the current loop, first exactly and then in the limit $b \gg a$; express the force in terms of the dipole moment $m' = m'\hat{k}$ of the loop (and other relevant parameters); and note how this force depends on $b$ and on the sign of $m'$. (b) Sketch a graph of the exact force versus $b$. For what value of $b$ is the force a maximum? This dipole-dipole interaction, explored in part here and somewhat more fully for the electric case in P4.54, is important in determining many molecular, atomic, and nuclear properties.

**P5.30.** Since $B = \nabla \times A$, Stokes’ theorem apparently combines with the flux law to give $\oint B \cdot dS = \oint \nabla \times A \cdot dS = \oint A \cdot dl = 0$. The final two parts of this equation, however, constitute the condition for deriving $A$ from a scalar via $A = \nabla \phi$. But then, $B = \nabla \times \nabla \phi = 0$ identically. All $B$-fields therefore vanish! Find the fallacy. This puzzler was presented by G. Arfken, *Am. J. Phys.* 27, 526 (1959).

**P5.31.** The force on a volume $V$ of a current distribution $J$ placed in a magnetic induction $B$ is given by $\int_V J \times B \, dv$ [Eq. (3.21)]. This force can also be expressed in terms of a magnetic pressure, $p_m(r)$. (a) Show that the force on $V$ is given in terms of $p_m(r)$ by $-\oint_{\Sigma} p_m \, dS$ where $\Sigma$ bounds $V$ and $dS$ is an outward normal. (b) Use Eq. (C.21) to write the surface integral as a volume integral and infer that $\nabla p_m = -J \times B$. (c) Assuming that $J = J(r)\hat{k}$
CHAPTER 5. THE MAGNETIC INDUCTION FIELD OF STEADY CURRENTS

and that $\mathbf{B} = B(\tau) \hat{\phi}$ in cylindrical coordinates, use the circuital law to show that

$$B(\tau) = \frac{\mu_0}{\tau} \int_0^\tau \tau' J(\tau') \, d\tau'$$

(d) Show that

$$p_m(\tau) - p_m(\tau_0) = \frac{B^2(\tau)}{2\mu_0} - \frac{B^2(\tau_0)}{2\mu_0} + \frac{1}{\mu_0} \int_0^\tau \frac{B^2(\tau')}{\tau'} \, d\tau'$$

where $\tau_0$ is an arbitrary reference point. Hint: Differentiate the result in part (c) with respect to $\tau$, solve for $J(\tau)$ in terms of $B(\tau)$, evaluate $\mathbf{J} \times \mathbf{B}$, and set the result equal to $-\nabla p_m = -(dp_m/d\tau)\hat{\tau}$. (c) Determine and sketch a graph of $p_m(\tau)$ as a function of $\tau$ for the field of a long wire of finite radius, Eq. (5.40). Choose $\tau_0 = 0$ and set $p_m(0) = 0$.

P5.32. One expression determining $\mathbf{A}$ directly from $\mathbf{B}$ is obtained by substituting $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ into the expression determining $\mathbf{A}$ from $\mathbf{J}$. From this expression, show that

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int \mathbf{B}(\mathbf{r'}) \times \frac{\mathbf{r} - \mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|^3} \, d\tau'$$

and state any assumptions made in your development. Optional: Seek an expression determining $\mathbf{A}$ from $\mathbf{B}$ that does not involve the function $1/|\mathbf{r} - \mathbf{r'}|$ or some power of it. Compare $-\int \mathbf{E} \cdot d\mathbf{r}$ for determining $V$ from $\mathbf{E}$.

P5.33. The magnetic induction produced at the point $(x, y, 0)$ by a circular current loop of radius $a$ lying in the $yz$ plane with its center at the origin and carrying a current $i$ counterclockwise as viewed from a point on the positive $x$-axis is given by

$$\mathbf{B}(x, y, 0) = \frac{\mu_0 ia}{2\pi} \int_0^\pi \frac{x \cos \phi' \hat{\phi} + (a - y \cos \phi') \hat{i}}{x^2 + y^2 + a^2 - 2ay \cos \phi' + a^2} \, d\phi'$$

where $\phi'$ is the angle between the positive $y$-axis and a point on the loop. (Compare P5.7.)

(a) Express this integral in dimensionless form by introducing $X = x/a_0$, $Y = y/a_0$, $A = a/a_0$, $I = i/i_0$, and $\mathbf{b} = \mathbf{B}/B_0$, where $a_0$ and $i_0$ are convenient values for length and current and $B_0 = \mu_0 i_0/2a_0$ is the field at the center of a loop of radius $a$ carrying current $i_0$. (b) Write a program for an available computer to evaluate $b_x$ and $b_y$ numerically. Assume that the current $I$, the radius $A$, and the position $X, Y$ are specified as input.

(c) Test your program by evaluating $\mathbf{b}$ at several points on the $x$-axis and comparing the results with the analytic result $b(X, 0, 0) = IA^2/(A^2 + X^2)^{3/2}$. (Compare the field in P5.5.) (d) Explore the field at representative points in the $xy$-plane off the axis of the loop. For example, examine $b(0, Y, 0)$ as a function of $Y$. Note: An alternative strategy for calculating the field of a loop numerically has been discussed by J. R. Merrill, Am. J. Phys. 39, 791 (1971). Optional: (1) modify your program to compute the field produced in the $xy$-plane by several loops, all of whose axes coincide with the $x$-axis. Let the current, radius, and location on the $x$-axis of each loop be specified as input. Use this program to study the field of the Helmholtz coil (two loops separated by their common radius and carrying the same current), the solenoid (many—say 10—closely spaced loops), and other configurations of your choosing. (2) Use one (or both) of the programs here written in a program similar to that of Fig. 4.12 to print out (or plot) points on the field lines of the magnetic induction in the $xy$-plane and study several configurations of loops. Can you invent a procedure to assure that field lines are drawn with the correct spacing? (3) Use one (or both) of these programs as part of a program based on Fig. 3.1 or 3.2 to print out (or plot) points on the trajectory of a particle moving in the field of one or more loops. (4) Write a program to compute the field established in the $xy$-plane by two loops of radius $a$ located in the $yz$-plane with their centers at $(0, d, 0)$ and $(0, -d, 0)$, $d > a$, and study the resulting field by printing its values or by printing (or plotting) points on selected field lines.
P5.34. Show that the vector potential $\mathbf{A}(\mathbf{r})$ established by a current distribution described by a surface current density $\mathbf{j}(\mathbf{r})$ (see P2.25) is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\Sigma} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \ dS'$$

where $\Sigma$ is the surface in which the current flows and $dS'$ is the (scalar) area of an element of the surface.

P5.35. A circular current loop of radius $a$ lies in the $xy$-plane with its center at the origin and carries a current $I'$ counterclockwise as viewed from a point on the positive $z$-axis. Show that the magnetic field at a point in the $xz$-plane is given by

$$\mathbf{B}(x, z) = \frac{\mu_0 I'/2\pi a}{a^2} \int_0^\pi \frac{z \cos \phi' \hat{i} + (a - x \cos \phi') \hat{k}}{\sqrt{x^2 + z^2 + a^2 - 2ax \cos \phi'}} \ d\phi'$$

and then use numerical integration to explore both components of this magnetic field as functions of $x/a$ for various values of $z/a$. 

---

**5.9. THE MULTPOLE EXPANSION OF $\mathbf{A}$**

155
Chapter 6

The Electromagnetic Field Produced by Time-Dependent Charge Distributions: Maxwell’s Equations in Vacuum

In this chapter, we shall develop relationships among time-dependent charge and current distributions and the resulting time-dependent fields. We have already defined these more general fields \( \mathbf{E}(\mathbf{r}, t) \) and \( \mathbf{B}(\mathbf{r}, t) \), in Chapter 3, essentially by the Lorentz force\(^1\)

\[
\mathbf{F}_q = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\] (6.1)
on a charge \( q \) moving in the fields with velocity \( \mathbf{v} \). Further, we have developed the equation of continuity

\[
\oint \mathbf{J} \cdot d\mathbf{S} = -\int \frac{\partial \rho}{\partial t} \, dv ; \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}
\] (6.2)
in Chapter 2 in a fully time-dependent form. In Chapters 4 and 5, however, we restricted our attention to the static electric fields of fixed charges and to the static magnetic induction fields of steady currents, finding (1) that the static \( \mathbf{E} \)-field has nonzero divergence and zero curl and satisfies Gauss’s Law

\[
\oint \mathbf{E} \cdot d\mathbf{S} = -\frac{1}{\epsilon_0} \int \rho \, dv ; \quad \nabla \cdot \mathbf{E} = -\frac{1}{\epsilon_0} \rho
\] (6.3)
and the restricted Faraday Law

\[
\oint \mathbf{E} \cdot d\mathbf{l} = 0 ; \quad \nabla \times \mathbf{E} = 0
\] (6.4)
(2) that the static \( \mathbf{B} \)-field has zero divergence and nonzero curl and satisfies the flux law

\[
\oint \mathbf{B} \cdot d\mathbf{S} = 0 ; \quad \nabla \cdot \mathbf{B} = 0
\] (6.5)

\(^{1}\)Hendrik Antoon Lorentz, see footnote 11 in Chapter 0, page 39.
and Ampere’s circuital law

\[ \oint B \cdot dl = \mu_0 \int J \cdot dS \quad ; \quad \nabla \times B = \mu_0 J \]  

(6.6)

and (3) that the static E-field and the static B-field are entirely independent. (Neither field occurs in the equations satisfied by the other.) In the first three sections of this chapter, we shall examine the applicability of Eqs. (6.3)–(6.6) to time-dependent fields, obtaining Maxwell’s equations for these fields\(^2\) and finding qualitatively (1) that the time-dependent E– and B–fields are interdependent and must therefore be thought of together as a single electromagnetic field, (2) that Eqs. (6.4) and (6.6) each require an additional term, and (3) that Eqs. (6.3) and (6.5) carry over to time-dependent fields with no change in form.\(^3\) In the remaining sections of the chapter, we shall develop a few very general consequences of Maxwell’s equations. Except for a few problems, the actual solution of these equations is postponed to later chapters.

6.1 Electromagnetic Induction: Faraday’s Law\(^4\)

The additional term needed to extend Eq. (6.4) to time-dependent fields was discovered early in the nineteenth century on the basis of direct experimental evidence acquired independently by Michael Faraday\(^5\) in England and by Joseph Henry\(^6\) in the United States. Qualitatively, the phenomenon of electromagnetic induction discovered by these scientists associates an induced electric field with a changing magnetic induction field. This connection and some of its consequences are the subjects of this section.

6.1.1 Electromotive Force

The basic law extending Eq. (6.4) to time-dependent fields is stated in terms of a quantity known as the electromotive force (emf) or the electromotance. To define this quantity, we imagine the following physical situation: In some region of space \( R \), let there exist general time-dependent electric and magnetic induction fields \( E(\mathbf{r},t) \) and \( B(\mathbf{r},t) \). Further, select a closed path \( \Gamma \)—perhaps but not necessarily defined by a wire—lying wholly in \( R \) and

---

\(^2\)Scottish mathematical physicist James Clerk Maxwell, b. 13 June 1831 in Edinburgh, Scotland; d. 5 November 1879 in Cambridge, England.

\(^3\)Absence of a change in form does not mean that there is no change in substance. These two equations look the same for both time-independent and time-dependent fields, but that identity in appearance can be accepted only after the more general equations have been subjected to experimental tests with time-dependent fields.


\(^5\)English scientist Michael Faraday, b. 22 September 1791 in Newington Buffs, England; d. 25 August 1867 in Hampton Court, Middlesex, England.

\(^6\)American scientist and first Secretary of the Smithsonian Institution Joseph Henry, b. 17 December 1797 in Albany, New York; d. 13 May 1878 in Washington, DC.
let that path (or portions of it) be moving—perhaps distorting—in some arbitrary way so that at time \( t \) the point \( r \) on the path is moving with velocity \( v(r, t) \). Finally, if there is a wire defining the path, include in \( \mathbf{E} \) and \( \mathbf{B} \) the contribution of any charge and current distributions on the wire and allow for the possibility of batteries in the wire, temperature differences, and perhaps other external influences. The positions of the path at times \( t \) and \( t + \Delta t \) are shown in Fig. 6.1. Now, consider a test charge of strength \( q \) at an arbitrary point on the path and moving with the path. This test charge experiences a total force \( \mathbf{f}_q(r, t) \) consisting not only of electromagnetic forces but also of effective (non-electromagnetic) forces from chemical effects in the batteries, thermal effects in the wire, etc. (The test charge is assumed to be at rest relative to its point on the path so that resistive forces arising from collisions of the test charge with the microscopic constituents of the wire do not arise.) The emf \( \mathcal{E}(t) \) about the selected path at time \( t \) is now defined as the work done by \( \mathbf{f}_q(r, t) \) on a unit charge moved once around the path, i.e.,

\[
\mathcal{E}(t) = \oint_{\Gamma} \mathbf{f}_q(r, t) \cdot d\mathbf{l} \tag{6.7}
\]

\[
= \oint_{\Gamma} \left[ \mathbf{E}(r, t) + \mathbf{v}(r, t) \times \mathbf{B}(r, t) \right] \cdot d\mathbf{l} + \oint_{\Gamma} \frac{\text{effective chemical forces}}{q} \cdot d\mathbf{l} + \oint_{\Gamma} \frac{\text{effective thermal forces}}{q} \cdot d\mathbf{l} + \cdots \tag{6.8}
\]

Although the definition of \( \mathcal{E} \) is very similar to the integral giving a (static) potential difference \( \Delta V \) about a closed path and both quantities are measured in volts, \( \mathcal{E} \) and \( \Delta V \) are physically very different; \( \mathcal{E} \) is in general not equal to zero and hence arises from nonconservative force fields while \( \Delta V \) is necessarily always zero and corresponds to a conservative force field.

Despite the abstractness of the definition in Eq. (6.7), \( \mathcal{E} \) is a measurable quantity, for it is related in a simple way to the current produced in a conducting wire that follows the
selected path. Thus, the dependence of $\mathcal{E}$ on various parameters can be studied experimentally in order to formulate a law (or laws) determining $\mathcal{E}$ from these parameters. In the present context, we are interested in the results of such experimental studies when batteries, temperature differences, and the like are not present. We therefore omit the chemical and thermal emf's from Eq. (6.8) and denote the term of interest by $\mathcal{E}_{em}$,

$$\mathcal{E}_{em} = \oint_{\Gamma} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \quad (6.9)$$

where the arguments $\mathbf{r}$ and $t$ have been suppressed.

### 6.1.2 Motional EMF

Under some circumstances, our previous theoretical framework permits a rewriting of Eq. (6.9) in a useful alternative form. Consider first a conducting rod of length $\ell$ that is moving through a uniform magnetic induction field. Before the rod is set into motion, the mobile electrons in the rod are uniformly distributed and the electrons are neutralized by the immobile positive ions that make up the rest of the conductor. On no portion of the rod is there an excess charge of either sign. When the rod is set into motion in the external field, the charges—both positive and negative—experience forces, and the electrons, which are free to move, respond by drifting away from their positions in the stationary rod. A charge imbalance is therefore established in the rod. This imbalance in turn produces an electric field both inside and outside the rod, and the charges in the rod now experience forces from this induced electric field as well as from the original magnetic induction. Ultimately, an equilibrium charge distribution is reached when sufficient charge separation has occurred so that the electric and magnetic forces on the electrons exactly cancel. All (macroscopic) motion of the electrons then ceases. The resulting equilibrium charge distribution is shown in Fig. 6.2.

Let us now evaluate the integral in Eq. (6.9) counterclockwise about the path shown in Fig. 6.3. We assume that, at every instant of time, the $\mathbf{E}$-field throughout the region containing the path is the static field appropriate to the instantaneous position of the rod, i.e., we assume that the $\mathbf{E}$-field follows changes in the position of the rod instantly. Since we shall later find that electromagnetic fields follow changes in their sources only after a delay determined by the speed of light and the distance from the source, our assumption here in effect requires that the speed of the rod be small compared to that of light and that the path not extend too far from the rod. When $\mathbf{E}$ can be regarded as static, however, $\oint \mathbf{E} \cdot d\mathbf{l} = 0$. Further, since $\mathbf{v}$ is zero everywhere except in the rod, Eq. (6.9), which now gives what is called a motional emf, reduces to

$$\mathcal{E}_{em}^{\text{mot}} = \int_{\text{lower end}}^{\text{upper end}} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = vB\ell \quad (6.10)$$

where the final evaluation follows because $\mathbf{v}$ and $\mathbf{B}$ are constant and mutually perpendicular at all points on the rod and $\mathbf{v} \times \mathbf{B}$ is parallel to $d\mathbf{l}$. Now, however, we note that

$$\frac{d\Phi_m(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Phi_m(t + \Delta t) - \Phi_m(t)}{\Delta t} = \lim_{\Delta t \to 0} \left( -\frac{(v \Delta t)B}{\Delta t} \right) = -vB \quad (6.11)$$

$^7\mathcal{E} = IR$, where $I$ is the current and $R$ is the resistance of the wire; see Chapter 9.

$^8$The time required for this equilibrium to be established depends on the properties of the conductor (see Chapter 9) but is not of concern here. For a perfect conductor, the equilibrium is established instantaneously.
Figure 6.2: A conducting rod moving at right angles to a constant magnetic induction field. Once a steady state has been reached, one end of the rod will be positively charged and the other end will be negatively charged.

\[ \Phi_m(t) \]

Figure 6.3: An imagined path for the evaluation of a line integral. The portion of the path lying inside the moving rod moves with the rod; the remainder of the path is at rest.

where \( \Phi_m(t) \) is the (magnetic) flux of \( \mathbf{B} \) across the surface bounded by the path \( \Gamma \) at time \( t \). The difference \( \Phi_m(t + \Delta t) - \Phi_m(t) \) is then just the flux across the area added to the path in time \( \Delta t \) and is negative because the added flux goes into the page in Fig. 6.3 while the positive direction defined by our counterclockwise traversal of the path is out of the page. Equations (6.10) and (6.11) together finally yield that

\[
\xi_{em}^{mot} = -\frac{d\Phi_m}{dt}
\]

which by its development is valid only when (1) the rod moves slowly enough and (2) the
change in flux comes about by motion of the rod in a static \( B \)-field. In words, the motional emf about the path in this situation is the negative rate of change of the magnetic flux across a surface bounded by the path.

The result in Eq. (6.12) for a motional emf can also be derived when the path moves in a more arbitrary way and the \( B \)-field is not uniform (but still static). We continue to assume that \( E \) is static, which means in particular that any conductors along the path must move slowly. With that assumption, \( \oint E \cdot dl = 0 \) for the more general path of Fig. 6.1, and Eq. (6.9) gives

\[
\varepsilon_{\text{em}}^{\text{mot}} = \oint \Gamma (v \times B) \cdot dl \tag{6.13}
\]

which is nonzero only if at least a portion of the path has a nonzero velocity. On the other hand, the rate of change of flux across a surface bounded by the path in Fig. 6.1 is given by

\[
\frac{d\Phi_m}{dt} \approx \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_i B(r_i) \cdot \Delta S_i \tag{6.14}
\]

where the path has been broken into segments \( \Delta l_i \), \( r_i \) is a point on the \( i \)-th segment, and \( \Delta S_i \) is the area added when the \( i \)-th segment moves a distance \( v(r_i, t) \Delta t \) to its position at time \( t + \Delta t \) (Fig. 6.4). Since \( v(r_i, t) \Delta t \times \Delta l_i \) agrees both in magnitude and direction with \( \Delta S_i \) (the direction of \( \Delta S_i \) must agree with the positive direction determined by applying the right-hand rule to the path at \( t + \Delta t \)), we can rewrite Eq. (6.14) as

\[
\frac{d\Phi_m}{dt} \approx \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_i B(r_i) \cdot [v(r_i, t) \Delta t \times \Delta l_i] \tag{6.15}
\]

\[
\approx \sum_i B(r_i) \cdot [v(r_i, t) \times \Delta l_i] \tag{6.16}
\]

In the limit as \( |\Delta l_i| \to 0 \) for all \( i \), Eq. (6.16) becomes

\[
\frac{d\Phi_m}{dt} = \oint \Gamma B \cdot (v \times dl) = -\oint \Gamma (v \times B) \cdot dl \tag{6.17}
\]

where the final form involves exchanging the dot and cross in the previous form (P0.7) and then reversing the order of the factors in the resulting cross product. By comparing Eqs. (6.13) and (6.17), however, we find that Eq. (6.12) in fact gives the motional emf under more general circumstances than its initial derivation implied, but even this more general development still restricts Eq. (6.12) to motional emf.

Although the minus sign in Eq. (6.12) has apparently been required merely to preserve certain sign conventions, the fact that a minus sign rather than a plus sign is needed expresses a physically significant aspect of motional emf’s. The direction assigned to \( dl \) in evaluating \( \varepsilon_{\text{em}}^{\text{mot}} \) defines the direction in which an induced current would flow if the path is replaced by a conducting wire and \( \varepsilon_{\text{em}}^{\text{mot}} > 0 \). (Why?) Under those circumstances, the induced current produces a magnetic induction field whose flux contributes positively to the flux across the surface bounded by the path. (Imagine a positive or counterclockwise current in the path of Fig. 6.3. What is the direction of the resulting \( B \)-field over the surface bounded by the path?) If, however, \( \varepsilon_{\text{em}}^{\text{mot}} > 0 \), Eq. (6.12) requires that \( d\Phi_m/dt < 0 \). Thus, a negative change in the flux induces a current whose magnetic induction contributes positively to the flux. The minus sign in Eq. (6.12) in effect states that the induced emf
brought about by a change in magnetic flux is always of such direction as to oppose the change in flux; it is the mathematical expression of what is called Lenz’s Law.

6.1.3 Faraday’s Law of Electromagnetic Induction

In the previous paragraphs, we have seen that an induced emf is generated if a path moves or distorts to encompass a different total magnetic flux. But the flux encompassed by a path also changes when the magnetic induction is time-dependent and the path itself remains stationary. These new circumstances occur, for example, when a current loop or bar magnet is moved toward or away from a fixed path in space. The observations of Faraday and Henry were that a time-dependent magnetic flux also induces an emf about a given path and, further, that Eq. (6.12), derived for motional emf, actually applies as well when the changing flux comes about through a time-dependent magnetic induction. Experimental evidence beyond any introduced earlier in this book thus supports the Faraday Law of electromagnetic induction, which we can write either as

\[ \mathcal{E}_{\text{em}} = -\frac{d\Phi_m}{dt} \]  

(6.18)

or, using Eqs. (6.9) and (5.19), as

\[ \oint_{\Gamma} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\Sigma} \mathbf{B} \cdot d\mathbf{S} \]  

(6.19)

where \( \Sigma \) is any surface bounded by \( \Gamma \). The minus sign expressing Lenz’s Law is still present and the fields may now have a general dependence on time. Further, Eq. (6.19) incorporates both motional emf’s and emf’s induced about fixed paths by time-dependent \( B \)-fields. We stress again that Eqs. (6.18) and (6.19) cannot be accepted solely on the basis of their agreement with the results obtained for motional emf. The law of induction has other aspects and these must be subjected to careful experimental test before accepting the law. This law cannot be derived from principles presented earlier in this book; it is inherently an experimental law and takes its justification from the agreement of its predictions with experimental observation.

We now rewrite Eq. (6.19) in a form that compares more directly with Eq. (6.4). Since both \( B \) and the surface of integration in Eq. (6.19) may depend on time, the total time derivative of the flux in general has two contributions; symbolically,

\[ \frac{d}{dt} \int_{\Sigma} \mathbf{B} \cdot d\mathbf{S} = \int_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_{\Gamma} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \]  

(6.20)

where the first term expresses the rate of change of flux arising from the explicit time dependence of \( B \) and the second term—obtained directly from Eq. (6.17)—gives the rate of change of flux arising from the motion of the boundaries of the surface of integration. Substituting Eq. (6.20) into Eq. (6.19), we find finally that

\[ \oint_{\Gamma} \mathbf{E} \cdot d\mathbf{l} = -\int_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \]  

(6.21)

---

9Russian physicist Heinrich Friedrich Lenz, b. 12 February 1804 in Dorpat, Russian Empire; d. 10 February 1865 in Rome, Papal State.
terms involving $\mathbf{v} \times \mathbf{B}$ having canceled. This form of Faraday’s Law is the preferred form for theoretical purposes. It reduces more obviously than Eq. (6.19) to Eq. (6.4) when $\mathbf{B}$ is static and can thus be viewed as the generalization of Eq. (6.4) to time-dependent fields. In words, Eq. (6.21) states that a time-dependent magnetic induction field is accompanied by an induced, nonconservative electric field.

Of all the basic laws of electromagnetism, Faraday’s Law is perhaps of greatest practical import, for this law expresses the essential physics of transformers and electric generators which in turn are crucial to the technological society we take so much for granted. To examine these applications here, however, would take us far afield; we relegate some elementary aspects to the problems and the rest of that subject to other books.

**PROBLEMS**

**P6.1.** In our derivation of Eq. (6.12) for motional emf, we restricted both $\mathbf{E}$ and $\mathbf{B}$ to be static. Subsequently, we set $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ and clearly utilized the assumption of a static $\mathbf{E}$-field. Where did we utilize the assumption of a static $\mathbf{B}$-field?

**P6.2.** Convince yourself that the expression substituted for $\Delta \mathbf{S}_i$ in going from Eq. (6.14) to Eq. (6.15) indeed has the correct direction for all possible values of $\mathbf{v}(r_i, t)$, not just for the direction we happen to have drawn in Fig. 6.4.

**P6.3.** A rectangular conducting loop of mass $m$ falls under gravity across the poles of a permanent magnet producing a constant field $\mathbf{B}$ out of the page (Fig. 6.5). Let the field extend over a region of width $w$. Find the terminal velocity $v$ that the loop acquires before falling out of the region occupied by the field. *Hint:* Let the loop have resistance $R$ and accept the relationship $\mathbf{E} = \mathbf{I} \cdot \mathbf{R}$ between the induced emf $\mathbf{E}$ and the current $\mathbf{I}$ in the loop.

**P6.4.** A conducting disc of radius $R$ rotates at constant angular speed $\omega$ about its axis, which is parallel to a constant magnetic induction $\mathbf{B}$. Calculate the motional emf induced in a path running along a radius from the center to the rim of the disc and then closing along a stationary path lying outside the disc. This device is called a Faraday disc dynamo.
6.1. ELECTROMAGNETIC INDUCTION: FARADAY’S LAW

P6.5. A conducting circular loop of area $A$ is arranged so that it can be rotated at constant angular speed $\omega$ about a diameter. (a) Let the axis of rotation be perpendicular to a constant magnetic induction $B$. Determine the induced emf as a function of time. What could you do to increase the maximum emf without changing $\omega$? This device is a simple generator. (b) Let the orientation of the axis be adjustable and suppose that you can display the induced emf as a function of time on an oscilloscope. Describe a means by which this device could be used to measure both the magnitude and the direction of an unknown field. This device is now called a search coil.

P6.6. A magnetic dipole has moment $m(t)\hat{k}$ and is located at the origin. Let the dipole moment be increasing ($dm/dt > 0$). Determine the direction of the emf induced about a circular path oriented as described in each of the following situations: (a) plane perpendicular to the $z$-axis, the center at $(0,0,b)$, $b > 0$; (b) in the $xy$-plane, the center at $(0,b,0)$; and (c) plane perpendicular to the $y$-axis, the center at $(0,0,0)$.

P6.7. A long straight wire carries a current $I(t)$ as shown in Fig. 5.9. (a) Determine (exactly) the magnitude and direction of the emf induced about the rectangular loop shown in the figure. *Hint:* The flux across the loop was calculated in P5.10. (b) Suppose the current in the straight wire changes with time as illustrated in Fig. 6.6, where $I > 0$ means a current in the positive direction indicated in Fig. 5.9. Sketch a qualitative graph of the current in the rectangular loop as a function of time, making sure that the direction of that current is clear.

P6.8. A conducting ring is placed on top of a solenoid (P5.14) that has its axis vertical. When
the current is turned on in the solenoid, the ring flies into the air. Explain this phenomenon for yourself and then read the discussion given by E. J. Churchill and J. D. Noble, *Am. J. Phys.* 39, 285 (1971).

**P6.9.** A uniform, time-dependent magnetic induction $\mathbf{B} = B(t) \hat{k}$ exists in space. Determine the emf induced about a fixed circular path of radius $r$ lying in the $xy$-plane with its center at the origin and then, assuming that the induced electric field $\mathbf{E}$ is tangent to the path at each point $(x, y)$ on the path, show that

$$\mathbf{E} = -\frac{r}{2} \frac{dB}{dt} \hat{\phi} = \frac{1}{2} \frac{dB}{dt} (y \hat{i} - x \hat{j})$$

Characteristics of this field are further explored in P6.36.

**P6.10.** In a betatron an evacuated toroidal doughnut, inside of which moves a beam of electrons, is placed between the poles of a large electromagnet. As the magnetic induction field is increased, an emf is induced about the doughnut and the electrons are accelerated. Simultaneously, the same changing field provides the centripetal force necessary to keep the electron in its circular orbit. Show that the field at the orbit must at all times be one-half the (space) average of the field over the surface bounded by the orbit if the electrons are to remain in an orbit of fixed radius.

**6.1.4 Energy in the Static Magnetic Induction Field**

The law of induction now makes possible a rigorous justification of Eq. (5.59) for the energy required to establish a steady current distribution. We ask first for the work required to establish a steady current $I$ in a closed circuit. Although the circuit can be of arbitrary shape, it is convenient to think of it as a simple loop, such as that shown in Fig. 6.7. Let the
current in this loop at time $t$ be $i(t)$, where $i(t)$ increases from 0 to $I$ in some time interval $0 < t < T$. Further, let the (magnetic) flux $\Phi_m(t)$ across the loop originate in part from the current in the loop and in part from other changing currents that are external to the loop. Finally, adopt the (consistent) conventions that $i(t) > 0$ means a counterclockwise current and $\Phi_m(t) > 0$ means a flux out of the page. Figure 6.7 shows the direction of the induced emf $E$ when $d\Phi/dt > 0$ (and hence $E = -|E|$). In the situation described, the current $i$ is flowing against the induced emf and whatever agent is maintaining the current must therefore be doing positive work on the charges as they move around the loop. In particular, if a charge $\Delta Q$ is carried around the loop at a time when the emf is $E$, the amount of work $\Delta W$ done on the charge is given by

$$\Delta W = -\mathcal{E} \Delta Q \quad (6.22)$$

(The minus sign must be inserted explicitly to assure that $\Delta W$ will be positive when $\Delta Q$ is positive and $\mathcal{E}$ is intrinsically negative.) If the charge transport represented by $\Delta Q$ takes place in a small time interval $\Delta t$, division of Eq. (6.22) by $\Delta t$ and passage to the limit $\Delta t \to 0$ yields the expression

$$P(t) = \frac{dW}{dt} = -\mathcal{E}(t) i(t) \quad (6.23)$$

for the instantaneous power output $P(t)$ of the agent maintaining the current. Thus, the total energy $W$ required to establish the current $I$ is given by

$$W = \int_0^T P \, dt = -\int_0^T \mathcal{E} i \, dt = \int_0^T i \frac{d\Phi_m}{dt} \, dt = \int_{\Phi_m(0)}^{\Phi_m(T)} i \, d\Phi_m \quad (6.24)$$

where the last two forms are obtained by using Faraday’s Law.

We next ask for the work required to establish steady currents simultaneously in each of several nearby loops. Let the current at time $t$ in the $r$-th loop be $i_r(t)$, where $i_r(0) = 0$ and $i_r(T) = I_r$, and let the flux across the $r$-th loop be $\Phi_{mr}(t)$. Then, summing the work required for each loop separately, we obtain for the total work $W_t$ the expression

$$W_t = \sum_r \int_0^T i_r \frac{d\Phi_{mr}}{dt} \, dt \quad (6.25)$$

Now, the flux across the $r$-th loop is the sum of independent contributions from all of the loops, including the $r$-th, i.e.,

$$\Phi_{mr} = \sum_s \int_{\text{rth loop}} (\mathbf{B} \text{ at rth loop due to sth loop}) \cdot d\mathbf{S} \quad (6.26)$$

From the Biot-Savart Law, however, the $\mathbf{B}$-field at the $r$-th loop due to the current $i_s$ in the $s$-th is directly proportional to $i_s$. This same proportionality thus applies also to the contribution of the $s$-th loop to the flux across the $r$-th loop. Further, since we assume that all the loops are fixed in space, the entire time dependence of this contribution to the flux is contained in $i_s$. We can therefore display the time dependence of $\Phi_{mr}$ explicitly and fully by writing Eq. (6.26) in the form

$$\Phi_{mr}(t) = \sum_s M_{rs} i_s(t) \quad (6.27)$$
where \( M_{rs} \) is a constant determined by the shape, size, and relative positions of loops \( r \) and \( s \) but independent of the current in either loop; it is called the mutual inductance of the \( r \)-th loop with respect to the \( s \)-th loop when \( r \neq s \) and the self-inductance of the \( r \)-th loop when \( r = s \), and \( M_{rs} = M_{sr} \) (P6.13). The mks unit of inductance is the henry (H), which may be alternatively expressed as a Wb/A or as \( \text{V} \cdot \text{s}/\text{A} \). Now, Eq. (6.27) yields not only the more explicit definition

\[
M_{rs} = \frac{\partial \Phi_{mr}}{\partial i_s}
\] (6.28)

for \( M_{rs} \) but also, on substitution into Eq. (6.25), yields the result

\[
W_t = \sum_{r,s} M_{rs} \int_0^T i_r \frac{di_s}{dt} \, dt
\] (6.29)

To deduce a simpler evaluation, let us write this sum in the alternative form

\[
W_t = \sum_{r,s} M_{sr} \int_0^T i_s \frac{di_r}{dt} \, dt
\] (6.30)

obtained by exchanging the indices. Better still, we obtain

\[
W_t = \frac{1}{2} \sum_{r,s} \left( M_{rs} \int_0^T i_r \frac{di_s}{dt} \, dt + M_{sr} \int_0^T i_s \frac{di_r}{dt} \, dt \right)
\] (6.31)

by adding Eq. (6.29) and Eq. (6.30) and dividing by two. Recalling the above mentioned symmetry of the matrix \( M \) (again see P6.13), however, we can rewrite Eq. (6.31) to obtain

\[
W_t = \frac{1}{2} \sum_{r,s} M_{rs} \left( i_r \frac{di_s}{dt} + i_s \frac{di_r}{dt} \right) \, dt = \frac{1}{2} \sum_{r,s} M_{rs} \int_0^T \frac{d(i_r i_s)}{dt} \, dt
\] (6.32)

where Eq. (6.33) follows from Eq. (6.32) on substitution of Eq. (6.27) with \( t = T \); \( \Phi_{mr} \) in Eq. (6.33) thus denotes the final steady flux across the \( r \)-th loop.

Finally, we transform Eq. (6.33) into a form more suited for calculating the energy required to establish a current distribution described by the steady current density \( \mathbf{J} \). Such a distribution can be regarded as a superposition of many current loops. Thus, in terms of the vector potential \( \mathbf{A} \), we find from Eq. (6.33) that

\[
W_t = \frac{1}{2} \sum_r I_r \int_{\text{rth loop}} \mathbf{B} \cdot d\mathbf{S} \quad \text{(see P5.21)}
\]

\[
= \frac{1}{2} \sum_r I_r \oint_{\text{rth loop}} \mathbf{A} \cdot d\mathbf{l}
\]

\[
= \frac{1}{2} \sum_r \int_{\text{volume occupied by rth loop}} \mathbf{J} \cdot \mathbf{A} \, dv \quad \text{(see P2.7)}
\]

\[
= \frac{1}{2} \int_{\text{entire distribution}} \mathbf{J} \cdot \mathbf{A} \, dv
\] (6.34)
and we have supplied the justification both for Eq. (5.59) and for the development in Section 5.8.

PROBLEMS

P6.11. Obtain an integral—probably multiple—for the self-inductance \( L \) of an isolated circular current loop of radius \( a \). *Hint:* The field of the loop is given in sufficient generality in P5.7. *Optional:* Express the result so that the integral is a pure number and then evaluate the integral by whatever means you can. Try `qsimp` in IDL, `quadl` in MATLAB, `quad` in OCTAVE, or `quad` in the `scipy.integrate` module in PYTHON.

P6.12. Calculate the mutual inductance \( M \) of the arrangement in Fig. 5.9.

P6.13. Show that the mutual inductance \( M_{12} \) of two current loops is given by

\[
M_{12} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{dl_1 \cdot dl_2}{|\mathbf{r}_1 - \mathbf{r}_2|}
\]

and hence that \( M_{12} = M_{21} \). *Hint:* See P5.21.

P6.14. In developing Eqs. (6.32) and (6.33) from Eq. (6.29), we selected a general but still not entirely arbitrary way to build up the currents to their final values. To what extent (if any) does the final result depend on that selection? Explain your answer fully.

P6.15. A coil is made by winding \( N \) turns of wire on a frame of nonmagnetic material. The frame consists of a portion of a hollow cylinder of inner radius \( a \), outer radius \( b \), and height \( h \) (Fig. 6.8). A current \( I \) flows through the wire. Assume that the magnetic induction \( \mathbf{B} \) is confined to the region interior to this frame and that at every point in this region \( \mathbf{B} \) is directed tangent to a circle through that point, which circle has its center on the axis of the frame. (a) Use the circuit law to determine the magnitude of the magnetic induction field at all points inside the frame. (b) Determine exactly the magnetic flux crossing a surface bounded by a single turn of the windings. (c) Show that the self-inductance of the arrangement is given by

\[
L = \frac{\mu_0}{2\pi} N^2 h \ln \frac{b}{a}
\]
Note specifically the dependence on the square of $N$; doubling the number of turns quadruples the self-inductance. (d) Determine the limit of the self-inductance when $w \ll a$, where $w = b - a$ is the thickness of the frame, and note the appearance of the area $wh$ of the cross section of the frame. *Hint:* When $\epsilon \ll 1$, $\ln(1 + \epsilon) \approx \epsilon$. (e) The energy required to establish a current $I$ in this coil was calculated in P5.23 by integrating the magnetic energy density over the volume of the coil. By a direct transformation of that result, show that this energy is given by $\frac{1}{2}LI^2$, in agreement with Eq. (6.32). (f) A second winding of $N'$ turns of wire is now wound on the frame pictured in this problem. Calculate the mutual inductance $M$ between the two coils. This arrangement of two coils is a crude transformer.

### 6.2 A Contradiction and its Resolution: Displacement Current

The second of the two terms needed to extend Eqs. (6.3)–(6.6) to time-dependent fields was predicted theoretically in the 1860’s by Maxwell but was not confirmed experimentally until some 20 years later. Qualitatively, this term associates an induced magnetic induction field with a changing electric field (magnetoelectric induction), a connection that we shall explore in this section.

We shall begin by demonstrating that the basic equations in their present form [Eqs. (6.1), (6.2), (6.3), (6.5), and (6.21)] are inconsistent. Specifically Eq. (6.6), whose divergence necessarily requires $\nabla \cdot \mathbf{J} = 0$ (Why?) contradicts Eq. (6.2) unless $\partial \rho / \partial t$ happens to be zero. To set the stage for the subsequent development, however, we elect to demonstrate this contradiction also by an argument based on the integral form of the laws. Consider, then, $\oint_{\Sigma} \mathbf{J} \cdot d\mathbf{S}$ over some closed surface $\Sigma$. Let an arbitrary closed curve $\Gamma$ be inscribed on $\Sigma$ and denote the two portions into which this curve divides the surface by $\Sigma_1$ and $\Sigma_2$ (Fig. 6.9). With $d\mathbf{S}$ chosen always to be the outward normal on $\Sigma$, Eq. (6.6) now gives

$$\mu_0 \oint_{\Sigma_1} \mathbf{J} \cdot d\mathbf{S} = \oint_{\Gamma_1} \mathbf{B} \cdot d\mathbf{l} ; \quad \mu_0 \oint_{\Sigma_2} \mathbf{J} \cdot d\mathbf{S} = \oint_{\Gamma_2} \mathbf{B} \cdot d\mathbf{l}$$

(6.35)

where $\Gamma_1$ and $\Gamma_2$ are the curves bounding $\Sigma_1$ and $\Sigma_2$, respectively, and each is traversed in the proper sense to agree with the conventions implicit in Eq. (6.6). Although $\Gamma_1$ and $\Gamma_2$ pass through the same points in space, they are traversed in opposite directions. Thus $\oint_{\Gamma_1} \mathbf{B} \cdot d\mathbf{l} = -\oint_{\Gamma_2} \mathbf{B} \cdot d\mathbf{l}$ and, on adding the two parts of Eq. (6.35), we find that Eq. (6.6) yields $\oint \mathbf{J} \cdot d\mathbf{S} = 0$, contradicting this time the integral form of Eq. (6.2) unless $\partial \rho / \partial t = 0$.

A fruitful way to approach the contradiction between Eq. (6.2) and Eq. (6.6) is to examine symmetries that one might expect to be present in the equations. Basing our intuition on terms already present, we might expect a generalization of the basic equations to have the appearance

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int \rho \, d\mathbf{v}$$

(6.36)

$$\oint \mathbf{B} \cdot d\mathbf{S} = \alpha \int \left( \text{magnetic charge density} \right) d\mathbf{v}$$

(6.37)
6.2. DISPLACEMENT CURRENT

Figure 6.9: A closed surface divided into two open surfaces by the curve $\Gamma$.

\[ \oint E \cdot dl = \beta \int \left( \text{magnetic current density} \right) \cdot dS - \int \frac{\partial B}{\partial t} \cdot dS \quad (6.38) \]

\[ \oint B \cdot dl = \mu_0 \int J \cdot dS + \gamma \int \frac{\partial E}{\partial t} \cdot dS \quad (6.39) \]

where $\alpha$, $\beta$, and $\gamma$ are as yet undetermined constants. We have already confirmed the correctness of the terms not multiplied by an undetermined constant. Of the remaining three terms, two—those multiplied by $\alpha$ and $\beta$—describe effects that would be attributed to isolated (perhaps microscopic) magnetic monopoles. To date, however, no experiments have given any suggestion that isolated magnetic monopoles can be found in nature. Although it would be an interesting exercise to keep these terms and explore their consequences (perhaps even predicting an experiment that would detect magnetic monopoles), we nonetheless elect to set $\alpha = \beta = 0$, not so much because these terms are known to be incorrect as because no experiments to date require them for an adequate accounting. Should some future experiment reveal magnetic monopoles, the terms are easily reinstated.

With the proper choice of the constant $\gamma$, however, Eq. (6.39) no longer contradicts Eq. (6.2). By the same arguments invoked in the sentences following Eq. (6.35), we find from Eq. (6.39) that

\[ \oint J \cdot dS = \frac{\gamma}{\mu_0} \frac{\partial}{\partial t} \int E \cdot dS = 0 \quad (6.40) \]

Substituting from Eq. (6.36), however, we find that

\[ \oint J \cdot dS + \frac{\gamma}{\mu_0 \epsilon_0} \frac{\partial}{\partial t} \int \rho \, dv = 0 \quad (6.41) \]
which agrees with Eq. (6.2) provided we set $\gamma = \mu_0 \epsilon_0$. Thus, if we replace Eq. (6.6) with

$$\oint B \cdot dl = \mu_0 \int \left( J + \epsilon_0 \frac{\partial E}{\partial t} \right) \cdot dS$$  \hspace{1cm} (6.42)

the contradiction is resolved. In effect we have identified in $\epsilon_0 \frac{\partial E}{\partial t}$ a new current density, called the displacement current density, and we have predicted theoretically that this new current density must be added to the current density $J$ arising from macroscopic charge transport. We shall call Eq. (6.42) the generalized circuital law.

**PROBLEMS**

**P6.16.** A current $I$ flows along a wire toward one of two parallel conducting plates and away from the second (Fig. 6.10). As a result a charge $Q(t)$ accumulates on the first plate and a charge $-Q(t)$ on the second, and a time-dependent electric field is established between the plates. For this situation, the surface integral on the right in Eq. (6.42) may in particular be evaluated either over the mouth of the “sack” shown or over the sack itself. Further, the two surface integrals must have the same value. (Why?) (a) Neglect fringing and show that in fact the right-hand side of Eq. (6.42) indeed does have the same value for both surfaces. Assume that the bottom of the sack is parallel to the plates. (b) Qualitatively, what factor would preserve the identity of the two surface integrals if the sack intersected the plate and the displacement current across its bottom were therefore reduced?

**P6.17.** A point sample of uranium located at the origin radiates (charged) alpha particles uniformly in all directions, thus generating a radial current density given by $J(r,t) = \left[ I(t)/4\pi r^2 \right] \hat{r}$, where $I(t)$ is the current crossing a sphere of radius $r$ with its center at the uranium. Let $Q(t)$ be the total charge on the uranium at time $t$. (a) Use the equation of continuity to show that $I = -dQ/dt$. (b) Use symmetry and the flux law to show that this current distribution produced no magnetic induction field. (c) Assuming that the electric field follows changes in the charge distribution instantaneously, use symmetry and Gauss’s Law to find $E(r,t)$. (d) Show explicitly that Eq. (6.39) is correct for this case only if $\gamma = \mu_0 \epsilon_0$. **Hints:** (1) Apply Eq. (6.39) to a path bounding a portion of a spherical surface of radius $r$. (2) Note that the integral $\int \hat{r} \cdot dS/r^2$ need not be explicitly evaluated in order to obtain the desired conclusion. (e) Identify any approximations or tacit assumptions made in this problem.
6.3 Maxwell’s Equations

With the addition of the term involving the displacement current, we have at last completed the development of the basic equations satisfied by the (time-dependent) electromagnetic field. Although these equations, called Maxwell’s equations, have emerged from an examination of experimental evidence and thus are grounded in experiment, they have been tested in such a wide variety of situations that we tend to regard them theoretically as irrefutable postulates that require no further experimental verification. In this regard, we must be cautious, for phenomena presently unknown could well require modification of these equations in the future.

Maxwell’s equations have two useful forms. In their integral form, which has been our primary concern so far, these equations are

1. Gauss’s Law, Eq. (6.36):
   \[ \oint E \cdot dS = \frac{1}{\varepsilon_0} \int \rho \, dv \]  \hspace{1cm} (6.43)

2. The magnetic flux law, Eq. (6.37) with \( \alpha = 0 \):
   \[ \oint B \cdot dS = 0 \]  \hspace{1cm} (6.44)

3. Faraday’s Law of induction, Eq. (6.38) with \( \beta = 0 \):
   \[ \oint E \cdot dl = - \int \frac{\partial B}{\partial t} \cdot dS \]  \hspace{1cm} (6.45)

4. The generalized circuital law, Eq. (6.39) with \( \gamma = \mu_0 \varepsilon_0 \):
   \[ \oint B \cdot dl = \mu_0 \int J \cdot dS + \mu_0 \varepsilon_0 \int \frac{\partial E}{\partial t} \cdot dS \]  \hspace{1cm} (6.46)

Throughout these equations, the various surfaces and curves are arbitrary, subject only to the convention relating the direction of \( dl \) to that of \( dS \) in Eqs. (6.45) and (6.46) by the right-hand rule.

Despite the generality of Eqs. (6.43)–(6.46), they are not particularly suited to the direct calculation of fields except when the sources \( \rho \) and \( J \) are known and exhibit considerable symmetry. Further awkwardness arises because each integral involves a field, a charge density, or a current density at every point on some curve, surface, or volume. The key to a reformulation that eliminates these objections is to allow the curves, surfaces, and volumes in Eqs. (6.43)–(6.46) to become indefinitely small. The argument transforming Eqs. (6.43) and (6.44) is therefore the same as that presented in Eqs. (4.34)–(4.36) except that we now must regard time as a parameter that remains fixed throughout the argument; we find that Eq. (6.43) becomes

\[ \nabla \cdot E = \frac{1}{\varepsilon_0} \rho \]  \hspace{1cm} (Gauss’s Law)  \hspace{1cm} (6.47)
and that Eq. (6.44) becomes

\[ \nabla \cdot \mathbf{B} = 0 \quad \text{(magnetic flux law)} \quad (6.48) \]

Equations (6.45) and (6.46) are transformed to differential form by applying the relationship in Eq. (0.71) and by regarding the integrand in the surface integral to be constant over the (now small) surface \( \Delta S \). From Eq. (6.45) we find that \( \nabla \times \mathbf{E} \cdot \Delta S \approx - (\partial \mathbf{B} / \partial t) \cdot \Delta S \). Since \( \Delta S \) is arbitrary, this expression in turn implies that

\[ \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday’s Law)} \quad (6.49) \]

A similar argument applied to Eq. (6.48) yields

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(generalized circuital law)} \quad (6.50) \]

and we have completed the transformation of Eqs. (6.43)–(6.46) to their differential form. The resulting equations are summarized in Table 6.1 both in vector form, which is independent of any particular coordinate system, and in the form satisfied by the Cartesian components of the fields. In contrast to the integral form, the differential form of Maxwell’s equations relates aspects of the fields at single points in space-time to the sources at that same point in space-time. The calculational difficulties associated with integrals over extended regions of space have therefore been replaced by whatever problems are associated with solving coupled partial differential equations. In particular, suitable boundary and initial conditions must be specified to supplement the differential equations. We shall see in subsequent chapters, however, not only that the necessary boundary and initial conditions are often easy to obtain but also that partial differential equations are often easily solved by at least one of several well-developed techniques, even when the sources \( \rho \) and \( \mathbf{J} \) are not fully known until after the fields have been found.

In the remainder of this book, uses of the differential form of Maxwell’s equations will predominate over uses of the integral form. These equations admit an immense variety of solutions, some of which we shall explore in the problems and in later chapters. In the remainder of this chapter, we shall treat energy and momentum in a general electromagnetic field and examine two useful reformulations of Maxwell’s equations.

**PROBLEMS**

**P6.18.** Stokes’ theorem and the divergence theorem can be used to convert Maxwell’s equations to differential form. Stokes’ theorem, for example, applied to Eq. (6.45) gives

\[ \int \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot dS = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot dS \]

Since this equation must be correct for an arbitrary path, it must also be correct for an arbitrary surface, which can be the case only if the integrands are equal at every point in space, and we infer Eq. (6.49). Derive the rest of Maxwell’s equations by similar arguments.

**P6.19.** Show that the solutions to Maxwell’s equations satisfy the principle of superposition. *Hint:* Let the field \( \mathbf{E}_1, \mathbf{B}_1 \) satisfy the equations with sources \( \rho_1 \) and \( \mathbf{J}_1 \) and let \( \mathbf{E}_2, \mathbf{B}_2 \) satisfy the equations with sources \( \rho_2 \) and \( \mathbf{J}_2 \). Then show that the field \( \mathbf{E}_1 + \mathbf{E}_2, \mathbf{B}_1 + \mathbf{B}_2 \) satisfies the equations with sources \( \rho_1 + \rho_2 \) and \( \mathbf{J}_1 + \mathbf{J}_2 \).
6.3. **MAXWELL’S EQUATIONS**

Table 6.1: Maxwell’s Equations in mks Units

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\n\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\n\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

**P6.20.** Let \( \rho \) and \( \mathbf{J} \) be zero. Then show that Maxwell’s equations are invariant to the transformation

\[
\begin{align*}
\mathbf{E}' &= \mathbf{E} \cos \theta + \frac{1}{\mu_0} \mathbf{B} \sin \theta \\
\mathbf{B}' &= -\varepsilon_0 \mathbf{E} \sin \theta + \mathbf{B} \cos \theta
\end{align*}
\]

where \( \theta \) is an arbitrary constant angle. Note particularly the form of this transformation when \( \theta = \frac{1}{2} \pi \)—in a sense, the fields \( \mathbf{E} \) and \( \mathbf{B} \) can be interchanged! **Hint:** Let \( \mathbf{E}, \mathbf{B} \) satisfy Maxwell’s equations and show that \( \mathbf{E}', \mathbf{B}' \) do also.

**P6.21.** Starting with Maxwell’s equations, show that there exist in a region free of charges and currents no static solutions for \( \mathbf{E} \) or \( \mathbf{B} \) that depend on only one Cartesian coordinate.

**P6.22.** The electrostatic field in some region of space has everywhere the same direction, say the \( z \) direction. Show that \( \mathbf{E} \) cannot depend on either \( x \) or \( y \). If there is no charge in this region of space, show that \( \mathbf{E} \) must be a constant field.

**P6.23.** Derive the equation of continuity in differential form directly from Maxwell’s equations in differential form.

**P6.24.** Starting with the Lorentz force and Maxwell’s equations in differential form, derive Coulomb’s Law. You may, of course, derive the integral form from the differential form and then use the integral form.

**P6.25.** Starting with the differential form of Maxwell’s equations, let \( \mathbf{B} = B(t) \hat{k} \) and derive the result in **P6.9** for an associated electric field. Carefully enumerate any assumptions made. Is it possible for \( \rho \) and/or \( \mathbf{J} \) to be zero? If not, what must they be?


6.4 Energy in the Electromagnetic Field

Maxwell’s equations can be manipulated to obtain a relationship that can be interpreted as a statement of energy balance. We begin by subtracting the dot product of Eq. (6.49) with \( B \) from the dot product of Eq. (6.50) with \( E \), obtaining

\[
E \cdot (\nabla \times B) - B \cdot (\nabla \times E) = \mu_0 J \cdot E + \mu_0 \frac{\partial}{\partial t} \left( \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) \tag{6.51}
\]

We next invoke Eq. (C.12) to find the differential statement

\[
-\nabla \cdot \left( E \times \frac{B}{\mu_0} \right) = J \cdot E + \frac{\partial}{\partial t} \left( \epsilon_0 E^2 + \frac{B^2}{2\mu_0} \right) \tag{6.52}
\]

which is equivalent to the integral statement

\[
-\oint_{\Sigma} \left( E \times \frac{B}{\mu_0} \right) \cdot dS = \int_V J \cdot E \, dv + \frac{\partial}{\partial t} \int_V \left( \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) \, dv \tag{6.53}
\]

obtained by integrating Eq. (6.52) over a fixed volume \( V \) and using the divergence theorem to express the left-hand side as an integral over the surface \( \Sigma \) bounding \( V \).

We now seek a physical interpretation of the terms in Eq. (6.53). In accordance with Eq. (3.39), \( \int J \cdot E \, dv \) is the rate at which the field does work on the particles composing the current in \( V \). We have also seen [Eqs. (4.78) and (5.60)] that the integral over \( V \) in the final term of Eq. (6.53) gives the energy stored in the fields when the fields are static. Consistent with these properties, we now simply assign an energy density

\[
u_{\text{EM}} = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \tag{6.54}
\]

to the time-dependent field as well. With that assignment, Eq. (6.53) has the semi-verbal expression

\[
-\oint_{\Sigma} \left( E \times \frac{B}{\mu_0} \right) \cdot dS = \left( \text{rate at which mechanical energy of particles increases} \right) + \left( \text{rate at which energy stored in fields increases} \right) = \left( \text{rate at which total energy in } V \text{ increases} \right) \tag{6.55}
\]

Now, in the absence of explicit sources of energy within \( V \), an increase in the total energy in \( V \) can come about only if energy flows into \( V \) over the surface \( \Sigma \). We are thus led to interpret the surface integral in Eq. (6.55) as the rate at which energy flows into \( V \) from the space outside \( V \). Verbally, and without the minus sign, we thus have that

\[
\oint_{\Sigma} \left( E \times \frac{B}{\mu_0} \right) \cdot dS = \left( \text{rate at which energy flows out of } V \text{ across the bounding surface } \Sigma \right) \tag{6.56}
\]

The vector

\[
S = E \times \frac{B}{\mu_0} \tag{6.57}
\]
that appears in Eq. (6.56) is called the Poynting vector:\textsuperscript{10} its direction is the direction in which the electromagnetic fields \( \textbf{E} \) and \( \textbf{B} \) are transporting energy and its magnitude is the rate at which energy is transported across a surface of unit area oriented perpendicular to the direction of energy flow. It is often stated that the only suitable interpretation of Eq. (6.56) is that the integral of the Poynting vector over a closed surface \( \Sigma \) represents energy transported out of the volume bounded by \( \Sigma \). In this view, the interpretation of \( \mathcal{S} \) as itself representing a point-by-point energy flow is regarded as a convenient fiction leading to inconsistencies if taken too literally.\textsuperscript{11} Whether \( \mathcal{S} \) does or does not represent a point-by-point energy flow remains a matter for debate.\textsuperscript{12}

Although we cannot be sure either that the interpretations given in the previous paragraphs are the only possible interpretations or that Eqs. (6.52) and (6.53) are the only expressions that are both consistent with Maxwell’s equations and interpretable as expressions of energy balance,\textsuperscript{13} these equations and the above interpretation are pleasing in their simplicity and experimentally adequate to all tests made of them to date. We therefore accept the interpretation not so much because of any inevitability we might like it to have as because of its suitability. In accepting that interpretation, we not only have allowed for general time-dependent fields to store energy but also have attributed to these fields a capacity to transport energy through space.

**PROBLEM**

\textbf{P6.26.} A straight cylindrical metal wire of radius \( b \) carries a current \( I \) along the \( z \)-axis in response to the application of an electric field \( \textbf{E} = E \hat{k} \) to points inside the wire. Determine the direction and magnitude of the Poynting vector at the surface of the wire, integrate the normal component of the Poynting vector over a segment of the wire of length \( L \), and compare your result with the Joule heat produced in this segment. Now, ponder what it means for the energy dissipated in the wire to enter from the space outside the wire rather than to propagate along the wire. \textit{Hint:} The Joule heat in a wire is given by \( IV \), where \( V \) is the potential difference between the two ends of the wire.

### 6.5 Momentum in the Electromagnetic Field

Maxwell’s equations can also be manipulated to obtain a relationship that can be interpreted as a statement of momentum balance. Suppose that we have a system of charges and currents distributed in space and that the electromagnetic field established by this system is \( \textbf{E}, \textbf{B} \). The fields, of course, satisfy Maxwell’s equations. The force \( \textbf{F} \) on the system of charges and currents, which is also the time rate of change of the (mechanical) momentum

---

\textsuperscript{10}English physicist John Henry Poynting, b. 9 September 1852 in Monton, Lancashire, England; d. 30 March 1914 in Birmingham, England.


\textsuperscript{12}See W. H. Furry, \textit{Am. J. Phys.} 37, 621 (1969) and the references given there.

\( P \) of the system, is then given by

\[
F = \frac{dP}{dt} = \int (\rho E + J \times B) \, dv \tag{6.58}
\]

where the integral of the *force density* \( \rho E + J \times B \) extends over the volume containing the distribution, which is equivalent to extending over all space, since \( \rho \) and \( J \) are zero outside the distribution. We now use Maxwell’s equations to rewrite Eq. (6.58). First, substitute for \( \rho \) and \( J \) from Eqs. (6.47) and (6.50) to find that

\[
\frac{dP}{dt} = \int \left[ (\epsilon_0 \nabla \cdot E) E + \frac{1}{\mu_0} (\nabla \times B) \times B - \epsilon_0 \frac{\partial E}{\partial t} \times B \right] \, dv \tag{6.59}
\]

Next, we seek an expression in which the time derivative under the integral can be removed from the integral. We write

\[
\int \left( \frac{\partial E}{\partial t} \times B \right) \, dv = \int \frac{\partial}{\partial t} (E \times B) \, dv - \int \left( E \times \frac{\partial B}{\partial t} \right) \, dv
\]

\[
= \frac{d}{dt} \int E \times B \, dv + \int [E \times (\nabla \times E)] \, dv \tag{6.60}
\]

where the second form follows partly on substitution from Eq. (6.49) and partly because the integrals extend over fixed volumes so that \( \partial/\partial t \) can be taken outside the integral and then written as a total derivative. Then we substitute Eq. (6.60) into Eq. (6.59) and rearrange the terms to obtain that

\[
\frac{d}{dt} \left( P + \int \epsilon_0 E \times B \, dv \right) = \int \left[ \epsilon_0 (\nabla \cdot E) E - \epsilon_0 E \times (\nabla \times E) - \frac{1}{\mu_0} B \times (\nabla \times B) \right] \, dv \tag{6.61}
\]

Finally, because the integral under the time derivative appears on the same footing as the mechanical momentum \( P \), it is appropriate to make the identification

\[
\left( \text{momentum in the field } E, B \right) = \int \epsilon_0 E \times B \, dv \tag{6.62}
\]

and to interpret the quantity

\[
G' = \epsilon_0 E \times B \tag{6.63}
\]

as a momentum density associated with the electromagnetic field. From Eq. (6.57) it is apparent that \( G' \) and \( S \) are here related by

\[
G' = \epsilon_0 \mu_0 S \tag{6.64}
\]

but this relationship applies *only* to fields in vacuum. When matter is introduced (Chapter 12), Maxwell’s equations assume a slightly different form and a significantly different degree of difficulty. The momentum density and the Poynting vector are then not even determined from the same pair of fields.

### 6.6 A Reformulation: Maxwell’s Equations for the Potentials

The existence of scalar and vector potentials for the general time-dependent field and also equations satisfied by these potentials can be inferred *directly* from Maxwell’s equations...
without reference to previous developments. We first note that the flux law, Eq. (6.48), implies the existence of a vector potential \( A \) in terms of which

\[
B = \nabla \times A
\]  

(6.65)

[See item (4) in Section 2.5] Since mixed second partial derivatives can be evaluated in either order, Faraday’s Law, Eq. (6.49), then becomes

\[
\nabla \times \left( E + \frac{\partial A}{\partial t} \right) = 0
\]  

(6.66)

which implies the existence of a scalar potential \( V \), in terms of which

\[
E + \frac{\partial A}{\partial t} = -\nabla V \quad \Rightarrow \quad E = -\nabla V - \frac{\partial A}{\partial t}
\]  

(6.67)

[See item (3) in Section 2.5.] Note that Eqs. (6.65) and (6.67) reduce correctly to their static counterparts, Eqs. (5.44) and (4.45), when the potentials are static. The most interesting aspect of the time-dependent expressions is the appearance of \( A \) in the equation giving \( E \).

The phenomenon of gauge invariance carries over from static to time-dependent potentials. Certainly, \( B \) as given by Eq. (6.65) is unchanged if an arbitrary gradient is added to \( A \); i.e., \( B \) is invariant to the transformation

\[
A_2 = A_1 + \nabla \Lambda
\]  

(6.68)

where \( \Lambda \) now may have both spatial and temporal dependence. The electric field \( E \) as given by Eq. (6.67) will be invariant to this change in \( A \), however, only if the transformation in \( A \) is accompanied by a transformation in \( V \) chosen so that

\[
-\nabla V_1 - \frac{\partial A_1}{\partial t} = -\nabla V_2 - \frac{\partial A_2}{\partial t}
\]  

(6.69)

Substituting for \( A_2 \) from Eq. (6.68), we find that

\[
-\nabla V_1 = -\nabla V_2 - \nabla \frac{\partial \Lambda}{\partial t} \quad \Rightarrow \quad V_2 = V_1 - \frac{\partial \Lambda}{\partial t}
\]  

(6.70)

except for a possible arbitrary constant. Thus, the time-dependent electromagnetic field \( E \), \( B \), is unchanged when the vector and scalar potentials are simultaneously transformed by Eqs. (6.68) and (6.70), where \( \Lambda \) is arbitrary, and this transformation is therefore a natural generalization of the gauge transformation introduced in Section 5.7. Here as there, one consequence of the gauge invariance of the fields is the arbitrariness of \( \nabla \cdot A \), which can be given any value that we find convenient (P6.27).

Two of Maxwell’s equations—the homogeneous pair, Eqs. (6.48) and (6.49)—are automatically satisfied when the fields are expressed in terms of the potentials. (Why?) The remaining two—the inhomogeneous pair, Eqs. (6.47) and (6.50)—generate differential equations whose solutions determine the potentials. Combined with Eq. (6.67), for example, Eq. (6.47) yields

\[
-\nabla^2 V - \frac{\partial}{\partial t} \nabla \cdot A = \frac{\rho}{\epsilon_0}
\]  

(6.71)
where the Laplacian, $\nabla^2 = \nabla \cdot \nabla$, is introduced initially in Section 2.5. Similarly, substituting Eqs (6.65) and (6.67) into Eq. (6.50) and using Eq. (C.19) to reexpress $\nabla \times (\nabla \times \mathbf{A})$, we find after rearranging terms that

$$
\left( \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \mathbf{J} + \nabla \left( \nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \right)
$$

(6.72)

Now, we can simplify Eq. (6.72) by imposing the so-called Lorentz condition

$$
\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0
$$

(6.73)
on the potentials, thereby stipulating the value of $\nabla \cdot \mathbf{A}$ and selecting what is called a Lorentz gauge for the potentials. In this gauge, Eq. (6.72) becomes

$$
\left( \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \mathbf{J}
$$

(6.74)

Further, on substituting $\nabla \cdot \mathbf{A}$ from Eq. (6.73), we find from Eq. (6.71) that

$$
\left( \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) V = -\frac{\rho}{\varepsilon_0}
$$

(6.75)

The similarity of these two equations is part of the reason for imposing the Lorentz condition. In Lorentz gauge, the time-dependent potentials both satisfy the same equation, an equation called the inhomogeneous wave equation, which we shall examine more fully in Chapter 14. When $\mathbf{A}$ and $V$ are time-independent, Eqs. (6.74) and (6.75) reduce to Poisson’s equation,$^{14}$

$$
\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \nabla^2 V = -\frac{\rho}{\varepsilon_0}
$$

(6.76)

or, when $\mathbf{J}$ and $\rho$ are zero, to Laplace’s equation,$^{15}$

$$
\nabla^2 \mathbf{A} = 0 \quad \nabla^2 V = 0
$$

(6.77)

both of which we shall study more fully in Chapter 8.

**PROBLEMS**

**P6.27.** Show that the arbitrariness of the gauge function relating two equivalent magnetic vector potentials means that the divergence of the vector potential is also arbitrary. *Hint:* See Section 5.7.

**P6.28.** Let $\mathbf{A}_1$ and $V_1$ be potentials in Lorentz gauge. Further, let the potentials $\mathbf{A}_2$ and $V_2$ be obtained from $\mathbf{A}_1$ and $V_1$ by a gauge transformation. Show that $\mathbf{A}_2$ and $V_2$ are also potentials in Lorentz gauge provided only that the gauge function satisfies the homogeneous wave equation. Comment on the uniqueness of the Lorentz gauge potentials.

**P6.29.** (a) Using the Poisson equation in spherical coordinates, find the electrostatic potential established by the charge distribution $\rho(r) = \rho_0$, $r < a$; $\rho(r) = 0$, $r > a$. (b) Find the electric field. *Hints:* (1) Symmetry rules out a dependence on the coordinates $\theta$ and $\phi$. (Why?) (2) Solve the problem in the two domains $r < a$ and $r > a$. The potential and its first derivative must be continuous everywhere, in particular at $r = a$, and the potential must be finite everywhere, in particular at $r = 0$. (Why?) (3) For definiteness, set the arbitrary constant so that $V(\infty) = 0$.

---

$^{14}$See footnote 14 in Chapter 4, page 110.

$^{15}$See footnote 15 in Chapter 4, page 110.
**P6.30.** An infinite conducting plate in the plane \( z = 0 \) is maintained at potential zero and a similar plate in the plane \( z = d \) is maintained at potential \( V_0 \). There is no charge between the plates. (a) Use symmetry to show that the potential can depend only on \( z \). (b) Solve the relevant form of Laplace’s equation and impose the conditions at \( z = 0 \) and \( z = d \) to show that \( V = V_0 z/d \). (c) Find the electric field in the region between the plates. (d) Find the charge density on each plate. *Hint:* See Eq. (4.33).

**P6.31.** Following the pattern illustrated more fully in P6.30, use Laplace’s equation and suitable boundary conditions to find the potential, field, and all relevant charge densities in the region between two infinitely long coaxial conducting cylinders, the inner of radius \( a \) being at potential \( V_a \) and the outer of radius \( b \) being at potential \( V_b \).

**P6.32.** Following the pattern illustrated more fully in P6.30, use Laplace’s equation and suitable boundary conditions to find the potential, field, and all relevant charge densities in the region between two concentric spherical conducting shells, the inner shell of radius \( a \) being at potential \( V_a \) and the outer of radius \( b \) being at potential \( V_b \). Express the field in terms of the charge on the inner sphere and comment.

### 6.7 Another Reformulation: Decoupling the Equations for the Fields

Maxwell’s equations as summarized in Section 6.3 are first-order equations and the equations for \( E \) and for \( B \) are coupled when the fields are time-dependent. At the expense of generating second-order equations, however, we can find apparently uncoupled equations for \( E \) and \( B \). For example, we can evaluate the curl of Eq. (6.49) and use Eq. (C.19) to expand \( \nabla \times (\nabla \times E) \), finding that

\[
\nabla \left( \nabla \cdot E \right) - \nabla^2 E = -\frac{\partial}{\partial t} (\nabla \times B) \tag{6.78}\n\]

We next substitute from Eqs (6.47) and (6.50) and rearrange the terms to obtain the equation

\[
\left( \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) E = \frac{1}{\varepsilon_0} \nabla \rho + \mu_0 \frac{\partial J}{\partial t} \tag{6.79}\n\]

A similar argument, beginning with the curl of Eq. (6.50), yields

\[
\left( \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) B = -\mu_0 \nabla \times J \tag{6.80}\n\]

Again, the inhomogeneous wave equation has arisen, this time for the fields. Neither of these equations is particularly simple except when \( \rho \) and \( J \) are zero. Further, the act of differentiating Maxwell’s equations in general introduces extraneous solutions—just as squaring an algebraic equation introduces extraneous solutions—and any solution obtained for Eqs. (6.79) and (6.80) must always be verified by substitution into the original first-order Maxwell equations. In that substitution, additional constraints relating constants and sometimes functions in the solution for \( E \) to similar entities in the solution for \( B \) will usually emerge. Thus the apparent separation of \( E \) from \( B \) in Eqs. (6.79) and (6.80) is illusory, as it must be since \( E \) and \( B \) are inextricably intertwined when they depend on time. Even so, we shall find that Eqs. (6.79) and (6.80) often can be exploited to advantage in solving some problems.
PROBLEM

P6.33. Evaluate the curl of Eq. (6.50) and derive Eq. (6.80).

SUPPLEMENTARY PROBLEMS

P6.34. In a sense, Maxwell’s equations are not entirely independent of one another. Show from Faraday’s Law that \( \partial (\nabla \cdot B) / \partial t = 0 \) and hence show that, if \( \nabla \cdot B \) ever was zero at some time in the past, it must still be zero. Can you find any other interconnections of this sort?

P6.35. Imagine that someone finds magnetic monopoles. Maxwell’s equations would then have the form

\[
\nabla \cdot E = \frac{1}{\epsilon_0} \rho \quad \nabla \times E = -\frac{\partial B}{\partial t} + \beta J_m \\
\nabla \cdot B = \mu_0 \rho_m \quad \nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} + \mu_0 J
\]

where \( \rho_m \) is the density of magnetic poles, \( J_m \) is the magnetic pole current density, \( \mu_0 \) is—as previously—the permeability of free space which, in its position here in one of these equations, simply defines the units in which \( \rho_m \) are measured, and \( \beta \) is an as yet undetermined constant. (a) One might expect that \( \rho_m \) and \( J_m \) should satisfy an equation of continuity of the form

\[
\nabla \cdot J_m + \frac{\partial \rho_m}{\partial t} = 0
\]

Show that such an equation follows from the embellished versions of Maxwell’s equations provided \( \beta = -\mu_0 \), and note that it is entirely separate from the equation involving \( \rho \) and \( J \). (b) Find an equation analogous to the energy equation, Eq. (6.52), and interpret its terms physically. (c) By substituting the assumed expressions

\[
E = -\nabla V - \nabla \times A_m - \frac{\partial A}{\partial t} \\
B = -\nabla V_m + \nabla \times A - \mu_0 \epsilon_0 \frac{\partial A_m}{\partial t}
\]

for the fields into the above new Maxwell equations (with \( \beta = -\mu_0 \)), determine appropriate translations of the Lorentz condition to the present case and find the equations satisfied by the four potentials \( V, A, V_m \) and \( A_m \).

P6.36. The uniform but time-dependent field \( B = B(t) \hat{k} \) is invariant to arbitrary translation and hence does not define any origin. Thus, we might equally well have centered the loop in P6.9 at \((x_0, y_0)\) rather than at the origin, and we would then have obtained

\[
E(r) = \frac{1}{2} \frac{dB}{dt} \left[ (y - y_0) \hat{i} - (x - x_0) \hat{j} \right] = \frac{1}{2} \frac{dB}{dt} (r - r_0) \times \hat{k}
\]

Show explicitly that \( \oint_\Gamma E \cdot dl = -\partial \Phi_m / \partial t \), where \( \Gamma \) is any path in the \( xy \)-plane—not necessarily a circle centered at \((x_0, y_0)\). In particular, since \( \Gamma \) may be a circle centered at the origin, this problem shows that Faraday’s Law can be satisfied for that path without having \( E \) everywhere tangent to the path. Optional: Use analog or digital methods (Section 3.2) to explore the trajectories of particles moving in the electromagnetic field of this problem. [See K. Shen, E. D. Alton, and H. C. S. Hsuan, Am. J. Phys., 38, 1133 (1970), and D. M. Cook, Am. J. Phys., 40, 210 (1972).]

P6.37. (a) Show that

\[
[(\nabla \cdot Q)Q - Q \times (\nabla \times Q)]_i = \sum_j \frac{\partial R_{ij}}{\partial x_j}
\]
6.7. DECOUPLING THE EQUATIONS FOR THE FIELDS

where

\[ R_{ij} = Q_i Q_j - \frac{1}{2} Q^2 \delta_{ij} \]

and \( \delta_{ij} = 1 \) if \( i = j \) and is zero otherwise. 

**Hint:** The notation of P0.33 may be useful.

(b) Show that the \( i \)-th component of the right-hand side of Eq. (6.61) can be expressed as a surface integral

\[ \sum_j \oint T_{ij} dS_j \]

where \( dS_j \) is a component of the vector \( dS \), provided \( T_{ij} \) is suitably identified. \( T_{ij} \) is known as the Maxwell stress tensor, the terminology surviving from the days when the electromagnetic field was viewed as a distortion or stress in an elastic, mechanical ether.

**P6.38.** Show that in a region free of charges and currents Maxwell’s equations are all satisfied if the fields are derived from a single potential \( Q \) by

\[ E = -\nabla \times \frac{\partial Q}{\partial t} \quad \quad B = \nabla \times (\nabla \times Q) \]

and \( Q \) satisfies

\[ \left( \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) Q = 0 \]

**P6.39.** Let the plate at \( z = 0 \) in P6.30 be the cathode of a simple vacuum diode and the plate at \( z = d \) be the anode. The cathode emits electrons into the region between the plates and the Poisson equation \( \frac{d^2 V}{dz^2} = -\rho(z)/\varepsilon_0 \), where \( \rho(z) \) is the charge density at the coordinate \( z \) between the plates, replaces the Laplace equation. We still require \( V(0) = 0 \) and \( V(d) = V_0 \), but the basic equation cannot be solved until more is known about \( \rho(z) \). Assume that the electrons start from rest at the cathode. (a) Show that \( \rho(z) = J(z)/v(z) = J(z) \sqrt{m/2eV(z)} \), where \( J(z) \) and \( v(z) \) are the current density and electron velocity at the coordinate \( z \) between the plates and \( -e \) and \( m \) are the charge and mass of the electron. (b) Assuming that charge is conserved and that a steady state has been reached between the plates, argue that \( J(z) \) in fact cannot depend on \( z \) and hence obtain an equation for \( V(z) \) that involves only one unknown function. (c) Show that for steady current flow, \( dV/dz = 0 \) at \( z = 0 \). (d) Solve the equation obtained in part (b) subject to all boundary conditions and derive Child’s Law that the current in this simple diode is proportional to the three-halves power of the anode to cathode potential. (e) Comment on the physical limitations of Child’s Law\(^{16}\) and sketch a graph of the expected current-voltage characteristics of this simple diode, taking into account whatever limitations you have pointed out. Can you infer the meaning of the phrase space-charge-limited from knowledge that the phrase is used to describe the current in the region to which Child’s Law applies? You may wish to try an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica with the differential equations that arise in this problem. **Note:** The derivation outlined in this problem is presented in some detail in Appendix 8 of K. R. Spangenberg, *Fundamentals of Electron Devices* (McGraw-Hill Book Company, New York, 1957). A treatment of the analogous problem in cylindrical coordinates is given by Ll. G. Chambers, *Am. J. Phys.*, **36**, 911 (1968).

---

\(^{16}\) American physicist and educator Clement D. Child, b. 1868 in Madison, Ohio; d. 15 July 1933 in Rochester, New York.
Interlude: A Change of View

Although we must yet consider numerous applications and at least one important generalization, the essential theory of the electromagnetic field is now complete. At this point in this book a marked change in point of view occurs. Initially the electric and magnetic induction fields were introduced in Chapter 3 to separate the interaction between a source distribution and a test distribution into two parts, and our initial motivation for examining the fields lay in a desire to treat the general interaction between two arbitrary distributions. Within this limited context, the fields were merely a useful contrivance invented specifically to simplify the study of that interaction. As the properties of the fields were further articulated in Chapters 4–6, however, the fields gradually acquired a broader significance. We assigned an energy content and attributed to the fields a capacity to transport this energy through space, and we assigned a momentum content and attributed to the fields a capacity to transport this momentum through space. With each additional attribute, the fields became more and more real in themselves and less and less a mere contrivance introduced for the sake of a limited initial concern. In the rest of this book, our interest remains focused on examining the properties of the fields in various circumstances and on determining the fields from their sources, but the motivation underlying that focus has changed from a desire to study the fields so that we can determine forces of interaction to a desire to study the fields because they have become real physical entities in their own right. Before proceeding to the remainder of this book, the reader may find it valuable to review the evolution of this change in viewpoint by studying again the upper portion of the flow chart following the Preface and by working the following problem.

PROBLEM

Starting with Coulomb’s Law and the analogous expression for the force per unit length on one of two parallel, current-carrying wires, repeat the essentials of the development in Chapters 3–6 to obtain the differential form of Maxwell’s equations in Gaussian units, namely,

\[
\nabla \cdot \mathbf{E} = 4\pi \rho \\
\n\nabla \cdot \mathbf{B} = 0 \\
\n\n\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \\
\n\n\n\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
\]

It is suggested that the reader hold Chapters 1–6 between her thumb and index finger and note the compactness of that theory.
Along the way, obtain also expressions in Gaussian units for (a) the electrostatic field established by a given charge distribution, (b) the electrostatic potential established by a given charge distribution, (c) the law of Biot-Savart, and (d) the magnetic vector potential established by a given steady current distribution. Then, arguing from Maxwell’s equations, find expressions in Gaussian units for (e) the energy density in an electromagnetic field, (f) the Poynting vector, (g) the momentum density in an electromagnetic field, (h) Poisson’s equation for the static scalar potential and the analogous equation for the static vector potential, and (i) the wave equation for the time-dependent electric field in vacuum. In writing a solution to this exercise, stress particularly the logical development. Make it clear where definitions are made, where reference is made to experimental results, where mathematical identities are used, etc. Describe each calculation briefly, but abbreviate the presentation of mathematical details as much as is consistent with clarity and use theorems such as the divergence theorem without proof. Note: The reader who is particularly interested in electromagnetic units is referred to several articles by R. T. Birge [Am. J. Phys., 2 41 (1934); 3, 102 (1935); and 3, 171 (1935)] and to the Appendix on Units and Dimensions in J. D. Jackson, Classical Electrodynamics (John Wiley & Sons, Inc., New York, 1999), Third Edition, Appendix on Units and Dimensions.
Chapter 7

Plane Electromagnetic Waves in Vacuum

In a region free of charges and currents ($\rho = 0$ and $J = 0$), Maxwell’s equations reduce to

\begin{align*}
\nabla \cdot \mathbf{E} &= 0 \quad (7.1) \\
\n\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad (7.2) \\
\n\nabla \cdot \mathbf{B} &= 0 \quad (7.3) \\
\n\n\nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (7.4)
\end{align*}

The solutions to these equations can be at least partially classified by examining their dependence on the three Cartesian coordinates $(x, y, z)$ and on the time $t$. Constant fields, which depend on none of these variables, obviously satisfy Eqs. (7.1)–(7.4), for every term contains a derivative. There are no solutions depending only on one of the four variables $(x, y, z, t)$. If, for example, we assume a solution in which $\mathbf{E}$ and $\mathbf{B}$ both depend only on $t$, we cannot satisfy either Eq. (7.2) or Eq. (7.4), for the left-hand side would be zero and the right-hand side would be nonzero. That no solutions depending only on $x$ or on $y$ or on $z$ can exist was proved in P6.21. Solutions depending on two or more of the four variables $(x, y, z, t)$ are less trivial. We postpone a detailed examination of some of these solutions to Chapters 8 and 14, confining our attention in this chapter to solutions that depend on $t$ and essentially on one Cartesian coordinate, which we initially take to be $z$. Further, we shall examine only some of the simpler solutions, for Eqs. (7.1)–(7.4) are linear and homogeneous and we can therefore superpose any number of simpler solutions to generate more complicated solutions.
7.1 Elementary Fields Depending on z and t; Plane Electromagnetic Waves

The essential features of solutions depending on (r, t) can be illustrated by considering solutions depending on (z, t). Suppose then that

\[ \mathbf{E} = E_x(z, t) \hat{i} + E_y(z, t) \hat{j} + E_z(z, t) \hat{k} \]
\[ \mathbf{B} = B_x(z, t) \hat{i} + B_y(z, t) \hat{j} + B_z(z, t) \hat{k} \]

where hereafter we shall leave the arguments (z, t) understood. For this electromagnetic field, Eqs. (7.1)–(7.4) reduce to

\[ \frac{\partial E_z}{\partial z} = 0 ; \quad \frac{\partial E_z}{\partial t} = 0 \quad \Rightarrow \quad E_z = \text{constant} \] (7.6)

\[ \frac{\partial B_z}{\partial z} = 0 ; \quad \frac{\partial B_z}{\partial t} = 0 \quad \Rightarrow \quad B_z = \text{constant} \] (7.7)

\[ \frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t} ; \quad \frac{\partial B_x}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \] (7.8)

\[ \frac{\partial E_x}{\partial z} = - \frac{\partial B_y}{\partial t} ; \quad \frac{\partial B_y}{\partial z} = - \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \] (7.9)

(See Table 6.1.) Since by superposition we can readily add a constant field to our solution at any time, we can take \( E_z \) and \( B_z \) to be zero without serious loss.

Equations (7.8) and (7.9) are more interesting than Eqs. (7.6) and (7.7), although even here there is a separation: Equation (7.8) involves \( E_y \) and \( B_x \) while Eq. (7.9) involves \( E_x \) and \( B_y \). Apart from an arbitrary constant field, the most general solution in the form of Eq. (7.5) therefore consists of two parts, the first a solution of Eq. (7.8) having only \( E_y \) and \( B_x \) nonzero and the second a solution of Eq. (7.9) having only \( E_x \) and \( B_y \) nonzero. We shall consider in detail only the second part, for which

\[ \mathbf{E} = E_x(z, t) \hat{i} \quad \mathbf{B} = B_y(z, t) \hat{j} \] (7.10)

and \( E_x \) and \( B_y \) satisfy Eq. (7.9). If the first member of Eq. (7.9) is differentiated with respect to \( z \) and the second with respect to \( t \) and then the equality of the two mixed second partial derivatives is recognized, we find that

\[ \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \] (7.11)

By a similar argument, we find also that

\[ \frac{\partial^2 B_y}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_y}{\partial t^2} \] (7.12)

Thus, the two components \( E_x \) and \( B_y \) both satisfy the one-dimensional wave equation. Since our objective is not to find the most general solution but merely to find a solution, let us try the function

\[ E_x(z, t) = E_{x0} \cos(\kappa z - \omega t + \phi) \] (7.13)
where the amplitude \( E_{x0} \) (in V/m), the wave number \( \kappa \) (in m\(^{-1}\)), the angular frequency \( \omega \) (in s\(^{-1}\)), and the phase \( \phi \) are constants to be determined. By direct substitution, we find that this function satisfies Eq. (7.11) only if \( \kappa \) and \( \omega \) are related by
\[
\kappa = \omega \sqrt{\mu_0 \varepsilon_0} \quad (7.14)
\]
but that \( E_{x0}, \phi, \) and either \( \kappa \) or \( \omega \) can be chosen arbitrarily. The function \( B_y(z, t) \) that must accompany Eq. (7.13) is now determined by substituting Eq. (7.13) into the first member of Eq. (7.9); we find that
\[
\frac{\partial B_y}{\partial t} = \kappa E_{x0} \sin(\kappa z - \omega t + \phi) \quad (7.15)
\]
which, on integration with respect to \( t \), gives
\[
B_y(z, t) = \frac{\kappa E_{x0}}{\omega} \cos(\kappa z - \omega t + \phi) + g(z) \quad (7.16)
\]
where \( g(z) \) is an arbitrary function of \( z \) (but cannot depend on \( t \)). Requiring that Eqs. (7.16) and (7.13) satisfy the second member of Eq. (7.9) gives \( dg/dz = 0 \), whence \( g(z) \) must in fact be constant. Since we can always add a constant solution by superposition, we take \( g = 0 \). One solution of Eq. (7.9) therefore is
\[
E = E_{x0} \cos(\kappa z - \omega t + \phi) \hat{i} \\
B = -\frac{\kappa}{\omega} E_{x0} \cos(\kappa z - \omega t + \phi) \hat{j} = \frac{1}{\omega} \kappa \times E \quad (7.17)
\]
where \( \kappa = \kappa \hat{k} \) and is known as the propagation vector. The two parts of Eq. (7.17) are inseparable: If the \( E \)-field is present, the \( B \)-field must be present, and vice versa. Note, in particular, that a \( B \)-field in the \( y \)-direction is associated with an \( E \)-field in the \( x \)-direction.

More general solutions to the original equations can be obtained by superposing Eq. (7.17) with other solutions of the same form but having different values for \( E_{x0}, \phi, \) and \( \kappa \) or by superposing Eq. (7.17) with solutions to Eq. (7.8), which are found in P7.2 to have the form
\[
E = E_{y0} \cos(Kz - \Omega t + \Phi) \hat{j} \\
B = -\frac{K}{\Omega} E_{y0} \cos(Kz - \Omega t + \Phi) \hat{i} = \frac{1}{\Omega} K \times E \quad (7.18)
\]
where \( K = K \hat{k} \) and \( K = \Omega \sqrt{\mu_0 \varepsilon_0} \). In Eq. (7.18), \( E_{y0}, \Phi, \) and \( K \) are arbitrary and independent of \( E_{x0}, \phi, \) and \( \kappa \) in Eq. (7.17). We shall treat these superpositions more fully in Sections 7.3 and 7.4.

We next examine the general properties of the solution expressed in Eq. (7.17). Graphs of \( E_x \) versus \( z \) for the three times \( t = 0, t = \pi/2\omega, \) and \( t = \pi/\omega \) are shown in Fig. 7.1. Apparently, \( E_x \) can be described as a sinusoidal wave propagating in the positive \( z \)-direction, a suggestion that is confirmed by the appearance of the combination \( z - (\omega/\kappa) t \) when \( E_x \) is written in the form
\[
E_x(z, t) = E_{x0} \cos \left[ \kappa \left( z - \frac{\omega}{\kappa} t \right) + \phi \right] \quad (7.19)
\]
From this form, we conclude that the wave propagates with the (constant) speed
\[
c = \frac{\omega}{\kappa} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad (7.20)
\]
Figure 7.1: $E_x(z, t)$ versus $z$ for several values of $t$. Here, (a) corresponds to $z = -\phi/\kappa$, (b) to $z = (\pi - \phi)/\kappa$, (c) to $z = (2\pi - \phi)/\kappa$, (d) to $t = 0$, (e) to $t = \pi/2\omega$, and (f) to $z = \pi/\omega$.

[see Eq. (7.14) and compare also P1.8], which—with sufficient knowledge of the properties of the wave equation—we could have inferred directly from the coefficient $\mu_0 \epsilon_0$ in Eqs. (7.11) and (7.12). Further, because of the periodicity of the cosine function, the value of $\cos(\cdot \cdot \cdot)$ in Eq. (7.19) is not changed if $\kappa z$ is changed by $2\pi$ or (equivalently) if $z$ is changed by

$$\lambda = \frac{2\pi}{\kappa} \quad (7.21)$$

Thus, $E_x(z, t)$ is the same at $z + \lambda$ as it is at $z$, where $z$ is arbitrary. The parameter $\lambda$ is called the wavelength of the wave and it measures the spatial separation between corresponding points on successive cycles of the wave at a given time; the mks unit of wavelength is the meter. Similarly (P7.3), the period $T$ (in seconds), which is the reciprocal of the (circular) frequency $\nu$ (in cycles per second, called a hertz, Hz), is given by

$$T = \frac{1}{\nu} = \frac{2\pi}{\omega} \quad (7.22)$$

and measures the temporal separation between successive occurrences of (say) a maximum of the field at a fixed point in space. The magnetic induction field $B$ associated with the electric field $E$ in Eq. (7.17) has the same analytic form as $E$ but is directed along the $y$-axis. In combination, the two fields constitute an electromagnetic wave, and the fields in this wave are shown at a particular instant of time in Fig. 7.2.

---

1Heinrich Hertz, see footnote 1 in Chapter 12, page 319.
2Note that it is the fields themselves that are “waving” as the wave propagates.
In brief, Maxwell’s equations predict the existence of electromagnetic waves in empty space \((\rho = 0, \mathbf{J} = 0)\). The speed of propagation of these waves is independent of frequency and, as given by Eq. (7.20), has the numerical value

\[
c = \left( \left( 8.85418 \pm 0.00002 \times 10^{-12} \frac{\text{C}^2}{\text{N} \cdot \text{m}^2} \right) \left[ 4\pi \times 10^{-7} \frac{\text{N}}{\text{A}^2} \right] \right)^{-1/2}
\]

\[
= (2.99793 \pm 0.00001) \times 10^8 \text{ m/s}
\]

which, within experimental uncertainty, is the speed of light! (See Table 1.1.)\(^3\) This unexpected occurrence cannot be explained as a pure coincidence; rather we conclude that light is—or at least (sometimes) behaves like—an electromagnetic wave. This prediction, which united two previously distinct areas of physics (optics and electromagnetism), was first made by Maxwell in 1861. It was not until the 1880’s that electromagnetic waves at frequencies outside the visible spectrum were first detected (by Hertz).

All of the solutions obtained in this section are referred to as plane waves because at every instant of time the surfaces over which the electric field has the same value are planes, in this case, perpendicular to the \(z\)-axis. (The magnetic induction field is, of course, also constant over these same planes.) The solutions discussed in this section are also said to be monochromatic because they are characterized by a definite wave number, which means in turn that they are characterized by a definite wavelength and hence by a single, well-defined color, where the term color may refer to wavelengths outside the visible portion of

---

\(^3\)Remember that, in consequence of changes adopted in 1983, eight years after the first edition of this book was published, the speed of light is now a defined number. Since \(\mu_0\) is also a defined number, \(\epsilon_0\) has also become a defined number. See Section 1.7.
Table 7.1: Names in Common Use for Regions of the Electromagnetic Spectrum

<table>
<thead>
<tr>
<th>Wavelength, $\lambda$ (m)</th>
<th>Name</th>
<th>Frequency, $\nu$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; \approx 10^{-10}$</td>
<td>Gamma rays</td>
<td>$&gt; \approx 3 \times 10^{18}$</td>
</tr>
<tr>
<td>$&lt; \approx 10^{-8}$</td>
<td>X-rays</td>
<td>$&gt; \approx 3 \times 10^{16}$</td>
</tr>
<tr>
<td>$4 \times 10^{-7} \approx 10^{-9}$</td>
<td>Ultraviolet</td>
<td>$7 \times 10^{14} \approx 3 \times 10^{17}$</td>
</tr>
<tr>
<td>$7 \times 10^{-7} \approx 10^{-7}$</td>
<td>Visible</td>
<td>$4 \times 10^{14} \approx 7 \times 10^{14}$</td>
</tr>
<tr>
<td>$\approx 10^{-4} \approx 7 \times 10^{-7}$</td>
<td>Infrared</td>
<td>$\approx 3 \times 10^{12} \approx 4 \times 10^{14}$</td>
</tr>
<tr>
<td>$\approx 10^0 \approx 10^{-4}$</td>
<td>Microwaves</td>
<td>$\approx 3 \times 10^8 \approx 3 \times 10^{12}$</td>
</tr>
<tr>
<td>$\approx 10^0$</td>
<td>Television</td>
<td>$\approx 3 \times 10^8$</td>
</tr>
<tr>
<td>$\approx 10^2$</td>
<td>Radio</td>
<td>$\approx 3 \times 10^6$</td>
</tr>
</tbody>
</table>

the electromagnetic spectrum. Names commonly applied to various regions of this spectrum are summarized in Table 7.1.

PROBLEMS

P7.1. Introduce the variable transformations $\xi = z + at$, $\eta = z - at$ into the scalar wave equation

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

and from the result show that $u(z, t) = f(z - at) + g(z + at)$, where $f$ and $g$ are arbitrary functions. Describe qualitatively the essential features of each term in this general solution.

P7.2. Obtain the fields in Eq. (7.18) by applying the arguments in Section 7.1 to Eq. (7.8).

P7.3. Present an argument based on the periodicity of the cosine function to derive Eq. (7.22) relating $T$ and $\omega$.

P7.4. Show that the relationship $\omega = \kappa c$ is equivalent to the more familiar statement $\lambda \nu = c$, where $\lambda$ is the wavelength and $\nu$ the frequency (in Hz) of the wave. Numerically, what is $\nu$ for visible light, $\lambda \approx 5000$ Å? · · · for microwaves, $\lambda \approx 10$ cm?

P7.5. Use Maxwell’s equations to determine the magnetic induction associated with the electric field $E = E_0 [i \cos(\kappa y - \omega t) + k \sin(\kappa y - \omega t)]$ where $E_0$ is a constant.

P7.6. Use Maxwell’s equations to find the magnetic induction field associated with an electric field of the form $E = f(z - ct) i$.

### 7.2 Energy and Momentum in Plane Waves

According to the interpretations in Sections 6.4 and 6.5, the electromagnetic field given by Eq. (7.17) carries both energy and momentum. We shall consider the energy first. For the
field in Eq. (7.17), the Poynting vector, which gives the rate at which energy is transported by the fields, is given by

$$ S = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{\kappa E_0^2}{\mu_0 \omega} \cos^2(\kappa z - \omega t + \phi) \hat{k} $$

(7.24)

[See Eq. (6.57).] As expected, $S$ is parallel to the direction of propagation of the wave. In addition, $S$ fluctuates both in space and in time. Since the period of microwaves and of waves of shorter wavelength (see Table 7.1) is macroscopically small ($< \approx 10^{-8} \text{ s}$), at least in these regions of the electromagnetic spectrum typical macroscopic measurements, which extend over many, many cycles of the wave, are sensitive not to the rapid fluctuations in the rate of energy transport but to the average rate of energy transport over many cycles. Although a macroscopic time interval may not contain exactly an integral number of cycles, a fraction of a cycle added to a large number of complete cycles is insignificant, and we can average over an integral number of cycles. Because of the periodicity of Eq. (7.24), however, that average is equivalent to the average over a single cycle. Thus, we find at a fixed point in space (fixed $z$) that

$$ \langle S \rangle = \frac{\kappa E_0^2}{\mu_0 \omega} \langle \cos^2(\kappa z - \omega t + \phi) \rangle \hat{k} = \frac{\kappa E_0^2}{2\mu_0 \omega} \hat{k} $$

(7.25)

where $\langle f(t) \rangle$ denotes the average of $f(t)$ over a single cycle,

$$ \langle f(t) \rangle = \frac{1}{T} \int_0^T f(t) \, dt \quad ; \quad T = \frac{2\pi}{\omega} $$

(7.26)

and we have recognized that $\langle \cos^2(\cdots) \rangle = \frac{1}{2}$ (P7.7). Further, as shown in P7.8, the average rate at which energy is transported through space by this plane wave is related to the average energy density in the fields in such a way as to justify thinking of the energy as moving rigidly through space with the wave.

The momentum transported by the plane wave in Eq. (7.17) leads to another interesting result. From Eq. (6.63), we find that the momentum density in this plane wave is

$$ G = \epsilon_0 \mathbf{E} \times \mathbf{B} = \frac{\epsilon_0 \kappa E_0^2}{\omega} \cos^2(\kappa z - \omega t + \phi) \hat{k} $$

(7.27)

Suppose this plane wave is incident on an absorbing surface located at $z = 0$ (Fig. 7.3). Since this surface absorbs momentum from the wave, it experiences a force $\mathbf{F}$ given by

$$ \mathbf{F} = \frac{\Delta (\text{momentum})}{\Delta t} $$

(7.28)

where $\Delta t$ is the time interval over which the absorption occurs. But the change in momentum of the electromagnetic field is equal to the amount of momentum in the volume of depth $c \Delta t$, since in time $\Delta t$ the radiation in that volume is absorbed. If $\Delta t$ is small enough, the momentum density at the surface can be assumed to be applicable to the entire volume. Thus, in magnitude,

$$ F = \frac{1}{\Delta t} \left[ \frac{\epsilon_0 \kappa E_0^2}{\omega} \cos^2(\omega t - \phi) \right] [Ac \Delta t] $$

(7.29)
where $A$ is the area of the surface and we have set $z = 0$. Defining the radiation pressure $p_r$ on the surface by $F/A$ and setting $\kappa c/\omega = 1$ [Eq. (7.20)], we find that

$$p_r = \epsilon_0 E_{x0}^2 \cos^2(\omega t - \phi) \quad (7.30)$$

On a macroscopic time scale, however, this instantaneous pressure fluctuates rapidly throughout a large portion of the electromagnetic spectrum. Thus, we can again connect macroscopic physical observations with an average over one cycle, and we find the expression

$$\langle p_r \rangle = \frac{1}{2} \epsilon_0 E_{x0}^2 \quad (7.31)$$

for the observed radiation pressure. (Remember that $\langle \cos^2(\cdot) \rangle = \frac{1}{2}$.) For a field $E_{x0} = 1$ V/m, this pressure is $4.4 \times 10^{-12}$ N/m$^2$ or about $10^{-17}$ atmospheres.

**PROBLEMS**

**P7.7.** Show by direct integration, as in Eq. (7.26), that $\langle \cos^2(\kappa z - \omega t + \phi) \rangle = \frac{1}{2}$ when $\kappa$, $z$, $\omega$, and $\phi$ are constants.

**P7.8.** Consider the wave in Eq. (7.17). (a) Show that $\langle u_E \rangle = \langle u_B \rangle$, where $u_E$ and $u_B$ are the electric and magnetic energy densities, respectively. (b) Show that $|\langle \delta \rangle| = c\langle (u_E) + (u_B) \rangle$, and interpret this equation physically. (c) Express the radiation pressure, Eq. (7.31), in terms of the average energy density $\langle u_{EM} \rangle = \langle u_E \rangle + \langle u_B \rangle$.

**P7.9.** The light from the sun delivers about 1300 W/m$^2$ to the surface of the earth. Assume the radiation to be a plane wave described by Eq. (7.17) and let it be incident normally on the earth’s surface. (a) Calculate the amplitude of the electric field in this wave and the resulting radiation pressure on the earth’s surface. *Hint:* Neither of these quantities depends specifically on the wavelength of the light; see Eq. (7.20). (b) The energy of a single photon of frequency $\nu$ is $h\nu$, where $h$ is Planck’s constant. Assume sunlight to have the wavelength 550 nm and calculate the number of photons incident per second on a square meter of the earth’s surface.
P7.10. Let the radiation impinging at normal incidence on a wall be reflected rather than absorbed. Determine the radiation pressure on the wall.

P7.11. (a) Determine the radiation pressure experienced by a perfectly reflecting plane surface bathed in monochromatic light whose incident direction makes an angle $\theta$ with the normal to the surface. Express your answer in terms of the average energy density $\langle u_{EM} \rangle$ in the incident wave. (b) Determine the pressure experienced by the same surface bathed in monochromatic light incident isotropically from all directions on one side of the surface. Hint: See F. K. Richtmeyer, E. H. Kennard, and T. Lauritsen, Introduction to Modern Physics (McGraw-Hill Book Company, New York, 1955), Fifth Edition, Chapter 4.

7.3 Superposition of Waves of the Same Frequency: Polarization and Interference

The electromagnetic wave is a transverse wave, which means that the oscillating electric and magnetic fields composing the wave are perpendicular to the direction of propagation. Since there are two linearly independent directions perpendicular to a given direction, there are two distinct plane waves with the same direction of propagation. These two waves are said to be in different states of polarization, the type of polarization being conventionally classified by the behavior of the electric field in the wave. A linearly polarized wave, for example, has an electric field that is always directed parallel to some fixed line in space. The wave in Eq. (7.17), for example, is linearly polarized in the $x$-direction.

More general states of polarization can be constructed by superposing two waves having the same frequency [and hence the same wave number; see Eq. (7.14)] but polarized at right angles to one another. The resulting electric field, which determines the state of polarization, is obtained by setting $K$ and $\Omega$ in Eq. (7.18) equal to $\kappa$ and $\omega$ and then adding the fields to those in Eq. (7.17); we find, for example, that

$$E = E_{x0} \cos(\kappa z - \omega t) \hat{i} + E_{y0} \cos(\kappa z - \omega t + \Phi) \hat{j} \quad (7.32)$$

where, for a simpler discussion, $\phi$ has been taken equal to zero. (Only the phase difference between the $x$- and $y$-components is important.) To determine the state of polarization of the wave in Eq. (7.32), we examine the field in the plane $z = 0$, where

$$E = E_{x0} \cos(\omega t) \hat{i} + E_{y0} \cos(\omega t - \Phi) \hat{j} \quad (7.33)$$

As a first example, let $\Phi = 0$. The two components of the wave are then said to be in phase, and

$$E = (E_{x0} \hat{i} + E_{y0} \hat{j}) \cos(\omega t) \quad (7.34)$$

Thus, as viewed from a point on the positive $z$-axis, the tip of the electric field vector in the $xy$-plane moves back and forth along a line having the direction of the constant vector $E_{x0} \hat{i} + E_{y0} \hat{j}$. This line makes an angle satisfying

$$\tan \theta = \frac{E_{y0}}{E_{x0}} \quad (7.35)$$

with the $x$-axis (Fig. 7.4). The wave described by Eq. (7.32) when $\Phi = 0$ is a linearly
Figure 7.4: Direction of the \( \mathbf{E} \)-field in a plane wave propagating along the \( z \)-axis and linearly polarized at an angle \( \theta \) to the \( x \)-axis. The \( \mathbf{E} \)-field oscillates in magnitude but is always directed one way or the other along the line shown. Its maximum amplitude is given by the vector \( E_x \hat{i} + E_y \hat{j} \).

A more interesting case occurs when the two components of Eq. (7.32) are \( 90^\circ \) out of phase, \( \Phi = \frac{1}{2} \pi \). In this case,

\[
\mathbf{E} = E_x \cos \omega t \hat{i} + E_y \sin \omega t \hat{j} \tag{7.36}
\]

More specifically, the electric field is given at successive times by

\[
\mathbf{E}(0) = E_x \hat{i}; \quad \mathbf{E} \left( \frac{\pi}{2\omega} \right) = E_y \hat{j}; \quad \mathbf{E} \left( \frac{\pi}{\omega} \right) = -E_x \hat{i}; \quad \mathbf{E} \left( \frac{3\pi}{2\omega} \right) = -E_y \hat{j} \tag{7.37}
\]

From the successive positions of \( \mathbf{E} \) shown in Fig. 7.5 we conclude that the \( \mathbf{E} \)-vector rotates counterclockwise as seen from a point on the positive \( z \)-axis, i.e., a point from which the wave will be seen as propagating towards the observer. If we eliminate \( t \) from the two equations, \( E_x(t) = E_{x0} \cos \omega t \) and \( E_y(t) = E_{y0} \sin \omega t \), we find that

\[
\left( \frac{E_x}{E_{x0}} \right)^2 + \left( \frac{E_y}{E_{y0}} \right)^2 = 1 \tag{7.38}
\]

which is the equation of an ellipse with semiaxes \( E_{x0} \) and \( E_{y0} \). Thus, the tip of the \( \mathbf{E} \)-vector in this case traces out an ellipse and the wave is said to be *left elliptically polarized*, the adjective *left* designating the counterclockwise direction of rotation. Sometimes the phrase *positive helicity* is used to describe this direction of the polarization. By similar arguments (P7.13) one can show that when \( \Phi = \frac{3}{4} \pi \), the electric field rotates clockwise, its tip again tracing out an ellipse; the result is a *right elliptically polarized* wave, sometimes said to have *negative helicity*. 

*polarized* plane wave, polarized in the direction shown. When \( \Phi = \pi \), the components of the wave are \( 180^\circ \) *out of phase*, and the composite wave is also linearly polarized, but in the direction making an angle \( -\theta \) with the \( x \)-axis (P7.12).
7.3. SUPERPOSITION OF WAVES OF THE SAME FREQUENCY

Figure 7.5: Successive values of $E$ for the elliptically polarized wave of Eq. (7.36). The vectors along the $x$-axis have magnitude $E_x0$ and those along the $y$-axis have magnitude $E_y0$.

If $E_x0 = E_y0$, then Eq. (7.38) is the equation of a circle and we have a special case of elliptic polarization called circular polarization. A left circularly polarized wave has an electric field given in the plane $z = 0$ by

$$E = E_x0 (\cos \omega t \hat{i} + \sin \omega t \hat{j})$$ (7.39)

and a right circularly polarized wave is characterized by an electric field given by

$$E = E_x0 (\cos \omega t \hat{i} - \sin \omega t \hat{j})$$ (7.40)

The various states of polarization are more easily related to one another if we introduce a complex representation for the fields.\(^4\) Recognizing the Euler identity

$$e^{i\theta} = \cos \theta + i \sin \theta$$ (7.41)

[Eq. (D.17)], we can then replace the fields in Eq. (7.17), for example, with the complex fields

$$\mathcal{E} = \mathcal{E}_x0 e^{i(\kappa z - \omega t)} \hat{i}, \quad \mathcal{B} = \frac{\kappa}{\omega} \mathcal{E}_x0 e^{i(\kappa z - \omega t)} \hat{j}$$ (7.42)

where the complex amplitude $\mathcal{E}_x0$ is defined by

$$\mathcal{E}_x0 = E_x0 e^{i\phi}$$ (7.43)

Whenever we need to, we can recover the physical fields $E$ and $B$ by taking the real part ($\Re$) of the complex fields, i.e.,

$$E = \Re(\mathcal{E}), \quad B = \Re(\mathcal{B})$$ (7.44)

A similar pair of complex fields can be introduced to represent the physical fields in Eq. (7.18). Since the complex fields corresponding to two or more physical fields can be added to give the complex field corresponding to the superposition of the physical fields

\(^4\)The properties of complex numbers are summarized in Appendix D.
\[ \mathcal{R}(z_1 + z_2) = \mathcal{R}(z_1) + \mathcal{R}(z_2) \], we then find that the complex electric fields for the basic states of polarization have the form

- linearly polarized, \( x \) direction: \( \mathcal{E} = E_0 i e^{i(\kappa z - \omega t)} \)
- linearly polarized, \( y \) direction: \( \mathcal{E} = E_0 j e^{i(\kappa z - \omega t)} \)
- right circularly polarized: \( \mathcal{E} = E_0 (i - j) e^{i(\kappa z - \omega t)} \)
- left circularly polarized: \( \mathcal{E} = E_0 (i + j) e^{i(\kappa z - \omega t)} \) \hspace{1cm} (7.45)

The complex fields in Eq. (7.42) also facilitate examining the effect of superposing two or more waves having the same frequency and the same polarization but different amplitudes and phases. Let the \( s \)-th such wave be represented by the complex field

\[ \mathcal{E}^{(s)} = E_0^{(s)} e^{i\phi_s} e^{i(\kappa z - \omega t)} i \quad ; \quad \mathcal{B}^{(s)} = \frac{\kappa}{\omega} E_0^{(s)} e^{i\phi_s} e^{i(\kappa z - \omega t)} j \] \hspace{1cm} (7.46)

where \( E_0^{(s)} \) is a real number. The complex fields \( \mathcal{E}, \mathcal{B} \) representing the superposition then are

\[ \mathcal{E} = \mathcal{E}_0 e^{i(\kappa z - \omega t)} i, \quad \mathcal{B} = \frac{\kappa}{\omega} \mathcal{E}_0 e^{i(\kappa z - \omega t)} j \] \hspace{1cm} (7.47)

where

\[ \mathcal{E}_0 = \sum_s E_0^{(s)} e^{i\phi_s} \] \hspace{1cm} (7.48)

We ask now for the rate at which energy is transported by this combined wave. The answer is given by the time average of the Poynting vector (Why?), but we must evaluate this quantity carefully. Its definition involves a product of the physical fields, and we cannot multiply the complex fields and then extract a real part; we must take the real parts first \([\mathcal{R}(z_1z_2) \neq \mathcal{R}(z_1)\mathcal{R}(z_2)]\). Thus, we must proceed as follows: \(^5\)

\[ \langle S \rangle = \frac{1}{\mu_0} \langle \mathbf{E} \times \mathbf{B} \rangle = \frac{1}{\mu_0} \langle \mathcal{R}(\mathcal{E}) \times \mathcal{R}(\mathcal{B}) \rangle = \frac{1}{4\mu_0} \langle (\mathcal{E} + \mathcal{E}^\ast) \times (\mathcal{B} + \mathcal{B}^\ast) \rangle \quad \text{[see Eq. (D.10)]} \]

\[ = \frac{1}{4\mu_0} \left[\langle (\mathcal{E} \times \mathcal{B}) + (\mathcal{E}^\ast \times \mathcal{B}^\ast) \rangle + \langle (\mathcal{E}^\ast \times \mathcal{B}) + (\mathcal{E} \times \mathcal{B}^\ast) \rangle\right] \] \hspace{1cm} (7.49)

Now with \( \mathcal{E} \) and \( \mathcal{B} \) in the form of Eq. (7.47), the time dependence of \( \mathcal{E} \times \mathcal{B} \) lies in a factor \( e^{-2i\omega t} \) and \( \langle \mathcal{E} \times \mathcal{B} \rangle = 0 \). Similarly, \( \mathcal{E}^\ast \times \mathcal{B}^\ast \) involves a factor \( e^{2i\omega t} \) and \( \langle \mathcal{E}^\ast \times \mathcal{B}^\ast \rangle = 0 \). The terms \( \mathcal{E} \times \mathcal{B}^\ast \) and \( \mathcal{E}^\ast \times \mathcal{B} \), however, contain no explicit time dependence. Each is therefore equal to its own time average and we have from Eq. (7.49) that

\[ \langle S \rangle = \frac{1}{4\mu_0} \langle \mathcal{E} \times \mathcal{B}^\ast + \mathcal{E}^\ast \times \mathcal{B} \rangle = \frac{1}{2\mu_0} \mathcal{R}(\mathcal{E} \times \mathcal{B}^\ast) = \frac{1}{2\mu_0} \mathcal{R}(\mathcal{E}^\ast \times \mathcal{B}) \] \hspace{1cm} (7.50)

[See Eq. (D.10)]. Finally, we obtain

\[ \langle S \rangle = \frac{\kappa}{2\mu_0 \omega} |\mathcal{E}_0|^2 \mathbf{k} \] \hspace{1cm} (7.51)

\[ = \frac{\kappa}{2\mu_0 \omega} \left| \sum_s E_0^{(s)} e^{i\phi_s} \right|^2 \mathbf{k} \] \hspace{1cm} (7.52)

---

\(^5\)Remember that a superscript "*" on a symbol indicates the complex conjugate of the flagged quantity. See Eq. (D.6) and the surrounding text; see also PD.10.
7.3. SUPERPOSITION OF WAVES OF THE SAME FREQUENCY

for the average Poynting vector of the fields in Eq. (7.47). Its magnitude depends on the
amplitudes and the phases of the component waves but is not just the sum of independent contributions from these components—that would be given by \((\kappa/2\mu_0\omega)\sum_s|E_0^{(s)}|^2\) (Why?)—so the individual components interact with one another to determine the rate at which energy is transported by the total wave. This phenomenon of interference between two or more waves that are simultaneously present is a characteristic property of waves and
is responsible for the useful functioning of such diverse apparatus as grating spectrometers
and arrays of antennas for radio and television broadcasting. The program antenna\(^6\) provides a way to draw polar graphs showing the intensity patterns produced by user-specified positions, phases, and amplitudes for up to 10 point sources.

As a more specific example of the consequences of Eq. (7.52), suppose there are \(N\) waves, all having the same amplitude \(E_0^{(s)} = E_0\). Further, let the phase of the \(s\)-th wave differ from that of the \((s-1)\)-st wave by an amount \(\delta\) that is independent of \(s\), so that \(\phi_s = (s-1)\delta\) if we set \(\phi_1 = 0\). In this case, Eq. (7.52) becomes

\[
\langle S \rangle = \frac{\kappa E_0^2}{2\mu_0\omega} f_N(\delta) \hat{k}
\]

(7.53)

where \(f_N(\delta)\) is defined by

\[
f_N(\delta) = \left| \sum_{s=1}^{N} e^{i(s-1)\delta} \right|^2
\]

(7.54)

Apart from a constant factor, \(f_N(\delta)\) gives the rate at which this composite wave transports energy, expressing that rate as a function of the phase difference \(\delta\) between consecutive individual waves.

The sum in Eq. (7.54) can be readily evaluated and plotted with the following commands
to IDL:

IDL> N = 3 ; Set number of sources
IDL> delta = 4*!Pi*findgen(201)/200.0 ; Establish array containing 201
; values of delta in interval
; 0 to 4*Pi
IDL> sum = complexarr(201) ; Create array for sums and
; initialize to zero

; *** Evaluate sum in Eq. (7.54) for all delta ***

IDL> for s=0,N-1 do sum = sum + complex( cos(s*delta), sin(s*delta) )
IDL> fN = abs(sum)^2 ; Calculate fN(delta)
IDL> plot, delta, fN, title='N = 3'

The result of this set of commands issued to IDL is shown in Fig. 7.6(b). Other segments
of that figure show the results for different values of \(N\).

\(^6\)See listing in Appendix F and instructions in Appendix G.
Figure 7.6: Graphs of $f_N(\delta)$ versus $\delta$ for selected values of $N$. Note particularly that the vertical scales are different for each graph. This graph was produced with IDL.

The sum on the right side of Eq. (7.54), however, can also be evaluated analytically. Writing that sum out in full, we find the geometric sum\(^7\)

$$
\sum_{s=1}^{N} e^{i(s-1)\delta} = 1 + e^{i\delta} + e^{2i\delta} + \cdots + e^{i(N-1)\delta}
$$

(7.55)

Invoking the standard approach for evaluating a geometric sum, we multiply Eq. (7.55) by the multiplier $e^{i\delta}$ to find that

$$
e^{i\delta} \sum_{s=1}^{N} e^{i(s-1)\delta} = e^{i\delta} + e^{2i\delta} + e^{3i\delta} + \cdots + e^{iN\delta}
$$

(7.56)

Next, we subtract Eq. (7.56) from Eq. (7.55) and rearrange the result, we obtain

$$
\sum_{s=1}^{N} e^{i(s-1)\delta} = \frac{1 - e^{iN\delta}}{1 - e^{i\delta}} = e^{i(N-1)\delta/2} \frac{\sin(N\delta/2)}{\sin(\delta/2)}
$$

(7.57)

where the last form follows after factoring $e^{iN\delta/2}$ and $e^{i\delta/2}$ out of the numerator and denominator, respectively, and then using the results of PD.13 in Appendix D. Finally, on

\(^7\)A geometric sum is a sum in which each successive term is obtained by multiplying the immediately preceding term by the same factor, e.g., $1 + a + a^2 + a^3 + \cdots$. Here $a = e^{i\delta}$.
substituting Eq. (7.57) into Eq. (7.54), we find that

\[ f_N(\delta) = \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)} \]  \hspace{1cm} (7.58)

Graphs of this function for several values of \( N \) have already been presented. For every \( N \) the function has principal maxima of height \( N^2 \) when \( \delta = 2n\pi \), where \( n \) is an integer, zeros when \( \delta = 2m\pi/N \), where \( m \) is an integer not equal to a multiple of \( N \), and secondary maxima at \( N - 2 \) points between adjacent principal maxima. As \( N \) becomes larger, the principal maxima become narrower (P7.17) and the secondary maxima become less significant.

To make the example of the previous paragraph even more meaningful, let us place it in a physical context. Suppose we have a linear array of \( N \) equally spaced sources oscillating in phase and all emitting electromagnetic radiation of the same frequency (Fig. 7.7). Further, let each source have the same amplitude and suppose each source emits energy equally in all directions. Finally, let the emitted radiation be observed on a distant viewing screen. Although strictly the sources do not emit plane waves, the radiation reaching the point \( P \) on the (distant) viewing screen from each source can be treated approximately as a plane wave propagating along the line from the source to \( P \). Because of the assumed distance of the viewing screen from the sources, however, lines from all of the sources to \( P \) are nearly parallel and all waves reaching \( P \) are therefore coming from essentially the same direction. Further, these waves have all traveled essentially the same distance from their respective sources, so any attenuation in amplitude will affect all of them nearly equally and the separate waves as they arrive at \( P \) will have nearly the same amplitude. Indeed, again because of the distance from the sources to the viewing screen, the amplitude of the individual waves is nearly independent of the position of \( P \) on the screen. Finally, the phase difference between the waves arriving at \( P \) from adjacent sources is determined by the difference in length between the two paths from \( P \) to the sources, a difference of one wavelength \( \lambda \) corresponding to a phase difference of \( 2\pi \). (Why?) Since \( \lambda \) is typically small, we cannot here ignore the small differences in these paths as we have done earlier in the paragraph. When a line from the sources—any source will do (Why?)—to \( P \) makes an angle \( \theta \) with a normal to the line of sources, the path difference for waves from adjacent sources is \( a \sin \theta \), where \( a \) is the separation of the sources. This path difference corresponds
to a phase difference \( \delta \) given by

\[
\delta = 2\pi \left( \frac{a \sin \theta}{\lambda} \right) = \kappa a \sin \theta
\]

which is the same for any two adjacent sources. Thus, with the various approximations made in this paragraph, the situation at \( P \) in Fig. 7.7 exactly duplicates the situation to which Eq. (7.58) applies, provided we identify \( \delta \) as in Eq. (7.59). Apart from the distortion of the horizontal scale caused by the nonlinear relationship between \( \delta \) and \( \theta \), the graph in Fig. 7.6 shows the intensity of the radiation received at \( P \) as a function of the position of \( P \) on the viewing screen. Some additional aspects of these radiation patterns are explored in P7.18, P7.19, and P7.33.

**PROBLEMS**

P7.12. Show that when \( \Phi = \pi \), Eq. (7.32) represents the electric field in a linearly polarized wave and determine the angle \( \alpha \) between the direction of polarization and the direction of increasing \( x \).

P7.13. Show that when \( \Phi = 3\pi/2 \), Eq. (7.32) represents the electric field in a right elliptically polarized wave.

P7.14. Show that for general \( \Phi \), Eq. (7.32) represents the electric field in an elliptically polarized wave and determine the angles that the axes of the ellipse make with the direction of increasing \( x \). The program lisjus,\(^8\) which permits tracing the path obtained by right-angle superposition of two sinusoidal waves with user-specified phases, may be useful in visualizing the various elliptical paths traced by the tip of the electric field vector in this problem.

P7.15. Show that the imaginary part of Eq. (7.42) satisfies Maxwell’s equations with \( \rho = 0 \) and \( J = 0 \). Thus, we can take either the real or the imaginary part of the complex field as a physical field.

P7.16. Verify the third and fourth members of Eq. (7.45).

P7.17. (a) Defining the width \( \Delta \) of the principal maximum as the distance (in \( \delta \)) from the peak to the nearest zero, find \( \Delta \) in terms of \( N \) for the pattern given by Eq. (7.58). (b) Estimate the height of the secondary maximum immediately adjacent to the principal maximum and compare this height to that of the principal maximum. **Hint:** (1) Approximately for what value of \( \delta \) does this secondary maximum occur? (2) Assume \( N \) fairly large.

P7.18. (a) Starting with Eq. (7.58), show that \( f_2(\delta) = 4\cos^2(\delta/2) = 2(1 + \cos \delta) \) and then use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to obtain a careful graph of \( f_2(\delta) \) versus \( \delta \) over the interval \(-4\pi < \delta < 4\pi\). (b) With \( \delta \) as given by Eq. (7.59), use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to draw polar graphs of \( f_2 \) as a function of \( \theta \) for enough values of \( a/\lambda \), say 0.5, 1, 2, \ldots to give you a feel for the dependence of this pattern on slit separation. Consider only \(-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi\). (c) Describe in words the effect of changes in \( a/\lambda \) on the pattern.

P7.19. (a) Show that \( f_4(\delta) = 8(1 + \cos \delta) \cos^2 \delta \). (b) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to obtain a careful graph of \( f_4(\delta) \) versus \( \delta \), find the positions of the two secondary maxima to three significant figures, and determine the height of these maxima. (c) With \( \delta \) as given by Eq. (7.59), use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to draw polar graphs of

\(^8\)See listing in Appendix F and instructions in Appendix G.
7.4. Superposition of Waves of Different Frequencies

7.4. Superposition of Waves of Different Frequencies: Spectral Decomposition

The complex fields also facilitate a study of superposition when the frequencies of the superposed waves are different. We shall consider here only the superposition of waves having the same linear polarization. Since the solutions expressed in Eq. (7.42) are valid for any $\mathcal{E}_{x0}$ and for any $\kappa$, provided $\omega = \kappa c$, we can build more general solutions by superposing these simpler solutions for a discrete spectrum of values $\{\mathcal{E}_{x0}(\kappa_i), \kappa_i, \omega_i; i = 1, 2, \ldots\}$, or, more usefully, by superposing simpler solutions for a continuous spectrum of $\kappa$ by integration, specifically

$$
\mathcal{E}(z, t) = \hat{i} \int_{0}^{\infty} \mathcal{E}_{x0}(\kappa) e^{i(\kappa z - \omega t)} d\kappa
$$

(7.60)

and

$$
\mathcal{B}(z, t) = \frac{1}{c} \hat{j} \int_{0}^{\infty} \mathcal{E}_{x0}(\kappa) e^{i(\kappa z - \omega t)} d\kappa
$$

(7.61)

where $\omega$ is function of $\kappa$ ($\omega = \kappa c$) and $\mathcal{E}_{x0}(\kappa)$ is an arbitrary (complex) function of the (real) variable $\kappa$. We shall also need expressions for the physical fields. For the physical $\mathbf{E}$-field, we obtain

$$
\mathbf{E}(z, t) = \Re(\mathcal{E}) = \frac{1}{2}(\mathcal{E} + \mathcal{E}^*)
$$

$$
= \frac{1}{2} \hat{i} \int_{0}^{\infty} \left[ \mathcal{E}_{x0}(\kappa) e^{i(\kappa z - \omega t)} + \mathcal{E}_{x0}^*(\kappa) e^{-i(\kappa z - \omega t)} \right] d\kappa
$$

$$
= \frac{1}{2} \hat{i} \left[ \int_{0}^{\infty} \mathcal{E}_{x0}(\kappa) e^{i(\kappa z - \omega t)} d\kappa + \int_{-\infty}^{0} \mathcal{E}_{x0}^*(-\kappa) e^{i(\kappa z - \omega t)} d\kappa \right]
$$

$$
= \hat{i} \int_{-\infty}^{\infty} \mathcal{A}(\kappa) e^{i(\kappa z - \omega t)} \frac{d\kappa}{2\pi}
$$

(7.62)

where the spectral function $\mathcal{A}(\kappa)$, which conveys how much of a particular wave number $\kappa$ is present in the general field, is defined by

$$
\mathcal{A}(\kappa) = \begin{cases} 
\pi \mathcal{E}_{x0}(\kappa), & \kappa > 0 \\
\pi \mathcal{E}_{x0}^*(-\kappa), & \kappa < 0 
\end{cases}
$$

(7.63)
This extension of $\kappa$ into the physically meaningless region $\kappa < 0$ should be noted. Mathematically this extension is useful, but it cannot be allowed to confuse any subsequent physical interpretations. A similar argument applied to Eq. (7.61) gives

$$B(z, t) = \frac{1}{c} \hat{j} \int_{-\infty}^{\infty} A(\kappa) e^{i(\kappa z - \omega t)} \frac{d\kappa}{2\pi}$$

(7.64)

for the physical $B$-field. The property

$$A(\kappa)^* = A(-\kappa)$$

(7.65)

which follows directly from Eq. (7.63), assures that both $E(z, t)$ and $B(z, t)$ as given by Eqs. (7.62) and (7.64) will in fact be real (P7.21). Further, the spectral function corresponding to a plane wave whose $E$-field is known at some initial time $t = 0$ can now be determined by Fourier inversion of Eq. (7.62); we find that

$$A(\kappa) = \int_{-\infty}^{\infty} E_x(z, 0) e^{-i\kappa z} dz$$

(7.66)

(See Appendix D.) Since $A(\kappa)$ in Eq. (7.66) is well-defined even if $E_x(z, 0) = 0$ except in some small range of $z$, we have now made it possible to express general plane waves that may be nonzero only in some finite range of $z$. Further, we have found expressions for calculating the physical fields from the spectral function [Eqs. (7.62) and (7.64)] and also for calculating the spectral function from the fields [Eq. (7.66)]. The formalism for describing these general superpositions is thus complete.

We shall now illustrate the use of this formalism to calculate the spectral distribution of energy in this general, linearly polarized, plane wave. Let us first calculate the total energy $W$ transported across a unit surface oriented with its plane perpendicular to the direction of propagation; we find

$$W = \int_{-\infty}^{\infty} |S| dt = \int_{-\infty}^{\infty} \left| \frac{1}{\mu_0} (E \times B) \right| dt$$

$$= \int_{-\infty}^{\infty} dt \left( \frac{1}{\mu_0} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} A(\kappa) e^{i(\kappa z - \omega t)} \right) \left( \frac{1}{c} \int_{-\infty}^{\infty} \frac{d\kappa'}{2\pi} A(\kappa') e^{i(\kappa' z - \omega' t)} \right)$$

(7.67)

Now, for the sake of the final result, let us change the sign of $\kappa'$ and recognize Eq. (7.65) to obtain

$$W = \frac{1}{\mu_0 c} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \int_{-\infty}^{\infty} \frac{d\kappa'}{2\pi} A(\kappa) A(\kappa')^* e^{i(\kappa - \kappa') z - (\omega - \omega') t}$$

(7.68)

For simplicity in the manipulation, let the origin of the coordinate system be taken in the surface across which the energy flux is computed. Then $z = 0$ and, since $\omega = \kappa c$ and $\omega' = \kappa' c$, Eq. (7.68) becomes

$$W = \frac{1}{\mu_0 c^2} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \int_{-\infty}^{\infty} \frac{d\kappa'}{2\pi} A(\kappa) A(\kappa')^* e^{-i(\kappa - \kappa')\xi}$$

(7.69)

where $\xi = ct$. Now, let us write this expression in a form to which the Fourier integral theorem (Appendix D) can be readily applied:

$$W = \frac{1}{\mu_0 c^2} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} A(\kappa) \left[ \int_{-\infty}^{\infty} d\xi e^{-i\kappa \xi} \left( \int_{-\infty}^{\infty} \frac{d\kappa'}{2\pi} A(\kappa')^* e^{i\kappa' \xi} \right) \right]$$

(7.70)
In this form, the innermost integral can be interpreted as giving that function of \( \xi \) whose
Fourier transform is \( A(\kappa')^* \). But then the middle integral expresses the Fourier transform of
that function. Together, the integrals on \( \kappa' \) and \( \xi \) merely produce \( A(\kappa)^* \) ![See Eqs. (D.30)
and (D.31).] Thus,

\[
W = \frac{1}{\mu_0 c^2} \int_{-\infty}^{\infty} |A(\kappa)|^2 \frac{d\kappa}{2\pi}
\]  
(7.71)

To extract a physical interpretation, however, we must remove all reference to the unphysical
negative wave numbers. From Eq. (7.65) it follows that \( |A(\kappa)|^2 = |A(-\kappa)|^2 \) (Why?); thus,
\( |A(\kappa)|^2 \) is an even function of \( \kappa \) and Eq. (7.71) may be written alternatively as

\[
W = \frac{1}{\pi \mu_0 c^2} \int_{0}^{\infty} |A(\kappa)|^2 d\kappa
\]  
(7.72)

from which it is now natural to interpret the integrand as giving the distribution \( u(\kappa) \) of
energy over all wave numbers,

\[
u(\kappa) = \frac{1}{\pi \mu_0 c^2} |A(\kappa)|^2
\]  
(7.73)

In essence, the function \( u(\kappa) \) predicts the reading of a spectral analyzer at wave number
\( \kappa \) when a wave defined by the spectral function \( A(\kappa) \) is incident on the instrument. The
transformation of Eq. (7.73) into expressions for the distribution of energy over frequency
\( \omega \) and over wavelength \( \lambda \) is the topic of P7.23.

PROBLEMS

**P7.21.** Verify that the condition \( A(\kappa)^* = A(-\kappa) \) assures that the fields given by Eqs. (7.62) and
(7.64) will be real.

**P7.22.** (a) Suppose \( E(0, t) \) is known. Find an expression for \( A(\kappa) \). (b) Let \( E(0, t) = E_0 \hat{i}, \)
\(-T \leq t \leq T \), and \( E(0, t) = 0 \) outside that time interval. Find \( A(\kappa) \) and \( u(\kappa) \) and sketch
a graph of the latter as a function of \( \kappa \). (c) Show that the width of \( A(\kappa) \) in \( \kappa \) is inversely
proportional to \( T \). This property is very closely related to the Heisenberg uncertainty
relations in quantum mechanics. **Optional:** Find and describe the \( B \)-field associated
with the \( E \)-field in part (b). **Hint:** In terms of the signum function \( \text{sgn}(\alpha) \) defined by
\( \text{sgn}(\alpha) = 1, \alpha > 0; = 0, \alpha = 0; = -1, \alpha < 0 \), note the integral

\[
\int_{-\infty}^{\infty} \frac{\sin \alpha x}{x} \, dx = \pi \text{sgn}(\alpha)
\]

**P7.23.** Starting with Eq. (7.72), obtain an expression giving the distribution of energy (a) in
frequency \( \omega \) and (b) in wavelength \( \lambda \).

### 7.5 Plane Waves in Three Dimensions

A three-dimensional plane wave solution to Maxwell’s equations can be inferred from the
one-dimensional solution already obtained. Instead of a coordinate \( z \), which measures the
displacement of the general point \( r \) from the \( xy \)-plane, we should have a coordinate measuring
the displacement of the point \( r \) from some more general plane whose normal vector is,
say, \( \hat{n} \). Thus, we expect that \( z \), which is equivalent to \( \hat{k} \cdot \mathbf{r} \), should be replaced by \( \hat{n} \cdot \mathbf{r} \), and further that the vector \( \hat{n} \) (as did \( \hat{k} \) before) should coincide with the direction of propagation of the wave. We are thus led to guess a more general solution of the form

\[
\mathbf{E}(\mathbf{r}, t) = E_0 e^{i(\kappa \cdot \mathbf{r} - \omega t)} \\
\mathbf{B}(\mathbf{r}, t) = B_0 e^{i(\kappa \cdot \mathbf{r} - \omega t)}
\]

(7.74)

where \( E_0 \) and \( B_0 \) will be assumed real and \( \kappa = \kappa \hat{n} \). It is shown in P7.25 that the fields in Eq. (7.74) in fact do satisfy Maxwell’s equations when \( \rho = 0 \) and \( \mathbf{J} = 0 \) provided that

\[
\kappa \cdot \mathbf{E}_0 = 0 \Rightarrow \kappa \perp \mathbf{E}_0 \\
\kappa \cdot \mathbf{B}_0 = 0 \Rightarrow \kappa \perp \mathbf{B}_0 \\
\omega = \kappa c \\
\mathbf{B}_0 = \frac{1}{\omega} \kappa \times \mathbf{E}_0
\]

(7.75)

Furthermore, the time average of the Poynting vector is given by

\[
\langle S \rangle = \frac{E_0^2}{2\mu_0 \omega} \kappa
\]

(7.76)

confirming our intuition that \( \kappa \) gives the direction of propagation of the wave. Consistent with Eq. (7.75), we can think of the plane wave propagating in three dimensions to be defined as follows:

1. Specify \( \kappa \), i.e., the direction of propagation and the wave number, *arbitrarily*.
2. Specify \( \mathbf{E}_0 \) *arbitrarily*, subject only to the constraint that \( \mathbf{E}_0 \) be perpendicular to \( \kappa \).
3. \( \mathbf{B}_0 \) and \( \omega \) are then determined by Eq. (7.75).

Plane waves in three dimensions are discussed in much greater detail in Chapter 13.

**PROBLEMS**

**P7.24.** Obtain expressions for the (complex) electric and magnetic induction fields in a plane wave that is (a) linearly polarized in the \( y \)-direction and propagating in the positive \( x \)-direction, (b) linearly polarized in the \( z \)-direction and propagating in a direction parallel to the \( xy \)-plane at an angle \( \theta \) to the positive \( x \)-axis, and (c) right circularly polarized and propagating in the positive \( x \)-direction.

**P7.25.** Because Maxwell’s equations are linear in the fields, the entire complex field can be required to satisfy the equations with the assurance that the real part alone then necessarily satisfies the equations by itself. (a) Substitute Eq. (7.74) into Maxwell’s equations with \( \rho = 0 \) and \( \mathbf{J} = 0 \) and derive the conditions expressed in Eq. (7.75). (b) Obtain Eq. (7.76) for the time-averaged Poynting vector of the fields in Eq. (7.74). *Caution:* The real parts of the complex fields must be taken before evaluating the cross product \( \mathbf{E} \times \mathbf{B} \). (Why?)

**P7.26.** For the wave in Eq. (7.74), show that

\[
\langle S \rangle = \frac{1}{2\mu_0} \Re(\mathbf{E}^* \times \mathbf{B}) = \frac{1}{2\mu_0} \Re(\mathbf{E} \times \mathbf{B}^*)
\]
\[ \langle u_E \rangle = \frac{\varepsilon_0}{4} \mathcal{R}(\mathcal{E}^* \cdot \mathcal{E}) \quad \langle u_B \rangle = \frac{1}{4\mu} (\mathcal{B}^* \cdot \mathcal{B}) \]

where \( \mathcal{S}, u_E, \) and \( u_B \) are the Poynting vector, the electric energy density, and the magnetic energy density, respectively.

**SUPPLEMENTARY PROBLEMS**

P7.27. Find the most general solution to the scalar wave equation

\[ \nabla^2 u = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} \]

in spherical coordinates if the solution is known to depend only on \( r \) and \( t \). *Hints:* Use \( \nabla^2 \) as given by Eq. (2.53). (2) Make the variable transformation \( u(r, t) = f(r, t)/r \). (3) See P7.1.

P7.28. Show that the wave equation

\[ \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \]

is invariant to the Lorentz transformation \( z' = \gamma (z - vt), \ t' = \gamma [t - (v/c^2)z], \) where \( \gamma = \sqrt{1 - (v/c)^2} \) and \( v \) is the speed of the primed coordinate system relative to the unprimed system. That is, show that a function \( u(z, t) \) satisfying the above equation also satisfies

\[ \frac{\partial^2 u}{\partial z'^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t'^2} \]

where \( z' \) and \( t' \) are determined from \( z \) and \( t \) by the Lorentz transformation.

P7.29. For the plane wave in Eq. (7.17), find a suitable *Lorentz gauge* vector potential \( \mathbf{A} \) when the scalar potential is taken to be zero.

P7.30. In experiments on nuclear magnetic resonance (nmr), the sample is often placed in a region free of charges and currents and subjected to a magnetic induction field of the form

\[ \mathbf{B}(t) = B_0 \mathbf{\hat{k}} + b \mathbf{\hat{i}} \cos \omega_0 t - b \mathbf{\hat{j}} \sin \omega_0 t \]

where \( B_0, b, \) and \( \omega_0 \) are constants. (a) Describe this field in words. (b) Show that there exists no electric field \( \mathbf{E} \) that can be added to \( \mathbf{B} \) so as to produce a pair of fields satisfying Maxwell’s equations and explain how the physicist doing experimental nmr obtains the specified field.

P7.31. Calculate the force experienced by a perfectly reflecting sphere of radius \( a \) placed in the path of a linearly polarized, plane, monochromatic electromagnetic wave. *Hints:* (1) Recall that the angle of incidence equals the angle of reflection, where both are measured with respect to the normal to the surface at the point of incidence. (2) See P7.11.

P7.32. If the common factors in Eq. (7.45) are ignored, we can make the correspondences

\[
\begin{align*}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \iff \mathbf{\hat{i}} \iff \text{wave linearly polarized in the } x \text{ direction and propagating in the } z \text{ direction} \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \iff \mathbf{\hat{j}} \iff \text{wave linearly polarized in the } y \text{ direction and propagating in the } z \text{ direction} \\
\begin{pmatrix} 1 \\ -\mathbf{\hat{i}} \end{pmatrix} & \iff \mathbf{\hat{i}} - \mathbf{\hat{j}} \iff \text{right circularly polarized wave propagating in the } z \text{ direction} \\
\begin{pmatrix} 1 \\ \mathbf{\hat{i}} \end{pmatrix} & \iff \mathbf{\hat{i}} + \mathbf{\hat{j}} \iff \text{left circularly polarized wave propagating in the } z \text{ direction}
\end{align*}
\]
These two-dimensional vectors, and others like them, are called Jones vectors. Within this representation, many optical devices can be put in correspondence with a two-by-two Jones matrix. The Jones vector corresponding to the output of such a device is then obtained by multiplying the Jones vector corresponding to the input by the Jones matrix of the device. (a) Show that the Jones vector representing a wave linearly polarized in the x-direction is a linear combination of the Jones vectors representing the two states of circular polarization. (b) Find the Jones vector representing light linearly polarized at an angle $\alpha$ to the x-axis. (c) An optical device is represented by the Jones matrix

$$\begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}$$

Describe the effect of this device on an incident plane wave linearly polarized in a direction making an angle $\theta$ with respect to the positive x axis. This so-called Jones calculus is developed in some detail in G. R. Fowles, Introduction to Modern Optics (Holt, Rinehart, and Winston, Inc., New York, 1968), Chapter 2.

P7.33. Relax the assumption that the sources in Fig. 7.7 are equally spaced and in phase, letting the s-th source be located more generally at $y_s$ on the y-axis. Further, let the s-th source have an intrinsic phase $\alpha_s$. (a) Using the same approximations as in the text, show that the phase of the s-th contribution to the amplitude at P is given by $\phi_s = \kappa y_s \sin \theta + \alpha_s$, where the phase reference is a (possibly imaginary) source at $y = 0$. (b) Using an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON, write a program that accepts the number of sources $N$, the emitted wavelength $\lambda$, the positions $y_s$, the intrinsic phases $\alpha_s$, and the amplitudes $E_s^0$ as input and then computes the square magnitude in Eq. (7.52) for a specified range of $\theta$ and plots a polar graph of intensity versus angle for the combination of two sources for which $E_1^0 = 1, y_1 = \lambda, \alpha_1 = 0$ and $E_2^0 = 1, y_2 = -\lambda, \alpha_2 = \frac{1}{2}\pi$. Optional: (1) Examine other arrays of sources of your choice. (2) Modify the technique to allow for positioning the sources in a plane but not necessarily on a line and use the resulting program to examine intensity versus angle for several such arrays. The program antena provides these capabilities, but you should try your hand at writing a suitable program on your own.

P7.34. In our discussion of interference, we tacitly assumed that the interfering waves were coherent, which means that the phase difference between any two individual waves remained constant so that at a particular point the two waves were, for example, always 147° out of phase or something similar. If the phase difference between any two individual waves changes rapidly and randomly on the time scale of a macroscopic observation any interference will be washed out because the waves are as much in phase as out of phase during the observation. In the latter case, which typically arises when the waves are derived from independent sources rather than by splitting the wave from a single source, the waves are said to be incoherent. Mathematically, the superposition of incoherent waves can be represented by averaging Eq. (7.52) over assumed random and independent fluctuations of all $\phi_s$. Recognizing that such an average of $e^{i\alpha}$, where $\alpha$ varies randomly, is zero, show that Eq. (7.52) reduces to the sum of independent contributions from each wave when the waves are incoherent.

P7.35. In this problem we shall explore a simple numerical technique for solving the wave equation

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

subject to the conditions $u(0, t) = u(L, t) = 0, u(z, 0) = f(z)$, and $u_t(z, 0) = 0$, where $a$ is a constant and $f(z)$ is a known function of $z$. [Think of a string stretched between

---

11 See listing in Appendix F and instructions in Appendix G.
7.5. PLANE WAVES IN THREE DIMENSIONS

Figure 7.8: Figure for Problem P7.35.

P7.36. The linearly polarized electromagnetic wave given by Eq. (7.42) is incident on an electron that, in the absence of the external field, undergoes damped harmonic motion about $z = 0$ and $z = L$ and released from rest with an initial displacement given by $f(z)$. We begin by superposing a rectangular grid on the domain $0 \leq z \leq L$, $0 \leq t < \infty$ of the problem (Fig. 7.8). Let the lines of the grid be separated by $\Delta z$ and $\Delta t$ in the two coordinate directions, where both $\Delta z$ and $\Delta t$ can be chosen as small as necessary. We now interpret solving this problem numerically to mean obtaining values for $u$ at all intersections on this grid. To facilitate writing subsequent equations, let

\[ u(i,j) \]

where $i = 0, 1, 2, \ldots$; $t_j = j \Delta t$, $j = 0, 1, 2, \ldots$; and $u(i,j) = u(z_i,t_j)$. (a) Approximate the derivatives in the above differential equation and show that

\[ u_{i,j+1} = \alpha (u_{i+1,j} + u_{i-1,j}) + 2(1 - \alpha) u_{i,j} - u_{i,j-1} \quad (1) \]

where $\alpha = a^2(\Delta t)^2/(\Delta z)^2$. Hint: Show first that

\[ d^2 f(\xi)/d\xi^2 \approx [f(\xi + \Delta \xi) - 2f(\xi) + f(\xi - \Delta \xi)]/(\Delta \xi)^2. \]

(b) Equation (1) permits a direct calculation of $u$ at points on the line $t = t_{j+1}$ provided values are known on the two previous lines $t = t_j$ and $t = t_{j-1}$. (Why?) To get started, we therefore need the values $u_{i,0}$ and $u_{i,1}$ for all $i$. The initial condition on $u$ gives $u_{i,0} = f(z_i)$. We obtain $u_{i,1}$ by introducing temporarily the values $u_{i,-1}$. Then, $u_{i} (z_i,0) \approx (u_{i,1} - u_{i,-1})/(2\Delta t)$. Thus, the initial condition on $u_i$ requires that $u_{i,1} = u_{i,-1}$. Now, set $j = 0$ in Eq. (1) and show that

\[ u_{i,1} = \frac{1}{2}[\alpha (u_{i+1,0} + u_{i-1,0}) + 2(1 - \alpha) u_{i,0}] \quad (2) \]

Thus, we calculate $u_{i,0}$ from the initial condition on $u$, then calculate $u_{i,1}$ from Eq. (2), and then use Eq. (1) to obtain $u_{i,2}, u_{i,3}, \ldots$ in succession. (c) Write a program to implement this algorithm on an available computer. Assume that $L, a, \Delta z,$ and $\Delta t$ are provided as input. (d) Use your program to find the solution at several successive times when $L = 10 \text{ cm}$, $a = 0.5 \text{ cm/s}$, $\Delta z = 1 \text{ cm}$, and $\Delta t = 1 \text{ S}$, and in consistent units $f(z) = z(10 - z)/25$. Try also some other cases of your choosing. Note, however, that this method is unstable (and hence unsatisfactory) unless $\alpha \leq 1$. Optional: (1) Modify your program to print $u_{i,j}$ only for every $N$-th value of $j$, thus permitting smaller time steps without generating excessive output. (2) Develop a means to extend this method to cases where $u_i(z,0) = g(z) \neq 0$. 

P7.36. See text.
a nominal equilibrium position, say the origin. Let the spring constant be \( k \) and the damping constant be \( b \). (a) Show that the force arising on the electron from the \( B \)-field in the wave is smaller than the force from the \( E \)-field by a factor of \( v/c \), where \( v \) is the electron speed, and hence argue that, when \( v/c \ll 1 \), the equation of motion of the electron is

\[
m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = -qE_{x0}e^{-i\omega t}
\]

where \( m \) and \(-q\) are the mass of and the charge on the electron, respectively, and \( E_{x0} \to E_{x0} \) has been taken to be real. Note that we have tacitly introduced a complex position \( x \) whose real part expresses the physical displacement of the electron along the \( x \)-axis. (b) Assume that \( x(t) = x_0 \exp(-i\omega t) \) and find a solution of Eq. (1) for \( x \) as a function of \( t \). (c) Discuss this solution, paying particular attention to the dependence of its amplitude and phase on the frequency of the incident wave. Note: The calculation in this problem is the first step in a classical calculation of the scattering of light by a bound electron, a phenomenon known as Compton scattering.\(^\text{12}\)

P7.37. Use \texttt{lisjus}\(^\text{13}\) to determine the path traced out by the tip of the electric field vector in the superposition expressed by Eq. (7.33) for various values of \( \Phi \). Optional: Explore the path when the frequencies bear a simple rational ratio to one another. You are, of course, generating Lissajous patterns.

P7.38. Use \texttt{antena}\(^\text{14}\) to explore the radiation pattern produced by antenna arrays of your choice. Try two in-phase antennae, two out-of-phase antennae, four equally spaced in-phase antennae, etc.

\(^{12}\)American physicist Arthur Holly Compton, b. 10 September 1892 in Wooster, Ohio; d. 15 March 1962 in Berkeley, California.

\(^{13}\)See listing in Appendix F and instructions in Appendix G.

\(^{14}\)See listing in Appendix F and instructions in Appendix G.
Chapter 8

Potential Theory

In Eq. (6.77), we found that, in a region where the charge and current densities are zero, both the scalar electrostatic potential and the components of the vector magnetostatic potential satisfy Laplace’s equation,

\[ \nabla^2 V = 0 \quad (8.1) \]

where the Laplacian operator \( \nabla^2 \) in different coordinate systems is given in Eqs. (2.51)–(2.53). The body of knowledge relating to methods for solving Eq. (8.1) and to properties of the resulting solutions is known as potential theory and is the topic of this chapter. We shall illustrate the general theory exclusively with problems in electrostatics, but the theory itself can be applied wherever Laplace’s equation appears. Such diverse areas of physics as steady-state heat flow, static deflections of elastic membranes, and irrotational fluid flow can thus be studied by the methods here treated. In Sections 8.1 and 8.2, we shall discuss how problems involving Laplace’s equation must be formulated in order to be unambiguous and completely stated. In the next six sections, we shall examine various direct and indirect methods for finding solutions to these problems. Finally, we shall devote a section to Poisson’s equation, obtained by replacing the right-hand side of Eq. (8.1) with a nonzero inhomogeneity. We shall leave a consideration of the equations satisfied by the time-dependent potentials to Chapter 14.

8.1 Boundary Conditions

No problem involving a differential equation is fully stated unless the equation is supplemented with conditions that stipulate the behavior of the solution at the boundaries of the region in which a solution is sought. One common boundary condition imposed on the electrostatic potential specifies the value of the potential at all points on the boundary and is called a Dirichlet boundary condition. If the boundary happens to be defined by the surface of a conductor (which is an equipotential surface when the field is static), then the potential must assume a constant value on the boundary. Sometimes it is sufficient to insist

---

1 Pierre-Simon Laplace; see footnote 15 in Chapter 4, page 110.
2 Siméon Denis Poisson; see footnote 14 in Chapter 4, page 110.
3 German mathematician Peter Gustav Lejeune Dirichlet, b. 13 February 1805 in Düren, French Empire; d. 5 May 1859 in Göttingen, Hanover, Prussia.
merely that the conductor be an equipotential, leaving the specific value of its potential to be determined as part of the problem.

A different boundary condition is imposed on a solution to Laplace’s equation when the charge density $\sigma$ at points on a conducting boundary is known. We found in P4.20 that this charge density is given by

$$\sigma = -\epsilon_0 \nabla V \cdot \hat{n} = -\epsilon_0 \frac{\partial V}{\partial n}$$  \hspace{1cm} (8.2)

where $\hat{n}$ is a unit vector normal to the boundary and directed into the volume in which the solution to Laplace’s equation is sought, and $\partial V/\partial n = \hat{n} \cdot \nabla V$ is the normal derivative of $V$. Equation (8.2) is meaningful only when evaluated at points on the boundary. When $\sigma$ is known initially, Eq. (8.2) determines values that must be imposed on the normal derivative of the solution at the boundary. Conditions imposed on the normal derivative are called Neumann boundary conditions.\(^4\) In reverse, Eq. (8.2) can be used to determine charge densities on a conducting surface if the potential has been found by other means.

Other types of boundary conditions are more involved. Mixed boundary conditions specify values on the boundary for the combination $aV + b \partial V/\partial n$, where $a$ and $b$ do not depend on $V$ but may vary from point to point on the boundary. Asymptotic boundary conditions require $V$ or perhaps something determined from $V$ to approach some given limiting form as the field point is moved to infinity. In most cases $V$ and its first derivatives must be everywhere finite, and this requirement sometimes plays the role of a boundary condition. Finally, when we extend our formalism to include matter (Chapters 9–13), internal boundaries appear at the interface between two different kinds of matter (or between matter and vacuum), and we shall need to develop boundary conditions relating the fields and potentials on one side of the interface to those on the other side of the interface.

We shall define two additional terms: (1) A boundary value problem is any problem whose statement involves a differential equation and associated boundary conditions, and (2) a homogeneous boundary condition is a boundary condition that requires some linear combination of $V$ and $\partial V/\partial n$, e.g., $V$, $\partial V/\partial n$, or $aV + b \partial V/\partial n$, to be (or approach) zero on the boundary.

### 8.2 Superposition and Uniqueness

In this section we shall describe two important general properties of solutions to Laplace’s equation. The first property is stated in the theorem of superposition, whose proof we leave to P8.1: If $V_1$ and $V_2$ satisfy Laplace’s equation and $a$ and $b$ are constants, then $V = aV_1 + bV_2$ also satisfies the equation. At a point $P$ on the boundary, $V(P) = aV_1(P) + bV_2(P)$. Thus, $V$ will not in general satisfy the same boundary conditions as $V_1$ and $V_2$. A sufficient (but not necessary) condition to assure that $V$ will satisfy the same boundary conditions as $V_1$ and $V_2$ is that homogeneous conditions of the same type be imposed on $V_1$ and $V_2$.

A second important property of solutions to Laplace’s equation is contained in the uniqueness theorem: The solution to Laplace’s equation subject to Dirichlet boundary conditions is unique; the solution to Laplace’s equation subject to Neumann boundary conditions

\(^4\) German professor of mineralogy and physics Carl Gottfried Neumann, b. 7 May 1832 in Königsberg, Prussia; d. 27 March 1925 in Leipzig, Germany.
is unique to within an additive constant. To prove this theorem, we assume two solutions $V_1$ and $V_2$ to a particular boundary value problem and then show that at worst the solutions can differ by an additive constant. Suppose (1) that
\[ \nabla^2 V_1 = 0 \quad ; \quad \nabla^2 V_2 = 0 \] (8.3)
in some region $R$ and (2) that $V_1$ and $V_2$ both satisfy either the Dirichlet conditions
\[ V_1 = f(r) \quad ; \quad V_2 = f(r) \quad ; \quad f \text{ given} \] (8.4)
or the Neumann conditions
\[ \nabla V_1 \cdot \hat{n} = g(r) \quad ; \quad \nabla V_2 \cdot \hat{n} = g(r) \quad ; \quad g \text{ given} \] (8.5)
at points $r$ on the boundary of $R$. Then the function $V = V_1 - V_2$ satisfies
\[ \nabla^2 V = 0 \] (8.6)
subject either to
\[ V = 0 \] (8.7)
or to
\[ \nabla V \cdot \hat{n} = 0 \] (8.8)
on the boundary. From Eq. (8.6) and Eq. (C.11), we now find that
\[ \nabla \cdot (V \nabla V) = |\nabla V|^2 + V \nabla^2 V = |\nabla V|^2 \] (8.9)
Thus, upon integrating Eq. (8.9) over the volume of the region $R$ and using the divergence theorem, we obtain finally that
\[ \oint V \nabla V \cdot dS = - \oint V \nabla V \cdot \hat{n} |dS| = \int |\nabla V|^2 dv \] (8.10)
where the minus sign appears because $dS$ has the direction of the outward normal to the surface bounding $R$ while $\hat{n}$ in Eq. (8.8) was tacitly taken to point inward. Whether Eq. (8.7) or Eq. (8.8) applies, however, the surface integral in Eq. (8.10) is zero, and we find that
\[ \int |\nabla V|^2 dv = 0 \] (8.11)
Since the integrand in Eq. (8.11) is necessarily nonnegative, the integral can be zero only if
\[ \nabla V = 0 \] (8.12)
at all points in $R$. Finally, we conclude from Eq. (8.12) that
\[ V = \text{constant} \implies V_1 = V_2 + \text{constant} \] (8.13)
since $V = V_1 - V_2$. [See Eq. (4.48) and the associated text.] Equation (8.13) establishes the uniqueness theorem for Neumann conditions; the theorem is established for Dirichlet conditions by noting that the condition $V_1 = V_2$ on the boundary requires the constant in Eq. (8.13) to be zero.

The assurance that the solution to a completely stated problem is unique is especially important to some methods of solving electrostatic problems. Sometimes we are forced to guess a solution and then show that this guessed solution satisfies Laplace’s equation and the associated boundary conditions. Without uniqueness, we could never be sure that such a guessed solution was the only solution. Furthermore, the uniqueness theorem helps us to understand the nature of the boundary conditions that must be specified in order that a problem in electrostatics be well stated.
PROBLEMS

P8.1. Prove the theorem of superposition. *Hint:* Assume \( V_1 \) and \( V_2 \) satisfy Laplace’s equation and show that \( aV_1 + bV_2 \), where \( a \) and \( b \) are constants, also satisfies the equation.

P8.2. (a) Prove that the solution to Laplace’s equation is unique if the solution is required to have a specified value over part of the boundary and a specified normal derivative over the remainder of the boundary. (b) A solution to the two-dimensional Laplace equation

\[
\nabla^2 V = 0
\]

is desired in the interior of the square \( |x| \leq a, |y| \leq a \). On the boundary, the solution is required to assume the values shown in Fig. 8.1(a). (1) Show that the solution is invariant to reflection in the \( x \) axis and also in the \( y \) axis. (2) Show that the solution to the problem in Fig. 8.1(b) is identical to the solution in the first quadrant in Fig. 8.1(a).

P8.3. In the presence of any static external field, the entire volume occupied by a solid conductor of arbitrary shape is an equipotential (Section 4.6), and the (constant) potential \( V \) in this region then satisfies \( \nabla^2 V = 0 \) subject to the requirement that \( V \) be constant on the surface of the conductor. Show that this same mathematical statement applies if the solid conductor is replaced by a thin conducting shell enclosing the same region of space and then use the uniqueness theorem in an argument showing that the region enclosed by the shell is field-free. The phenomenon of this problem makes possible the electrostatic shielding of a region of space by enclosing it in a conducting shell. Indeed, a conducting shell made of screen wire is often adequate.

P8.4. Show that maximum and minimum values of the solution to \( \nabla^2 V = 0 \) can occur only on the boundaries of the region \( R \) to which the solution applies. *Hint:* Assume a maximum or minimum in \( V \) at the point \( P \) in the interior of \( R \). What would the electric field look like in the vicinity of \( P \)? Now review P4.14.
8.3 One-Dimensional Problems

For some problems, symmetry arguments (or perhaps other arguments) can demonstrate that the electrostatic potential depends only on one of the three coordinates in the relevant coordinate system. Such a problem is essentially a one-dimensional problem, and stipulation of the boundary conditions reduces, for example, to specification of the value of \( V \) at two values of the one coordinate on which \( V \) depends. A potential depending only on one Cartesian coordinate \( x \) satisfies

\[
\frac{d^2V}{dx^2} = 0 \quad \Rightarrow \quad V = ax + b
\]  

(8.14)

while a potential depending only on the cylindrical variable \( \tau \) satisfies

\[
\frac{1}{\tau} \frac{d}{d\tau} \left( \tau \frac{dV}{d\tau} \right) = 0 \quad \Rightarrow \quad V = a \ln \tau + b
\]  

(8.15)

and a potential depending only on the spherical coordinate \( r \) satisfies

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = 0 \quad \Rightarrow \quad V = \frac{a}{r} + b
\]  

(8.16)

In each solution, the constants \( a \) and \( b \) are determined by the boundary conditions. Verification of these solutions is left to the reader. (See also P6.30, P6.31, and P6.32.)

8.4 Two-Dimensional Problems by Separation of Variables

Two-dimensional problems, in which dependence on only one of the three spatial variables can be ruled out \textit{ab initio}, are more interesting. The most common of these are problems in Cartesian coordinates that involve only \( x \) and \( y \), problems in cylindrical coordinates that involve only \( \tau \) and \( \phi \), problems in cylindrical coordinates that involve only \( \tau \) and \( z \), and problems in spherical coordinates that involve only \( r \) and \( \theta \). The treatment in this section is illustrative rather than exhaustive.

8.4.1 Laplace’s Equation in 2D Cartesian Coordinates \( x, y \)

In Cartesian coordinates, Laplace’s equation in the two variables \( x \) and \( y \) assumes the form

\[
\frac{\partial^2V}{\partial x^2} + \frac{\partial^2V}{\partial y^2} = 0
\]  

(8.17)

[See Eq. (2.51).] Without having any initial assurance that it will work, we adopt the method of \textit{separation of variables} and try a solution of the form

\[
V(x, y) = X(x) Y(y)
\]  

(8.18)

i.e., a product of functions of each variable separately. If the function in Eq. (8.18) is to satisfy Eq. (8.17), the functions \( X(x) \) and \( Y(y) \) must satisfy the condition

\[
\frac{1}{X} \frac{d^2X}{dx^2} = -\frac{1}{Y} \frac{d^2Y}{dy^2}
\]  

(8.19)
obtained by substituting Eq. (8.18) into Eq. (8.17) and rearranging the result. In Eq. (8.19), however, the two independent variables \( x \) and \( y \) have been separated, each side of Eq. (8.19) depending on only one of these variables. Since variation of \( x \), for example, cannot change the value of the side depending on \( y \), the side depending on \( x \) must in fact be a constant.

Thus, we can extract the two ordinary differential equations

\[
\frac{1}{X} \frac{d^2X}{dx^2} = k \quad ; \quad \frac{1}{Y} \frac{d^2Y}{dy^2} = -k
\]

from Eq. (8.19). These two equations are coupled only because the same separation constant \( k \) appears in both equations. The solution of Eq. (8.21) is now immediate, viz.,

\[
X = A e^{\lambda x} + B e^{-\lambda x} \quad ; \quad Y = A' \sin \lambda y + B' \cos \lambda y
\]

(8.22)

where \( \lambda = \sqrt{k} \). Finally, on substitution of Eq. (8.22) into Eq. (8.18), we find that

\[
V_\lambda(x, y) = (A e^{\lambda x} + B e^{-\lambda x})(A' \sin \lambda y + B' \cos \lambda y)
\]

(8.23)

where the subscript \( \lambda \) has been added as a reminder of the dependence on a parameter that can assume any value whatever (including complex values) without destroying the basic property that Eq. (8.23) is a solution of Eq. (8.17). Our tentative assumption of a solution in product form has thus yielded not just a single solution but an entire family of solutions.

A more explicit evaluation of the integration constants \( A, A', B, \) and \( B' \) and of the separation constant \( \lambda \) appearing in Eq. (8.23) requires a more specific problem, for these constants are determined by the boundary conditions. Sometimes the boundary conditions can be satisfied by a single value of \( \lambda \). More often several values of \( \lambda \) are consistent with a portion of the boundary conditions and several “product” solutions corresponding to different values of \( \lambda \) must be superposed to find a solution fitting the rest of the boundary conditions. Suppose, for example, that a solution to Laplace’s equation is sought in the region interior to the rectangle of Fig. 8.2 and that \( V \) is to assume the indicated values on the edges of this rectangle. Consider the three homogeneous boundary conditions first; we find that \( V_\lambda(x, 0) = 0 \), which implies \( B' = 0 \), and that \( V_\lambda(0, y) = 0 \), which implies \( A = -B \).

Thus, Eq. (8.23) reduces to

\[
V_\lambda(x, y) = A'' \sinh \lambda x \sin \lambda y
\]

(8.24)

The remaining homogeneous boundary condition requires that \( V_\lambda(x, b) = 0 \), which requires in turn that \( \sin \lambda b = 0 \) and hence that \( \lambda b = n\pi \), with \( n \) a positive integer (not including zero). We have therefore found a countable infinity of solutions

\[
V_n(x, y) = A_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}, \quad n = 1, 2, \cdots
\]

(8.25)

all of which are consistent with the three homogeneous boundary conditions. A more general solution \( V(x, y) \) is now obtained by superposing these simpler solutions; we find that

\[
V(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}
\]

(8.26)

\[5\] The hyperbolic function \( \sinh \lambda x \) is defined to be \( \frac{1}{2}(e^{\lambda x} - e^{-\lambda x}) \). We may also need the hyperbolic function \( \cosh \lambda x \) defined by \( \cosh \lambda x = \frac{1}{2}(e^{\lambda x} + e^{-\lambda x}) \).
which clearly satisfies the three homogeneous boundary conditions and which will satisfy the final (inhomogeneous) boundary condition if

\[ V(a, y) = V_0 = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} \]  

(8.27)

This series, however, is the Fourier sine series expansion for the constant \( V_0 \) over the interval \( 0 < y < b \). (See PD.16 in Appendix D.) Hence, the constant \( A_n \sinh(n\pi a/b) \) must match the coefficients in that Fourier series.\(^6\) Once these constants have been found (P8.5), Eq. (8.26) expresses a solution to Laplace’s equation subject to the given boundary conditions. By uniqueness, this series therefore expresses the solution to the problem.

### 8.4.2 Laplace’s Equation in 2D Cylindrical Coordinates \( \tau, \phi \)

For solutions that depend only on \( \tau \) and \( \phi \) in cylindrical coordinates, Laplace’s equation assumes the form

\[
\frac{1}{\tau} \frac{\partial}{\partial \tau} \left( \tau \frac{\partial V}{\partial \tau} \right) + \frac{1}{\tau^2} \frac{\partial^2 V}{\partial \phi^2} = 0
\]  

(8.28)

[See Eq. (2.52).] If we employ the method of separation of variables and further insist that the solutions be periodic in \( \phi \) with period \( 2\pi \), we find ultimately—see P8.8—that the most general solution can be expressed as the infinite series

\[
V(\tau, \phi) = a_0 \ln \tau + b_0 + \sum_{n=1}^{\infty} \tau^n (a_n \cos n\phi + b_n \sin n\phi) \\
+ \sum_{n=1}^{\infty} \tau^{-n} (c_n \cos n\phi + d_n \sin n\phi)
\]  

(8.29)

\(^6\)French mathematician and physicist Jean-Baptiste Joseph Fourier, b. 21 March 1768 in Auxerre, Burgundy, France (now Yonne, France); d. 16 May 1830 in Paris, France.
where the constants $a_0, b_0, a_n, b_n, c_n,$ and $d_n$ are determined by the boundary conditions. The requirement of periodicity imposed on this solution restricts its utility to problems in which the full angular range $0 \leq \phi \leq 2\pi$ is assumed; modifications will be necessary if the solution need not have the periodicity required of Eq. (8.29). Further, if the origin happens to be in the domain of the problem, the constants $c_n$ and $d_n$ for all $n$ and also the constant $a_0$ must be zero to suppress terms that diverge at $r = 0$. Note finally that Eq. (8.29) with all constants except $a_0$ equal to zero reduces to the potential of a line charge (P4.19).

8.4.3 Laplace’s Equation in 2D Cylindrical Coordinates $\tau, z$

For solutions that depend only on $\tau$ and $z$ in cylindrical coordinates, Laplace’s equation assumes the form

$$\frac{1}{\tau} \frac{\partial}{\partial \tau} \left( \tau \frac{\partial V}{\partial \tau} \right) + \frac{\partial^2 V}{\partial z^2} = 0 \quad (8.30)$$

[see Eq. (2.52)] and, again using the method of separation of variables, we find ultimately that a basic product solution has the form

$$V_\lambda(\tau, z) = [A e^{\lambda z} + B e^{-\lambda z}] \left[ C J_0(\lambda \tau) + D Y_0(\lambda \tau) \right] \quad (8.31)$$

where $\lambda$ is a separation constant that may assume any value (including complex values); $A$, $B$, $C$, and $D$ are integration constants; and $J_0(x)$ and $Y_0(x)$ are two conventionally defined, linearly independent solutions to the zeroth order Bessel equation,

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + xy = 0 \quad (8.32)$$

The functions $J_0(x)$ and $Y_0(x)$ are called the zeroth-order Bessel and Neumann functions, respectively, and are shown graphically in Fig. 8.3 for real values of $x$. In most problems, $V$ must approach zero as $z \to 0$ so the increasing exponential in Eq. (8.31) must be suppressed. Further, if the origin $\tau = 0$ is included in the domain of the problem, the Neumann function $Y_0$ must be eliminated because it diverges at $\tau = 0$, thereby giving an unphysical potential. Under these additional conditions, the most general acceptable solution in product form is

$$V_\lambda(\tau, z) = C' e^{-\lambda |z|} J_0(\lambda \tau) \quad (8.33)$$

A possible superposition of these solutions is then expressed by the integral

$$V(\tau, z) = \int_0^\infty C'(\lambda) e^{-\lambda |z|} J_0(\lambda \tau) \, d\lambda \quad (8.34)$$

where $C'(\lambda)$ would be determined by additional boundary conditions. Equation (8.31) is derived in P8.11, and one application of Eq. (8.34) is explored in P8.13.

8.4.4 Laplace’s Equation in 2D Spherical Coordinates $r, \theta$

Consider now solutions depending on the spherical variables $r$ and $\theta$. The relevant form of Laplace’s equation is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0 \quad (8.35)$$

German mathematician Friedrich Bessel, b. 22 July 1784 in Minden, Minden-Ravensberg (now Germany); d. 17 March 1846 in Königsberg, Prussia (now Kaliningrad, Russia).
Figure 8.3: $J_0(x)$ and $Y_0(x)$ versus $x$. Both functions are oscillatory with decaying amplitude as $x$ increases. The first few zeroes of $J_0(x)$ occur at $x = 2.405, 5.520, 8.654, 11.792, 14.931,$ and $18.071$.

[see Eq. (2.53)] and separation of variables by writing $V(r, \theta) = R(r) \Theta(\theta)$ leads ultimately to the two equations

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - kR = 0 \tag{8.36}
\]
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + k\Theta = 0 \tag{8.37}
\]

(See P8.14.) Consider Eq. (8.37) first. The variable transformation $x = \cos \theta$ reduces this equation to the form

\[
(1 - x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} + k\Theta = 0 \tag{8.38}
\]

which is Legendre’s equation. As with every second-order linear differential equation, Legendre’s equation has two linearly independent solutions. For Eq. (8.38), however, it happens that both of these solutions diverge at $x = \pm 1$ ($\theta = 0, \pi$) unless $k$ has one of the values given by $n(n+1)$, where $n$ is a nonnegative integer. Even when $k$ has one of the specified values, one of the two solutions still diverges at $x = \pm 1$ ($\theta = 0, \pi$); those solutions that do not diverge are the Legendre polynomials, conventionally denoted $P_n(x)$. (The first few of these polynomials are plotted in Fig. 8.4 and several of their properties are summarized in Table 8.1. Thus, whenever the values $\theta = 0$ and/or $\theta = \pi$ are included in the domain of the physical problem, we conclude (1) that the only solutions to Eq. (8.37) that are finite everywhere in the domain of the problem are the Legendre polynomials and (2) that the physically meaningful values of the separation constant $k$ are given in terms of a nonnegative integer $n$ by $n(n+1)$. The radial equation then becomes

\[
\frac{d}{dr} \left( r^2 \frac{dR_n}{dr} \right) - n(n+1)R_n = 0 \tag{8.39}
\]

---

8 French mathematician Adrien-Marie Legendre, b. 18 September 1752 in Paris, France; d. 10 January 1833 in Paris, France.
Figure 8.4: $P_n(\cos \theta)$ versus $\theta$ for several values of $n$.

which has the solution

$$R_n = a_n r^n + \frac{b_n}{r^{n+1}} \tag{8.40}$$

(P8.14). Combining each solution for $R_n$ with the corresponding Legendre polynomial and superposing these products for all possible values of $n$, we find the sum

$$V(r, \theta) = \sum_{n=0}^{\infty} \left( a_n r^n + \frac{b_n}{r^{n+1}} \right) P_n(\cos \theta) \tag{8.41}$$

which expresses a general solution to Laplace’s equation for the conditions assumed in this paragraph. Modifications will be necessary if the values $\theta = 0$ and $\pi$ are actually excluded from the domain of the problem. Further, if the origin happens to be in the domain of the problem, $b_n$ must be zero for all $n$ to suppress terms that diverge at $r = 0$.

Equation (8.41) reduces to several familiar results if the constants are given particular values. The constant potential is obtained by setting all constants except $a_0$ equal to zero; with all constants except $b_0$ equal to zero and $b_0 = Q/4\pi\epsilon_0$, Eq. (8.41) reduces to the potential of a point charge; setting $b_1 = p/4\pi\epsilon_0$ and all other constants equal to zero produces the potential of a point dipole with dipole moment $p = P \hat{k}$; the potential of a linear quadrupole emerges if all constants except $b_2$ are zero; the potential of a uniform field emerges if only $a_1$ is nonzero; and so on.

As an example of the use of Eq. (8.41) to determine the electrostatic potential under given circumstances, let us determine the potential established when an uncharged, conducting sphere is placed in a uniform external electric field. If we take the direction of the electric field to define the polar axis so that

$$E = E_0 \hat{k} \tag{8.42}$$
Table 8.1: Legendre Polynomials and Their Properties.

<table>
<thead>
<tr>
<th>$P_0(x)$</th>
<th>$P_1(x)$</th>
<th>$P_2(x)$</th>
<th>$P_3(x)$</th>
<th>$P_4(x)$</th>
<th>$P_5(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$x$</td>
<td>$\frac{1}{2}(3x^2 - 1)$</td>
<td>$\frac{1}{2}(5x^2 - 3x)$</td>
<td>$\frac{1}{8}(35x^4 - 30x^2 + 3)$</td>
<td>$\frac{1}{8}(63x^5 - 70x^3 + 15x)$</td>
</tr>
<tr>
<td>$P_0(x) = 1$</td>
<td>$P_1(x) = x$</td>
<td>$P_2(x) = \frac{1}{2}(3x^2 - 1)$</td>
<td>$P_3(x) = \frac{1}{2}(5x^2 - 3x)$</td>
<td>$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$</td>
<td>$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$</td>
</tr>
</tbody>
</table>

\[
\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n
\]

\[
P_n(-x) = (-1)^n P_n(x)
\]

\[
(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0
\]

\[
(1 - x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n + 1)P_n = 0
\]

\[
\int_{-1}^{1} P_n(x) P_m(x) dx = \int_{0}^{\pi} P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = \frac{2}{2n + 1} \delta_{nm}
\]

$P_n(1) = 1$  $P_n(-1) = (-1)^n$

$P_1(x) = 0$  at  $\left\{ \begin{array}{l} x = 0.000 \\ \theta = 90.0^\circ \end{array} \right.$

$P_2(x) = 0$  at  $\left\{ \begin{array}{l} x = 0.577 \\ \theta = 54.7^\circ \end{array} \right.$  $\left\{ \begin{array}{l} x = -0.577 \\ \theta = 125.3^\circ \end{array} \right.$

$P_3(x) = 0$  at  $\left\{ \begin{array}{l} x = 0.774 \quad 0.000 \quad -0.774 \\ \theta = 39.2^\circ \quad 90.0^\circ \quad 140.8^\circ \end{array} \right.$

$P_4(x) = 0$  at  $\left\{ \begin{array}{l} x = 0.861 \quad 0.340 \quad -0.340 \quad -0.861 \\ \theta = 30.6^\circ \quad 70.1^\circ \quad 109.9^\circ \quad 149.4^\circ \end{array} \right.$

$P_5(x) = 0$  at  $\left\{ \begin{array}{l} x = 0.906 \quad 0.538 \quad -0.000 \quad -0.538 \quad -0.906 \\ \theta = 25.0^\circ \quad 57.4^\circ \quad 90.0^\circ \quad 122.6^\circ \quad 155.0^\circ \end{array} \right.$

and further if we place the origin at the center of the sphere, the entire problem is invariant to rotation about the $z$ axis and the potential therefore cannot depend on the spherical coordinate $\phi$. Several conditions must be satisfied by an acceptable solution: (1) In the region far from the sphere, the sphere will have little effect on the potential and hence

\[
\lim_{r \to \infty} V(r, \theta) = -E \cdot r = -E_0 r \cos \theta = -E_0 r P_1(\cos \theta)
\]

which is the potential of a uniform field (P4.18); (2) the potential on the surface of the sphere must be constant; (3) the sphere must be uncharged; (4) there can be no dependence on $\phi$; (5) the solution must be finite everywhere, in particular at $\theta = 0$ and $\theta = \pi$; and (6) the
potential in the region exterior to the sphere must satisfy Laplace’s equation. Conditions (4)–(6) restrict the most general potential in the region outside the sphere to be given by Eq. (8.41). Equation (8.41) will match Eq. (8.43) at large \( r \), however, only if \( a_1 = -E_0 \) and the rest of the \( a_n \)'s (including \( a_0 \)) are zero. Thus, the asymptotic boundary condition reduces the most general solution to

\[
V(r, \theta) = b_0 r + \left( \frac{b_1}{r^2} - E_0 r \right) P_1(\cos \theta) + \sum_{n=2}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos \theta) \tag{8.44}
\]

where the terms multiplied by \( b_0 \) and \( b_1 \) have been written explicitly. If \( V \) is to be constant on the sphere, say \( r = a \), we must then require that

\[
V(a, \theta) = b_0 a + \left( \frac{b_1}{a^2} - E_0 a \right) P_a(\cos \theta) + \sum_{n=2}^{\infty} \frac{b_n}{a^{n+1}} P_n(\cos \theta) \tag{8.45}
\]

be independent of \( \theta \), which in turn requires that \( b_1 = E_0 a^3 \) and \( b_n = 0 \) for \( n \geq 2 \). (Why? \textit{Hint:} The \( P_n \)'s are linearly independent.) Thus, Eq. (8.44) reduces even further to

\[
V(r, \theta) = \frac{b_0}{r} + E_0 r \left( \frac{a^3}{r^3} - 1 \right) \cos \theta \tag{8.46}
\]

The condition that the conducting sphere be uncharged now determines the remaining constant \( b_0 \). From Eq. (8.2), we find that the charge density \( \sigma \) at points on the surface of the sphere is given by

\[
\sigma(\theta) = -\epsilon_0 \nabla V \cdot \hat{r} \bigg|_{r=a} = -\epsilon_0 \frac{\partial V}{\partial r} \bigg|_{r=a} = \epsilon_0 \left( \frac{b_0}{a^2} + 3E_0 \cos \theta \right) \tag{8.47}
\]

The total charge \( Q \) on the sphere is then given by integrating \( \sigma \) over the sphere,

\[
Q = \int_{\text{sphere}} \sigma \, dS = \int_0^{2\pi} \int_0^\pi \epsilon_0 \left( \frac{b_0}{a^2} + 3E_0 \cos \theta \right) a^2 \sin \theta \, d\theta \, d\phi = 4\pi \epsilon_0 b_0 \tag{8.48}
\]

and will be zero only if \( b_0 = 0 \). Thus, Eq. (8.47) reduces to

\[
\sigma(\theta) = 3\epsilon_0 E_0 \cos \theta \tag{8.49}
\]

and Eq. (8.46) becomes

\[
V(r, \theta) = -E_0 r \left( 1 - \frac{a^3}{r^3} \right) \cos \theta \tag{8.50}
\]

Further properties of this field are explored in P8.16.

The method of separation of variables can also be applied to solve problems in three dimensions, but the mathematics of such problems in most coordinate systems is considerably more involved than that for two-dimensional problems. A few examples may be found in P8.39, P8.40, and P8.41.

**PROBLEMS**

P8.5. Complete the solution of the problem in Fig. 8.2 by finding the coefficients \( A_n \) in Eq. (8.27). \textit{Hint:} See PD.16 in Appendix D.
P8.6. Use the method of separation of variables to find a series solution for the problem defined by Fig. 8.2 when, however, the potential on the right-hand edge \((x = a)\) increases linearly from zero at \((a, 0)\) to \(V_0\) at \((a, \frac{1}{2}b)\) and then decreases linearly back to zero at \((a, b)\). The solution in Eq. (8.27), of course, applies also to this case but you are asked to write out a solution completely starting from Laplace’s equation, the boundary conditions, and the assumption of a solution in a product form. (Please think carefully about the entire process.) Be sure to find values of the constants in the series expansion and then use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to obtain both contour plots and a surface plot of the potential in the \(xy\) plane.

P8.7. Find the electrostatic potential in the interior of a square region of side \(a\) if two adjacent sides are at zero potential and the other two sides are maintained at a potential \(V_0\). 
*Hint:* Regard the given problem as a superposition of two problems, each of which has homogeneous boundary conditions on three sides of the square.

P8.8. Find the general solution, Eq. (8.29), for a potential depending on the cylindrical coordinates \(r\) and \(\phi\) by applying separation of variables to Eq. (8.28).

P8.9. An infinitely long, uncharged, conducting cylinder of radius \(a\) is placed in a uniform external electric field directed perpendicular to the axis of the cylinder. Find the resulting electrostatic potential and determine the charge density on the surface of the cylinder. 
*Hint:* Let the axis of the cylinder coincide with the \(z\) axis and let the field be in the \(x\) direction.

P8.10. Determine the electrostatic potential in the region interior to a \(90^\circ\) conducting wedge if the two straight sides are at zero potential and the \(90^\circ\) arc at radius \(a\) is maintained at a constant potential \(V_0\).

P8.11. Find the product form, Eq. (8.31), for a potential depending on the cylindrical coordinates \(r\) and \(z\) by applying separation of variables to Eq. (8.30).

P8.12. Assume a power series solution of the form \(\sum_{n=0}^{\infty} \alpha_n x^n\) to the zeroth-order Bessel equation, Eq. (8.32), find the conditions imposed on the \(\alpha_n\)'s by Eq. (8.32), and find a solution if \(y(0)\) is required to be one. The result is a power series representation of \(J_0(x)\). Can you explain why this method does not yield two linearly independent solutions to the zeroth-order Bessel equation?

P8.13. A circular conducting disc of radius \(a\) lies in the \(xy\) plane with its center at the origin and is charged to a potential \(V_0\). Find the potential at all points in space, find and sketch the charge distribution on the disc as a function of \(r\), and find the total charge \(Q\) on the disc. 
*Hints:* (1) Use Eq. (8.34) and employ the identity

\[
\int_0^\infty J_0(\alpha x) \frac{\sin \beta x}{x} \, dx = \begin{cases} \pi, & \alpha < \beta \\ \frac{\pi}{\alpha}, & \alpha > \beta \end{cases}
\]

Note the amazingly simple value for this apparently complicated integral. In particular, the integral is constant for \(\alpha < \beta\). (2) The similar identity

\[
\int_0^\infty J_0(\alpha x) \sin \beta x \, dx = \begin{cases} \frac{1}{\sqrt{\beta^2 - \alpha^2}}, & \alpha < \beta \\ 0, & \alpha > \beta \end{cases}
\]


P8.14. (a) Find the two equations, Eq. (8.36) and (8.37), into which Laplace’s equation for a potential depending on the spherical coordinates \(r\) and \(\theta\) separates when a solution in
product form is assumed. (b) Accepting the separation constant $k = n(n + 1)$ as determined from the angular equation, solve the radial equation to find Eq. (8.40). Hint: Guess a solution $R_n$ for the form $r^p$. Why might you expect this guess to work? Optional: Assume a power series solution of the form $\sum_{n=0}^{\infty} \alpha_n x^{n+c}$ to Eq. (8.38); find the conditions that Eq. (8.38) imposes on $c$ and the $\alpha_n$'s; find power series representing two basic solutions $\Theta_{\text{odd}}$ and $\Theta_{\text{even}}$ defined by $\Theta_{\text{odd}}(0) = 0$, $d\Theta_{\text{odd}}(0)/dx = 1$ and $\Theta_{\text{even}}(0) = 1$, $d\Theta_{\text{even}}(0)/dx = 0$; show that these series in general diverge at $x = \pm 1$ but that at least one of them terminates (and hence converges) when $k = n(n + 1)$; and finally show that the terminating series is proportional to the corresponding Legendre polynomial.

P8.15. Use the recurrence relation in Table 8.1 to obtain $P_5(x)$ from $P_3(x)$ and $P_4(x)$.

P8.16. (a) Show that the potential in Eq. (8.50) is the potential of a dipole plus that of the uniform field and infer the dipole moment to assign to the conducting sphere. (b) Calculate the dipole moment of the sphere directly from the charge distribution of Eq. (8.49) and compare your result with that obtained in part (a). (c) Determine the amount of charge $Q$ on the positively charged hemisphere of this sphere and calculate the separation $s$ between two charges of this magnitude, one positive and the other negative, in order that the combination will have the same dipole moment as the actual sphere. (d) Calculate the field from the potential and show explicitly that the field is perpendicular to the sphere at points on the surface of the sphere. (e) Sketch a graph showing the field lines and equipotential surfaces.

P8.17. Find the potential established when a conducting sphere is placed in a uniform electric field if the sphere carries charge $Q$. Find also the charge density on the surface of the sphere.

P8.18. Consider a circular ring of radius $a$ carrying charge $Q$ distributed uniformly about its perimeter and lying in the $xy$ plane with its center at the origin. In spherical coordinates, the potential established by this ring is independent of $\phi$, $V = V(r, \theta)$ (Why?), and in particular on the axis ($\theta = 0$), $V(r, 0) = Q/(4\pi \varepsilon_0 \sqrt{a^2 + r^2})$ (P4.57). Further, except at points on the ring, $\nabla^2 V$ is zero. (Why?) (a) Find the first four terms in the binomial expansion of $V(r, 0)$ in powers of $a/r$. (b) Show that Eq. (8.41) applies and find the values of the coefficients $a_n$ and $b_n$ by requiring Eq. (8.41) to agree with part (a) at points on the $z$ axis. The resulting series is a multipole expansion of the potential established by the ring and is valid both on and off the $z$ axis. Similar techniques can be used to find off-axis values for the potential established by a uniformly charged circular disc (P4.26) and for the magnetic scalar potential (P5.28) of a circular current loop, discussed by D. R. Corson and P. Lorrain, Introduction to Electromagnetic Fields and Waves (W. H. Freeman and Company, San Francisco, 1962), pp. 207-209.

8.5 Two-Dimensional Problems Using Complex Variables

Another method for solving two-dimensional potential problems exploits the properties of what are called analytic functions of a complex variable, $z = x + iy$.\(^9\) Any function $f(z)$ of the variable $z$ is also a (complex) function of the real variables $x$ and $y$ and can itself be separated into real and imaginary parts, i.e.,

$$f(z) = V_1(x, y) + iV_2(x, y)$$  \hfill (8.51)

\(^9\)The properties of complex numbers are summarized in Appendix D. Do not confuse the complex variable $z$ with the Cartesian coordinate $z$. 
where \( V_1 \) and \( V_2 \) are both \textit{real} functions of the \textit{real} variables \( x \) and \( y \). Thus, for example,

\[
f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi
\]

\[
\implies V_1(x, y) = x^2 - y^2; \quad V_2(x, y) = 2xy \tag{8.52}
\]

or

\[
f(z) = \ln z = \ln(ze^{i\phi}) = \ln r + i\phi = \ln \sqrt{x^2 + y^2} + i \tan^{-1} \frac{y}{x}
\]

\[
\implies V_1(x, y) = \ln \sqrt{x^2 + y^2}; \quad V_2(x, y) = \tan^{-1} \frac{y}{x} \tag{8.53}
\]

and so on. The function \( f(z) \) is said to be \textit{analytic} if \( V_1 \) and \( V_2 \) satisfy the \textit{Cauchy-Riemann conditions},\(^{10,11}\)

\[
\frac{\partial V_1}{\partial x} = \frac{\partial V_2}{\partial y}; \quad \frac{\partial V_1}{\partial y} = -\frac{\partial V_2}{\partial x} \tag{8.54}
\]

Most reasonable functions, including \( z^n \) for any (positive or negative) integer \( n \), \( \sin z \), \( \ln z \), \( e^z \), products of analytic functions, sums of analytic functions, and analytic functions of analytic functions in fact \textit{are} analytic, except possibly at isolated points or along particular lines.

Analytic functions are important to two-dimensional potential theory because both the real and imaginary parts of an analytic function \textit{automatically} satisfy Laplace’s equation in two dimensions. We need merely differentiate one member of Eq. (8.54) with respect to \( x \) and the other with respect to \( y \), obtaining

\[
\frac{\partial^2 V_1}{\partial x^2} = \frac{\partial^2 V_2}{\partial x \partial y}; \quad \frac{\partial^2 V_1}{\partial y^2} = -\frac{\partial^2 V_2}{\partial x \partial y} \tag{8.55}
\]

and then recognize the equality of the two mixed second partial derivatives to find that

\[
\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0 \tag{8.56}
\]

The proof for \( V_2 \) is similar. Thus, every analytic function contains \textit{two} solutions to Laplace’s equation. Furthermore, again because of the Cauchy-Riemann conditions,

\[
\nabla V_1 \cdot \nabla V_2 = \left( \frac{\partial V_1}{\partial x} \mathbf{i} + \frac{\partial V_1}{\partial y} \mathbf{j} \right) \cdot \left( \frac{\partial V_2}{\partial x} \mathbf{i} + \frac{\partial V_2}{\partial y} \mathbf{j} \right)
\]

\[
= \frac{\partial V_1}{\partial x} \frac{\partial V_2}{\partial x} + \frac{\partial V_1}{\partial y} \frac{\partial V_2}{\partial y} = 0
\]

\[
\implies \nabla V_1 \perp \nabla V_2 \tag{8.57}
\]

Since the equipotentials of \( V_2 \) are perpendicular to \( \mathbf{E}_2 = -\nabla V_2 \) (Section 4.6), we conclude from Eq. (8.57) that these equipotentials are also tangent to \( \mathbf{E}_1 = -\nabla V_1 \). Similarly, the equipotentials of \( V_1 \) are tangent to \( \mathbf{E}_2 = -\nabla V_2 \). Thus, not only are \( V_1 \) and \( V_2 \) solutions.

\(^{10}\) French mathematician Augustin-Louis Cauchy, b. 21 August 1789 in Paris, France; d. 23 May 1857 in Sceaux, France.

\(^{11}\) German mathematician Bernhard Riemann, b. 17 September 1826 in Breselenz, Hanover (now Germany); d. 20 July 1866 in Selasca, Italy.
to Laplace’s equation in two dimensions, but also they are conjugate solutions in the sense that the field lines of one potential coincide with the equipotentials of the other. It is indeed unfortunate that these elegant properties in two dimensions cannot be extended to three dimensions.

We shall now illustrate one way in which the above properties can be used effectively.\(^{12}\) Suppose \(f(z) = z^2\). The resulting two potentials are then given by Eq. (8.52), and, in particular, the equipotential curves for these two potentials satisfy

\[
x^2 - y^2 = \alpha \quad ; \quad 2xy = \beta
\]

respectively. Here \(\alpha\) and \(\beta\) are constants. In both cases, the equipotentials are hyperbolas. In the first case each hyperbola is bisected by the \(x\) or \(y\) axis, and in the second case each has the axes as asymptotes; the equipotentials for selected values of \(\alpha\) and \(\beta\) are shown in Fig. 8.5(a). Once the equipotentials for a given analytic function are known, we can associate one or more physical situations with the potentials. If, for example, the regions exterior to the hyperbolas for \(\alpha = \pm 4\) are occupied by conductors maintained at the potentials given by \(\alpha\), then the hyperbolas for values of \(\alpha\) satisfying \(|\alpha| \leq 4\) are the equipotentials in the region \(R\) between the conductors, and the hyperbolas for various values of \(\beta\) give the field lines in \(R\) [Fig. 8.5(b)]. The resulting field \(\mathbf{E}_1\) in \(R\) is given by

\[
\mathbf{E}_1 = -\nabla V_1 = -2x \hat{i} + 2y \hat{j}
\]

and is linear in both coordinates; it is used in the so-called quadrupole lens for focusing beams of charged particles. Because the \(x\) and \(y\) axes, which correspond to \(\beta = 0\), are at zero potential, the first quadrant in Fig. 8.5(a) shows the field lines (values of \(\alpha\)) and the equipotentials (values of \(\beta\)) in a 90° conducting wedge maintained at zero potential. It is left as an exercise for the reader to find still other physical problems described by this set of equipotentials.

The approach described in the above paragraphs in a sense attacks problems involving Laplace’s equation by first finding a solution and then seeking the problem that it solves. We could assemble a table of such solutions simply by exploring several analytic functions. We would find that \(f(z) = \ln z\) relates physically to an infinite line charge, \(f(z) = \sqrt{z}\) to a semi-infinite conducting plate, and so on. The method, however, is certainly not a general method and is made only a little more general by the theory of conformal mapping, which is beyond the scope of this book. (See P8.42.) Thus, although this technique is powerful, perhaps even beautiful, when it works, it is not useful for all problems or even for all two-dimensional problems.

**PROBLEMS**

**P8.19.** Verify that the two examples in Eqs. (8.52) and (8.53) satisfy the Cauchy-Riemann conditions, Eq. (8.54).

**P8.20.** Find the real and imaginary parts of \(z^n\) in cylindrical coordinates and recognize that you have generated some of the terms in Eq. (8.29). \textit{Hint}: Write \(z\) in polar form.

\(^{12}\) The treatment in this paragraph is patterned after a similar treatment in *The Feynman Lectures on Physics* by R. P. Feynman, R. B. Leighton, and M. Sands (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1964), Volume II, Lecture 7, and is used here by permission of Addison-Wesley Publishing Company, Inc.
Figure 8.5: Solutions to Laplace’s equation obtained from the analytic function \( f(z) = z^2 \). Part (a) shows the two sets of equipotentials with those for \( \alpha (\beta) = \text{constant} \) being shown dashed (solid); part (b) shows the fields and equipotentials in a quadrupole lens.

**P8.21.** Find the charge density on the conductors in the quadrupole lens of Fig. 8.5(b) at the points where the conducting surfaces intersect the axes.

**P8.22.** Explore the equipotentials and field lines arising from \( f(z) = \ln z \). Include an evaluation of the field corresponding to \( V_1 \) and compare the result with Eq. (4.32) for the field of a line charge. Describe at least one other physical problem that would give rise to these fields.

**P8.23.** Explore the equipotentials and field lines arising from \( f(z) = \sqrt{z} \) and describe at least one physical problem that would give rise to these fields.

### 8.6 The Method of Images

Solutions to Laplace’s equation for systems consisting of conductors and point charges can sometimes be found by the method of images, which works in one, two, and three dimensions. In essence this method involves identifying a second system that contains only simple, known charge distributions (usually point charges) but that establishes equipotentials on surfaces coinciding with conducting surfaces in the first system. The simpler system is chosen so that its potential can be easily written down. Because of the way the simpler distribution is chosen, however, its potential also satisfies the boundary conditions appropriate to the original system. Since solutions to Laplace’s equations under those conditions
are unique, the potential of the simpler system is therefore everywhere identical to that of the original system, at least in the region outside the conductors (which is the only region of interest anyway), and we have an immediate solution to the original problem. In general, however, the method is successful only if we happen to know a simpler system that establishes the necessary equipotentials. There is no systematic way to find the simpler system corresponding to an arbitrarily given set of equipotentials. The procedure is therefore much like that in the previous section: We compile a table of the equipotentials established by various simple charge distributions and then refer to this table whenever we think the method of images might be successful. In the remainder of this section, we shall develop a few entries for that table.

Suppose first that the potential established by a point charge \( q \) placed in front of an infinite plane conducting sheet is desired (Fig. 8.6). Let the conductor be grounded (i.e., at zero potential). We therefore seek a solution to Laplace’s equation subject to the requirements (1) that the potential be zero in the \( yz \) plane, (2) that there be a point charge \( q \) at \((d, 0, 0)\), and (3) that the potential go to zero as \( x \to 0 \). The domain of the problem is that region for which \( x > 0 \), exclusive of a small volume surrounding the point charge. Now the charge distribution shown in Fig. 8.7 has the right characteristics in the region \( x > 0 \) and has a zero potential surface in the \( yz \) plane. (Why?) By the uniqueness theorem, the potential for \( x > 0 \) in Fig. 8.7 is then the same as that in Fig. 8.6 and the potential for the point charge in front of the grounded conductor is therefore given by

\[
V(x, y, z) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{\sqrt{(x - d)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x + d)^2 + y^2 + z^2}} \right]
\]  

(8.60)

where \((x, y, z)\) are the coordinates of the observation point. The first entry in our table of images then is the image of a point charge in an infinite, plane conductor at zero potential, which image is a charge of opposite sign located behind the conductor the same distance as the original charge is in front of the conductor. Here as always, image charges will turn out to be located inside conductors and these added charges are therefore always in a region of
8.6. THE METHOD OF IMAGES

Figure 8.7: A distribution of point charges equivalent to the charge and plate in Fig. 8.6. The equivalence is valid only in the region $x > 0$.

Figure 8.8: Coordinates for evaluating the potential established by two point charges.

space that is not part of the region of interest.

A second useful set of point charges is shown in Fig. 8.8. The potential established by this distribution at a point $(r, \theta, \phi)$ in spherical polar coordinates is

$$ V(r, \theta, \phi) = \frac{q'}{4\pi \varepsilon_0} \frac{1}{|r - b\hat{k}|} + \frac{q}{4\pi \varepsilon_0} \frac{1}{|r - d\hat{k}|} $$

$$ = \frac{q'}{4\pi \varepsilon_0} \frac{1}{\sqrt{r^2 + b^2 - 2rb \cos \theta}} + \frac{q}{4\pi \varepsilon_0} \frac{1}{\sqrt{r^2 + d^2 - 2rd \cos \theta}} $$

(8.61)

The surface on which $V(r, \theta, \phi) = 0$ therefore satisfies the condition

$$ -\frac{q'}{q} = \sqrt{\frac{r^2 + b^2 - 2rb \cos \theta}{r^2 + d^2 - 2rd \cos \theta}} $$

(8.62)

which in general expresses a rather complicated relationship between $r$ and $\theta$. If, however,
we introduce a length \( a \) defined by

\[ bd = a^2 \]  
(8.63)

and take the charge \( q' \) to have the value

\[ q' = -q \frac{a}{d} \]  
(8.64)

we find that the equation \( r = a \) of a sphere of radius \( a \) reduces Eq. (8.62) to an identity. Thus, for this charge distribution (and actually for any pair of point charges of opposite sign; P8.31) there exists a spherical equipotential surface on which the potential is zero. Further, the potential can be changed to any other constant value by placing a third charge at the center of the sphere. (Why?) This second entry for our table of images can be used, for example, to find the potential established when a point charge \( q \) is placed a distance \( d \) from the center of a grounded conducting sphere of radius \( a \) (Fig. 8.9). In the region outside the conducting sphere, the potential is the same as that established by \( q \) and its image \( q' \) located at \((0, 0, b)\), where \( q' \) is given by Eq. (8.64) and \( b \) is given by Eq. (8.63). (See P8.28.)

A third useful entry for our table of images is the potential of two infinitely long, oppositely charged, parallel line charges. The equipotential surfaces are cylinders (P8.32). Thus, some potential problems involving line charges outside of conducting cylinders are readily solved by the method of images.

**PROBLEMS**

**P8.24.** (a) Determine the charge density \( \sigma(y, z) \) on the plate in Fig. 8.6. (b) Calculate the total charge \( Q \) on the plate by direct integration over the plate and compare the result with the image charge. (c) Calculate the force \( \mathbf{F} \) experienced by the charge \( q \) in Fig. 8.6 by integration over the plate and observe how the answer could have been more easily obtained using the image charge.
8.7  **NUMERICAL SOLUTION OF LAPLACE’S EQUATION**

We have now explored several techniques for obtaining *analytic* solutions to Laplace’s equation. None of these techniques, however, is perfectly general. Separation of variables works only if the bounding surfaces coincide with the natural surfaces in a coordinate system in which Laplace’s equation is separable. Even when it does work, the method usually yields an infinite series whose physical meaning is difficult to visualize. The method of complex

---

**P8.25.** Determine the image distribution for an arbitrary static charge distribution placed in front of an infinite, grounded, conducting plate. *Hint:* Consider first a distribution consisting of two point charges in front of the plate.

**P8.26.** Let a grounded conductor occupy all of space except the region for which \( x > 0, \ y > 0 \). Find the potential established in that region when a point charge \( q \) is placed at coordinates \((a, b, 0)\).

**P8.27.** A point charge \( q \) is placed midway between two infinite, parallel, plane conducting plates, both of which are grounded. Let the plates be separated by a distance \( a \). Use the method of images to find an infinite series for the potential in the region between the plates and examine carefully whether this series converges. Interesting discussions and several references may be found in papers by J. J. G. Scanio, *Am. J. Phys.*, 41, 415 (1973) and by B. G. Dick, *Am. J. Phys.*, 41, 1289 (1973).

**P8.28.** (a) Find the potential established by the system in Fig. 8.9 by eliminating \( b \) and \( q' \) from Eq. (8.61). (b) Show explicitly that \( V = 0 \) when \( r = a \). (c) Find the charge density \( \sigma(\theta) \) on the surface of the conducting sphere, and show by direct integration that the total charge on the sphere is equal to \( q' \). Sketch a graph of \( \sigma(\theta) \) versus \( \theta \). *Optional:* Determine the force of attraction between the charge \( q \) and the sphere by two different methods.

**P8.29.** A point charge \( q \) is placed a distance \( b \) from the center of a spherical cavity of radius \( a \) in a grounded conductor, with \( b < a \). Determine the potential in the cavity and the charge density on its walls.

**P8.30.** Apply the method of images to find the charge density on an uncharged conducting sphere placed in a uniform electric field. *Hint:* Generate a uniform field by imagining the sphere between two point charges \( q \) and \(-q\), each placed a distance \( L \) from the center of the sphere on an extension of a single diameter. Then, let \( L \to \infty \), keeping \( q/L^2 \) constant.

**P8.31.** Two point charges \( q \) and \( q' \), with \( q/q' < 0 \), are placed a distance \( s \) apart. Describe a way to determine which charge lies inside the resulting *spherical* equipotential and find the location of the center and the radius of that equipotential.

**P8.32.** Two infinite line charges lie in the \( xy \) plane parallel to the \( x \) axis. The line at \( y = d \) carries a linear charge density \( \lambda \) and the line at \( y = -d \) carries a linear charge density \(-\lambda\). Find an equation for the equipotential curves in the \( yz \) plane, and show that these curves are circles centered on the \( y \) axis. *Hint:* The potential of a single line charge is given in P4.19. Sketch a graph showing these equipotentials for several (positive and negative) values of the potential.

---

**8.7  Numerical Solution of Laplace’s Equation**

We have now explored several techniques for obtaining *analytic* solutions to Laplace’s equation. None of these techniques, however, is perfectly general. Separation of variables works only if the bounding surfaces coincide with the natural surfaces in a coordinate system in which Laplace’s equation is separable. Even when it does work, the method usually yields an infinite series whose physical meaning is difficult to visualize. The method of complex

---

\(^{13}\)Much of the material in this section appears also in one of the author’s contributions to *Computer-Oriented Physics Problems* edited by J. W. Robson (Commission on College Physics, College Park, Md., 1971) and is used here in slightly revised form by permission of the Commission on College Physics. A more detailed treatment may be found, for example, in B. Carnahan, H. A. Luther, and J. O. Wilkes, *Applied Numerical Methods* (John Wiley & Sons, Inc., New York, 1969), Chapter 7.
variables works only in two dimensions and then only if we happen (essentially accidentally) to find the right analytic function. The method of images works only when we know a priori a simple charge distribution that establishes the correct equipotentials. In this section, we shall discuss a simple method to avoid these restrictions by solving Laplace’s equation numerically.

We shall first derive a basic property of solutions to Laplace’s equation in Cartesian coordinates in two dimensions. Assume \( V(x, y) \) satisfies \( \nabla^2 V = 0 \). Now, from a Taylor series expansion (Appendix B), we find that

\[
V(x + d, y) = V + d \frac{\partial V}{\partial x} + \frac{1}{2} d^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{6} d^3 \frac{\partial^3 V}{\partial x^3} + O(d^4) \tag{8.65}
\]

\[
V(x - d, y) = V - d \frac{\partial V}{\partial x} + \frac{1}{2} d^2 \frac{\partial^2 V}{\partial x^2} - \frac{1}{6} d^3 \frac{\partial^3 V}{\partial x^3} + O(d^4) \tag{8.66}
\]

where functions with unspecified arguments are evaluated at the point \((x, y)\) and \(O(d^4)\) stands symbolically for terms of order \(d^4\) and higher that have been omitted. Adding Eqs. (8.65) and (8.66), we find that

\[
V(x + d, y) + V(x - d, y) = 2V + d^2 \frac{\partial^2 V}{\partial x^2} + O(d^4) \tag{8.67}
\]

Similarly,

\[
V(x, y + d) + V(x, y - d) = 2V + d^2 \frac{\partial^2 V}{\partial y^2} + O(d^4) \tag{8.68}
\]

Finally, adding Eqs. (8.67) and (8.68), dividing the result by 4, and setting \( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \) (\(V \) satisfies \( \nabla^2 V = 0 \) by hypothesis), we find that

\[
V(x, y) = \frac{1}{4} \left[ V(x + d, y) + V(x - d, y) + V(x, y + d) + V(x, y - d) \right] \tag{8.69}
\]

To within terms of order \(d^4\), the solution to Laplace’s equation at a particular point \((x, y)\) is therefore given by the average of the solutions at four neighboring points, each displaced from the point \((x, y)\) in a direction parallel to one of the coordinate axes by a distance \(d\); i.e., the solution at point \(P\) in Fig. 8.10 is the average of the solutions at the four points \(P_1, P_2, P_3, \) and \(P_4\).

To illustrate a simple procedure by which Eq. (8.69) can be used to obtain approximate solutions to Laplace’s equation, we consider again the problem in Fig. 8.2, taking \(a = b\) and \(V_0 = 100\) in whatever units are appropriate. Let the region be divided by a square grid having four points in the interior of the region of interest (Fig. 8.11). An approximate solution to Laplace’s equation is then obtained by finding values satisfying Eq. (8.69) at each interior point. The unknowns are represented by question marks in Table 8.2(a).\(^{14}\) We start by guessing values at the interior points, as, for example, in Table 8.2(b). This guess is then improved by progressing systematically through the grid of interior points, replacing the value at each point with the average of the values at the four nearest neighboring points, using each new value as soon as it becomes available. First the 30 in the upper left corner of Table 8.2(b) is replaced by \((0 + 0 + 50 + 30)/4 = 20\) [Table 8.2(c)]; next

\(^{14}\)Fortunately, the procedure by which these unknowns are found does not involve the values at the four corners, two of which are ambiguous.
8.7. NUMERICAL SOLUTION OF LAPLACE’S EQUATION

Figure 8.10: A square grid superimposed on a region within which a solution to Laplace’s equation is sought.

Figure 8.11: A grid of four interior points superimposed on a square region in the $xy$ plane.

the 50 in the upper right corner of Table 8.2(c) is replaced by $(20 + 0 + 100 + 50)/4 = 42.5$ [Table 8.2(d)]; then the 30 in the lower left corner of Table 8.2(d) is replaced by $(0 + 20 + 50 + 0)/4 = 17.5$ [Table 8.2(e)]; and finally the 50 in the lower right corner of Table 8.2(e) is replaced by $(17.5 + 42.5 + 100 + 0)/4 = 40$, yielding Table 8.2(f) and completing the first iteration of a process that can be repeated indefinitely. Table 8.2(g) shows the approximate solution obtained by repeating this procedure starting with the numbers in Table 8.2(f), and Table 8.2(h) shows the approximate solution resulting after the tenth successive application of this procedure. It is readily verified that the solution in Table 8.2(h) satisfies Eq. (8.69) exactly at all points; continued iterations merely regenerate
Table 8.2: Successive Steps in the Numerical Solution of Laplace’s Equation for the Problem in Fig. 8.11. (a) The unknowns. Each entry corresponds to an intersection of two lines of the grid in Fig. 8.11. Note the ambiguity at the two right-hand corners. (b) An initial guess. (c) The first step toward an improved approximation. (d) The second step toward an improved approximation. (e) The third step toward an improved approximation. (f) The final step toward an improved approximation (completion of the first iteration). (g) Approximate solution after the second iteration. (h) Approximate solution after the tenth iteration.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0,100 \\
0 & ? & ? & 100 \\
0 & ? & ? & 100 \\
0 & 0 & 0 & 0,100 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0,100 \\
0 & 30 & 50 & 100 \\
0 & 30 & 50 & 100 \\
0 & 0 & 0 & 0,100 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0,100 \\
0 & 20 & 42.5 & 100 \\
0 & 30 & 50 & 100 \\
0 & 0 & 0 & 0,100 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0,100 \\
0 & 20 & 42.5 & 100 \\
0 & 17.5 & 50 & 100 \\
0 & 0 & 0 & 0,100 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0,100 \\
0 & 15.000 & 38.750 & 100 \\
0 & 13.750 & 38.125 & 100 \\
0 & 0 & 0 & 0,100 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0,100 \\
0 & 12.500 & 37.500 & 100 \\
0 & 12.500 & 37.500 & 100 \\
0 & 0 & 0 & 0,100 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0,100 \\
0 & 12.500 & 37.500 & 100 \\
0 & 12.500 & 37.500 & 100 \\
0 & 0 & 0 & 0,100 \\
\end{array}
\]

Note that at each calculation the most recently obtained value of the solution is used in the right-hand side of Eq. (8.69) and also that a poor initial guess requires a large number of iterations to achieve a desired accuracy but does not undermine the ultimate convergence of the process. The results of solving this same problem numerically on a finer grid are shown in Table 8.3 and in Fig. 8.12. (See P8.34.)

Although we have here considered only a two-dimensional problem in a simple geometry subject to Dirichlet boundary conditions, the method illustrated is not so restricted. It can be easily extended to three dimensions (see P8.36), and it is confined neither to Dirichlet boundary conditions nor to simple geometries. Certainly, application of the technique to these more general situations is more involved, but the basic idea remains unaltered. This simple method yields readily interpreted solutions for any unambiguously stated problem involving Laplace’s equation. Indeed, numerical methods are the only methods that can be counted on to work for all problems.

PROBLEMS

P8.33. Verify the results given in Table 8.2(g) for the approximate solution to the example in the text after the second iteration.

P8.34. A simple algorithm for solving Laplace’s equation numerically in a rectangular region is shown in Fig. 8.13. (a) Decide on an explicit criterion for determining when the current approximation has converged adequately to the true solution. A possible criterion would

---

Note that satisfaction of the difference equations exactly does not mean that the values also satisfy the original differential equations exactly, since the difference equations are themselves an approximation to the differential equations.
Table 8.3: Numerical Solution of the Example in the Text on a Grid with 225 Interior Points.

<p>| | | | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.4</td>
<td>0.9</td>
<td>1.4</td>
<td>1.9</td>
<td>2.5</td>
<td>3.2</td>
<td>4.1</td>
<td>5.2</td>
<td>6.6</td>
<td>8.3</td>
<td>10.8</td>
<td>14.2</td>
<td>19.7</td>
<td>29.4</td>
<td>49.6</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.7</td>
<td>2.7</td>
<td>3.7</td>
<td>4.9</td>
<td>6.3</td>
<td>8.0</td>
<td>10.1</td>
<td>12.7</td>
<td>16.1</td>
<td>20.4</td>
<td>26.5</td>
<td>35.1</td>
<td>48.3</td>
<td>68.9</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.2</td>
<td>2.5</td>
<td>3.9</td>
<td>5.4</td>
<td>7.1</td>
<td>9.2</td>
<td>11.6</td>
<td>14.5</td>
<td>18.2</td>
<td>22.7</td>
<td>28.5</td>
<td>36.1</td>
<td>46.1</td>
<td>59.7</td>
<td>77.7</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.5</td>
<td>3.2</td>
<td>4.9</td>
<td>6.8</td>
<td>9.0</td>
<td>11.6</td>
<td>14.6</td>
<td>18.3</td>
<td>22.7</td>
<td>28.1</td>
<td>34.8</td>
<td>43.2</td>
<td>53.6</td>
<td>66.6</td>
<td>82.3</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.8</td>
<td>3.7</td>
<td>5.7</td>
<td>8.0</td>
<td>10.6</td>
<td>13.5</td>
<td>17.0</td>
<td>21.2</td>
<td>26.3</td>
<td>32.2</td>
<td>39.4</td>
<td>48.2</td>
<td>58.6</td>
<td>70.9</td>
<td>84.9</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>2.0</td>
<td>4.1</td>
<td>6.4</td>
<td>8.9</td>
<td>11.7</td>
<td>15.0</td>
<td>18.8</td>
<td>23.3</td>
<td>28.7</td>
<td>35.0</td>
<td>42.6</td>
<td>51.5</td>
<td>61.8</td>
<td>73.5</td>
<td>86.4</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>2.1</td>
<td>4.4</td>
<td>6.7</td>
<td>9.4</td>
<td>12.4</td>
<td>15.8</td>
<td>19.8</td>
<td>24.6</td>
<td>30.2</td>
<td>36.7</td>
<td>44.4</td>
<td>53.3</td>
<td>63.5</td>
<td>74.9</td>
<td>87.2</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>2.2</td>
<td>4.4</td>
<td>6.9</td>
<td>9.6</td>
<td>12.6</td>
<td>16.1</td>
<td>20.2</td>
<td>25.0</td>
<td>30.6</td>
<td>37.3</td>
<td>45.0</td>
<td>53.9</td>
<td>64.1</td>
<td>75.3</td>
<td>87.4</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>2.1</td>
<td>4.4</td>
<td>6.7</td>
<td>9.4</td>
<td>12.4</td>
<td>15.8</td>
<td>19.8</td>
<td>24.6</td>
<td>30.2</td>
<td>36.7</td>
<td>44.4</td>
<td>53.3</td>
<td>63.5</td>
<td>74.9</td>
<td>87.2</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>2.0</td>
<td>4.1</td>
<td>6.4</td>
<td>8.9</td>
<td>11.7</td>
<td>15.0</td>
<td>18.8</td>
<td>23.3</td>
<td>28.7</td>
<td>35.0</td>
<td>42.6</td>
<td>51.5</td>
<td>61.8</td>
<td>73.5</td>
<td>86.4</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.8</td>
<td>3.7</td>
<td>5.7</td>
<td>8.0</td>
<td>10.6</td>
<td>13.5</td>
<td>17.0</td>
<td>21.2</td>
<td>26.2</td>
<td>32.2</td>
<td>39.4</td>
<td>48.2</td>
<td>58.6</td>
<td>70.9</td>
<td>84.9</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.5</td>
<td>3.2</td>
<td>4.9</td>
<td>6.8</td>
<td>9.0</td>
<td>11.6</td>
<td>14.6</td>
<td>18.3</td>
<td>22.7</td>
<td>28.1</td>
<td>34.8</td>
<td>43.2</td>
<td>53.6</td>
<td>66.6</td>
<td>82.3</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.2</td>
<td>2.5</td>
<td>3.9</td>
<td>5.4</td>
<td>7.1</td>
<td>9.2</td>
<td>11.6</td>
<td>14.5</td>
<td>18.2</td>
<td>22.7</td>
<td>28.5</td>
<td>36.1</td>
<td>46.1</td>
<td>59.7</td>
<td>77.7</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.8</td>
<td>1.7</td>
<td>2.7</td>
<td>3.7</td>
<td>4.9</td>
<td>6.3</td>
<td>8.0</td>
<td>10.1</td>
<td>12.7</td>
<td>16.1</td>
<td>20.4</td>
<td>26.5</td>
<td>35.1</td>
<td>48.3</td>
<td>68.9</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.4</td>
<td>0.9</td>
<td>1.4</td>
<td>1.9</td>
<td>2.5</td>
<td>3.2</td>
<td>4.1</td>
<td>5.2</td>
<td>6.6</td>
<td>8.3</td>
<td>10.8</td>
<td>14.2</td>
<td>19.7</td>
<td>29.4</td>
<td>49.6</td>
<td>100.0</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>xxxxx</td>
</tr>
</tbody>
</table>
Figure 8.12: Equipotentials obtained from a numerical solution of the example in the text. Each equipotential is labeled with the corresponding value of $V$.

Figure 8.13: An algorithm for solving Laplace’s equation numerically.

```
BEGIN_PROGRAM
  DIMENSION V(,) for solution matrix
  SET boundary values and initial trial solution in V(,)
  INITIALIZE COUNTER I TO 0
  LOOP
    ITERATE once through solution matrix
    I <- I+1
    EXIT_LOOP WHEN converged OR too many iterations
  END_LOOP
  PRINT I, V(,)
END_PROGRAM
```

terminate the iterations when the value assigned to each interior point in the grid differs by no more than some predetermined amount $\epsilon$ from the value assigned at that point in the previous iteration. To avoid a potentially infinite loop in case of non-convergence, it would be prudent to include a limit on the number of iterations. (b) Translate the algorithm in Fig. 8.13, including your convergence criterion, into a FORTRAN or C computer program and execute your program to find an approximate solution to Laplace’s equation when the boundary conditions are those in Fig. 8.11 and the grid has 25 interior points (a $7 \times 7$ grid). Then recast your program to solve the problem on a grid of 529 interior points ($25 \times 25$ grid) and write its output to a file, running that refined program to create a file containing that solution. Take the dimensionless coordinates $x$ and $y$ ($0 \leq x, y \leq 1$) to measure coordinates in units of the side $a$ of the square and take the dimensionless $V$ to measure voltage in units of $V_0$. (c) Using an available numeric processing program like
IDL, MATLAB, OCTAVE, or PYTHON, create both a wire mesh plot of the solution and a contour plot of the solution, using the solution on a 25 \times 25 grid obtained in part (b). (d) Assuming that the right side is insulated from the top and bottom sides of the region of interest and that all sides are conductors maintained at the specified potentials. Calculate the charge density on the surface of the conducting boundary and then graph the charge density. Hint: This charge density is given by

\[ \sigma = -\varepsilon_0 \nabla V \cdot \hat{n} \]  

where \( \hat{n} \) is the direction normal to the boundary and increasing in the direction into the region of interest. Evaluate the derivative numerically, noting the forward difference formulae

\[ \frac{df(x)}{dx} \approx \frac{f(x+a) - f(x)}{a} ; \quad \frac{df(x)}{dx} \approx \frac{-3f(x) + 4f(x+a) - f(x+2a)}{2a} \]

and the backward difference formulae

\[ \frac{df(x)}{dx} \approx \frac{f(x) - f(x-a)}{a} ; \quad \frac{df(x)}{dx} \approx \frac{3f(x) - 4f(x-a) + f(x-2a)}{2a} \]

where the second form in each case is more accurate than the first form. Optional: (1) Modify your program to start with a grid of 1 interior point and then move automatically in succession to grids with 9, 49, 225, \ldots interior points using the solution on each grid to obtain an initial guess for the solution on the next grid. (2) Find solutions for other boundary conditions, such as those shown in Fig. 8.14.

P8.35. (a) Develop a numerical method for finding a solution to the problem in Fig. 8.1(b). Think in terms of the dimensionless variables \( V/V_0, x/a, \) and \( y/a \). Hint: At points on the boundaries where Neumann conditions are imposed, Eq. (8.69) cannot be used directly. We circumvent this problem at points on the \( y \) axis, for example, by imagining an extension of the domain into the region \( x < 0 \) and noting that the Neumann condition then requires that \( [V(d, y) - V(-d, y)]/2d \) be zero, or \( V(d, y) = V(-d, y) \). Thus, when Eq. (8.69) is evaluated at \( x = 0 \), the term in \( V(-d, y) \) that arises can be replaced with \( V(d, y) \) and the result involves only points lying within or on the boundaries of the problem. Similar arguments apply to the Neumann condition on the \( x \) axis and to the origin (which must be treated separately). Thus, there are four different classifications for grid points (interior, \( y \) axis, \( x \) axis, origin), each of which must be treated differently in the iterative “march” through the grid. (b) Either by hand or by machine, explore the solution to the problem in Fig. 8.1(b), find the equipotentials, and determine the charge densities on the conducting boundaries. Hint: See the hint in P8.34(c).
P8.36. Develop the relationship analogous to Eq. (8.69) for a solution to Laplace’s equation in three-dimensional Cartesian coordinates \((x, y, z)\).

8.8 Solution of Laplace’s Equation by Experiment: The Method of Analogy

Because Laplace’s equation describes many different phenomena in many different areas of physics by mathematically identical statements, a solution to Laplace’s equation for one of these phenomena can be taken over intact to the analogous phenomenon in another area of physics. In some cases, the physical situation corresponding to a given mathematical statement as applied to one area of physics can be easily constructed experimentally. The solution to Laplace’s equation for this situation, and hence for analogous situations in other areas of physics as well, can then be measured. The most common example of this method of analogy in electrostatics is the (two-dimensional) field tray, in which electrodes having the desired shapes and maintained at the desired potentials are placed in a thin layer of a conducting solution. Because both are described mathematically by the same boundary value problem (P9.4), the electrostatic potential in the field tray is identical to the potential established by a two-dimensional system of conductors located in free space but having the same geometry as those placed in the solution. The potential in the field tray, however, is the more easily measured potential. Deflections of elastic membranes, steady-state temperature distributions, flow patterns in fluids when the velocity field is steady and irrotational (i.e., “curl-less”), and many other physical situations can be used to obtain solutions to Laplace’s equation for analogous electrostatic problems, but a detailed development of these techniques would lead us away from the primary objectives of this book.

8.9 Poisson’s Equation

When charge is present in the region of interest, the electrostatic potential satisfies Poisson’s equation,

\[
\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{8.71}\n\]

[See Eq. (6.76).] If the charge density \(\rho\) is known, then the general solution has the form

\[
V(r) = \left(\text{any solution to } \nabla^2 V = 0\right) + \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{|r - r'|} \, dv' \tag{8.72}\n\]

where the added solution to the homogeneous equation provides the freedom to satisfy boundary conditions that may not be satisfied by the integral alone. [Compare Eq. (4.57) and the paragraph following Eq. (4.57).] Thus, if \(\rho\) is known, Eq. (8.71) is reducible to Laplace’s equation and its solution can be obtained by any of the methods already described. Alternatively, if \(\rho\) is known, the single term \(\rho(x, y)d^2/4\epsilon_0\) added to Eq. (8.69) provides a starting point for a direct numerical solution of Poisson’s equation in two dimensions (P8.38).
When the charge density is not known explicitly, we need another equation relating \( V \) and \( \rho \) before Eq. (8.71) can be solved. The derivation of Child’s Law (P6.39)\(^{16}\) included finding such an equation. We conclude this section with another example by applying Eq. (8.71) to determine the character of the electron cloud attracted to a positive test charge \( q \) in an initially neutral ionized gas or plasma (which contains mobile positive and negative charges). Choose a coordinate system with its origin at the charge \( q \). In this coordinate system, the physical system is spherically symmetric, so the equilibrium charge density \( \rho \) and the electrostatic potential \( V \) in the plasma can depend only on the spherical coordinate \( r \), and Poisson’s equation reduces in spherical coordinates to

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = -\frac{\rho}{\epsilon_0} = -\frac{q_e}{\epsilon_0} (n_+ - n_-) \quad (8.73)
\]

where \( n_-(r) \) is the density of electrons in the plasma, \( n_+(r) \) is the density of ions (assumed singly ionized), and \( q_e \) is the charge on the ions. [See Eq. (2.10).] The particle densities \( n_+ \) and \( n_- \), however, can be related to the potential by using a theorem from statistical mechanics: If the density of particles is \( n_0 \) in a region where the potential energy is \( U_0 \), then the density \( n \) in a region where the potential energy is \( U \) is given by

\[
n = n_0 e^{-U-U_0)/kT} \quad (8.74)
\]

where \( k \) is Boltzmann’s constant\(^{17}\) and \( T \) is the (absolute) temperature. To apply Eq. (8.74) to the plasma, we take the reference point at \( r = \infty \), where we assume the electrostatic potential \( V(r) \) has the value zero. Further, we assume that the plasma remains electrically neutral in the region remote from the test charge so that \( n_+(\infty) = n_- (\infty) \), and we symbolize both by \( n_0 \), which is the density of particles of either charge in the undisturbed plasma. Remembering finally that potential energy is potential times charge, we find from Eq. (8.74) that

\[
n_+ = n_0 e^{-q_e V/kT} \quad ; \quad n_- = n_0 e^{q_e V/kT} \quad (8.75)
\]

and hence from Eq. (8.73) that \( V \) by itself satisfies

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = -\frac{q_e n_0}{\epsilon_0} \left[ e^{-q_e V/kT} - e^{q_e V/kT} \right] \quad (8.76)
\]

Actually solving this equation to find \( V \) is left to P8.37. When \( q_e V/kT \ll 1 \) (high temperature), the result is

\[
V = \frac{q}{4\pi\epsilon_0 r} e^{-r/D} \quad ; \quad D^2 = \frac{\epsilon_0 kT}{2n_0 q_e^2} \quad (8.77)
\]

where the length \( D \), called the Debye shielding length,\(^{18}\) measures the scale of the exponential decay of this shielded Coulomb potential (sometimes called the Yukawa potential).\(^{19}\) For \( n_0 = 10^{21} \) particles/m\(^3\) and \( T = 10^8 \) K, which are typical of plasmas produced in controlled thermonuclear research, the Debye length is about \( 2 \times 10^{-4} \) m. In a plasma described by

---

\(^{16}\) See footnote 16 in Chapter 6, page 183.

\(^{17}\) Austrian physicist Ludwig Eduard Boltzmann, b. 20 February 1844 in Vienna, Austrian Empire; d. 5 September 1906 in Trieste, Austria-Hungary (now Duino, Italy).

\(^{18}\) Dutch-American physicist and physical chemist Peter Joseph William Debye, b. 24 March 1884 in Maastricht, Netherlands; d. 2 November 1966 in Ithaca, New York.

\(^{19}\) Japanese physicist Hideki Yukawa, b. 23 January 1907 in Tokyo, Japan; d. 8 September 1981 in Kyoto, Japan.
PROBLEMS

P8.37. Solve Eq. (8.76) when \( q_c V / kT \ll 1 \) subject to the requirements that \( V(r) \to 0 \) as \( r \to \infty \) and that \( V \to q / 4\pi\epsilon_0 r \) as \( r \to 0 \). Hints: (1) \( e^\alpha \approx 1 + \alpha \) when \( \alpha \) is small. (2) Introduce a new dependent variable \( V' = r V \).

P8.38. Repeat the development of Eq. (8.69) when \( \nabla^2 V = -\rho / \epsilon_0 \), describe a numerical algorithm similar to that in Section 8.7 for solving Poisson’s equation, and then apply this algorithm (either by hand or by computer) to solve the equation \( \nabla^2 V = -2 \) in the interior of a square 10 cm on a side when \( V \) is required to be zero everywhere on the boundary. Physically, this problem relates more directly to the twisting of a square shaft under torsion than to a problem in electrostatics.

SUPPLEMENTARY PROBLEMS

P8.39. A rectangular parallelepiped is bounded by the coordinate planes and the planes \( x = a \), \( y = b \), and \( z = c \). All surfaces of this parallelepiped are maintained at zero potential except the surface at \( z = c \) and that surface is maintained at a constant potential \( V_0 \). Apply the method of separation of variables to find the potential in the interior of the parallelepiped.

P8.40. Apply the method of separation of variables to Laplace’s equation in cylindrical coordinates when all three coordinates appear, solve the resulting equations, and write the most general solution subject only to the requirements that the origin be in the domain of the problem and the angle \( \phi \) assume its full range. Hints: (1) Seek exponential solutions in \( \phi \). (2) The equation \( x^2 y'' + xy' + (x^2 - n^2)y = 0 \), which reduces to Eq. (8.32) when \( n = 0 \), is the \( n \)-th order Bessel equation. The only solution that is finite for all values of \( x \) including the origin is called the \( n \)-th order Bessel function, \( J_n(x) \), and is tabulated, for example, in E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945).

P8.41. Apply the method of separation of variables to Laplace’s equation in spherical coordinates when all three coordinates appear, solve the resulting equations, and write the most general solution subject only to the requirements that \( \theta = 0 \) and \( \theta = \pi \) be in the domain of the problem and the angle \( \phi \) assume its full range. Hints: (1) Seek trigonometric solutions in \( \phi \). (2) Set \( x = \cos \theta \) in the equation for the \( \theta \) dependence. The resulting equation, \((1 - x^2)y'' - 2xy' + [k - m^2/(1 - x^2)]y = 0\), which reduces to Eq. (8.38) when \( m = 0 \), is the associated Legendre equation. Here \( m \) is an integer that arises from solution of the equation for the \( \phi \) dependence and \( k \) is another separation constant. Solutions finite for all \( x \) in the range \(|x| \leq 1\) can be found only when \( k \) has one of the values \( n(n+1) \), where \( n \) is a nonnegative integer. One such solution exists for each value of \( n \); it is called an associated Legendre function, \( P_n^m(x) \), and is tabulated, for example, in E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945).

P8.42. A conformal mapping from the complex \( z \) plane to the complex \( w \) plane associates points in the two planes by the transformation \( w = f(z) \), where \( f \) is an analytic function of \( z \). For some two-dimensional problems, proper selection of this transformation can convert a complicated geometry in the \( z \) plane to a simpler geometry in the \( w \) plane. A solution to Laplace’s equation can then be found in the simpler geometry of the \( w \) plane and transformed back to the \( z \) plane to yield a solution to the original problem. This problem illustrates this very powerful but also quite specialized technique. Suppose a solution to Laplace’s equation in the half-plane \( y > 0 \) is desired when a conducting plate at zero potential lies along the positive \( x \) axis and a conducting plate at 100 V lies along the
negative $x$ axis. (a) Show that the transformation $w = \ln z$ [Eq. (8.53)] transforms the region of interest in the $z$ plane to the region $-\infty < u < \infty$, $0 < v < \pi$ in the $w$ plane, where $u = \text{Re}(w)$ and $v = \text{Im}(w)$. Further, show that the positive $x$ axis is transformed to the line $v = 0$ and the negative $x$ axis to the line $v = \pi$. (b) Show that the desired solution in the $w$ plane is the imaginary part of the analytic function $q(w) = 100u/\pi$. (c) Since an analytic function of an analytic function is analytic, the solution $V(x, y)$ in the $z$ plane is then $\text{Im}(q(w(z)))$. Show that $V(x, y) = (100/\pi) \tan^{-1}(y/x)$, sketch the equipotentials and field lines, and determine the charge density on the conducting plates. Optional: Show that the transformation $z' = [(i - z)/(i + z)]$ transforms the region $y > 0$ in the $z$ plane to the interior of the unit circle $(x')^2 + (y')^2 = 1$ in the $z'$ plane and find a solution to Laplace’s equation in the interior of a unit circle when the bounding arc $0 < \phi' < \pi$ is kept at zero potential and the arc $\pi < \phi' < 2\pi$ is kept at 100 V. Sketch the equipotentials and field lines.

**P8.43.** Find an integral giving the solution $V(x, y)$ to Laplace’s equation in the two-dimensional domain $-\infty < x < \infty$, $0 < y < \infty$ when $V$ and its derivatives approach zero as $|x| \to \infty$ and as $y \to \infty$, and $V(x, 0) = 0$ except in $|x| \leq 1$, where $V(x, 0) = 1$. Hint: Take the Fourier transform (Appendix D) of Laplace’s equation with respect to $x$ (i.e., multiply by $e^{-i\kappa x}$ and integrate on $x$), note that the transform of $\partial^2 V/\partial x^2$ is $-\kappa^2 \tilde{V}(\kappa, y)$ (Why?), solve the resulting ordinary differential equation for $\tilde{V}(\kappa, y)$ but let the “constants” of integration depend on $\kappa$, and then invert the transform to return to $V(x, y)$, determining any “constants” by the boundary conditions.

**P8.44.** Obtain the integral in Eq. (8.72) by taking a three-dimensional Fourier transform (Appendix D) of Poisson’s equation, solving for $\tilde{V}(\kappa)$, and then inverting the transform. Hints: (1) The transform of $\partial^2 V/\partial x^2$ is $-\kappa^2 \tilde{V}$. (Why?) (2) After writing the formal inversion integral, reexpress $\tilde{\rho}(\kappa)$ as a Fourier transform of $\rho(r)$ and interchange orders of integration to obtain

$$V(r) = \frac{1}{8\pi^4 \epsilon_0} \int \rho(r') \left[ \int \frac{e^{i\mathbf{K} \cdot (r-r')}}{\kappa^2} d\kappa_x d\kappa_y d\kappa_z \right] d'v$$

(3) Evaluate the integrals on the components of $\kappa$ by introducing spherical coordinates in $\kappa$-space, taking the polar axis in the direction of $r - r'$. (4) Don’t be afraid of integral tables.

**P8.45.** (a) Prove Green’s theorem,\(^{20}\) Eq. (C.27). (b) Show that

$$\int \Phi(r) \nabla^2 \left( \frac{1}{|r - r'|} \right) dv = -4\pi \Phi(r')$$

Hint: $\nabla^2 [1/|r - r'|] = 0$ everywhere except where $|r - r'| = 0$. (c) Apply these two results to show that a function $V$ that satisfies Laplace’s equation also satisfies the integral equation

$$V(r) = \frac{1}{4\pi} \oint_{\Sigma} \left[ \frac{\nabla' V(r')}{|r - r'|} - V(r') \frac{\nabla' \left( \frac{1}{|r - r'|} \right)}{|r - r'|} \right] \cdot dS'$$

where $\Sigma$ is any closed surface enclosing the point $r$. This identity relates the solution to Laplace’s equation at a single point $r$ to the solution and its normal derivative over a surface enclosing $r$. Similar integral equations are important to the Kirchhoff theory of diffraction.

**P8.46.** Consider a rectangular plate occupying the region $-1.0 < x < 1.0$, $0.0 < y < 2.0$, i.e., $x_a = -1.0$, $x_b = 1.0$, $y_c = 0.0$, and $y_d = 2.0$. The right and left sides of the plate are

\(^{20}\)British mathematical physicist George Green, b. 14 July 1793 in Sneinton, Nottinghamshire, England; d. 31 May 1841 in Nottingham, Nottinghamshire, England.
insulated so that no heat can flow through them and the top of the plate is maintained at a constant temperature of 0°C. The bottom of the plate is heated by the flame of a candle, the center of which is located at the origin (0,0). Translation of these physical conditions into mathematical statements leads us to the mixed (Neumann on the right and left; Dirichlet on the top and bottom) boundary conditions

\[
\frac{\partial u(x_a, y)}{\partial x} = 0 \quad \frac{\partial u(x_b, y)}{\partial x} = 0 \quad u(x, y_d) = 0.0 \quad (8.78)
\]

and

\[
u(x, y_c) = \begin{cases} 
0 & |x| > 0.2 \\
\cos(2.5\pi x) & |x| \leq 0.2 
\end{cases} . \quad (8.79)
\]

Adapt the numerical approach described in Section 8.7 and use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to find the steady state temperature distribution on the plate. To do so, you will need to solve the equation for steady state heat flow

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (8.80)
\]

in the region subject to the described boundary conditions.
Chapter 9

Properties of Matter I: Conduction

Whether they be in the solid, liquid, gas, or plasma phase, macroscopic samples of matter exhibit several different responses when placed in external electric and magnetic induction fields. In Chapters 9, 10, and 11 we shall examine (electric) conduction, dielectric polarization, and magnetization\(^1\) when the external fields are static, and in Chapter 12, we shall extend that treatment to include time-dependent external fields. Each of these responses can be approached either from a macroscopic or a microscopic viewpoint. In the macroscopic approach, we accept the properties of matter as exhibited in the macroscopic world, introduce appropriate descriptive concepts analogous to the charge and current densities of Chapter 2, and seek experimentally determined relationships among these descriptive concepts and the external fields. In the microscopic approach, we postulate a microscopic structure for a particular type of matter and seek to understand—probably through the application of quantum mechanics—the observed macroscopic properties by relating them to the presumably simpler behavior of the assumed microscopic constituents. Although we shall not ignore either approach altogether, we shall place greater emphasis on the macroscopic approach.

Qualitatively and microscopically, a conducting material or a conductor is a sample of matter that contains microscopic charged particles or charge carriers that are free to move macroscopic distances through the material. Steady-state conduction is the response observed when a conducting material is placed in a static electric field, e.g., when the two poles of a battery are connected to opposite ends of a metallic or semiconducting rod or when an ionized gas is placed between two parallel plates maintained at different potentials. Under these circumstances, the motion of the individual carriers has two components. Whether a field is present or not, these carriers move about erratically, collide frequently with one another and with any relatively immobile particles (e.g., “fixed” ions in the crystal lattice of the conducting material) that may be present, and experience random changes in velocity with every collision. Since this random component of the velocity of each particle averages to zero over time intervals that are macroscopically short but microscopically long, it results in no net macroscopic migration of charge through the conducting material. When the

\(^1\)A fourth possible phenomenon—magnetic conduction—will be of interest only after free magnetic monopoles have been found, for only then will macroscopic magnetic currents, i.e., a flow of magnetic monopoles, be physically realizable.
conducting material is placed in an external field, however, its constituent charges experience non-random forces and—in most materials—the charge carriers will be accelerated parallel or antiparallel to the field, depending on the sign of their charge. Further, the frequent collisions, which continue to occur even in the presence of the field, prevent the carriers from accelerating indefinitely. Once any transient effects resulting when the field is first turned on have decayed away (and they do so quite rapidly in “good” conducting materials; see P9.5), a balance between steady acceleration and frequent abrupt changes in velocity is achieved, and the overall microscopic effect of the field is the addition of a steady (non-random) drift velocity to the total velocity of each carrier. In the macroscopic world, a steady drift velocity of the carriers produces a steady macroscopic migration of charge (i.e., a steady current) through the conducting material, which is then said to be conducting.

9.1 Macroscopic Description: Conductivity and Ohm’s Law

In the qualitative terms of the previous paragraphs, the macroscopic response of a conductor to an electric field is the appearance of a macroscopic current.Crudely, we can view the electric field as a cause and the current as its effect. Now a quantitative description of any phenomenon is built on quantitative definitions for concepts measuring its cause(s) and its effect(s). In the case of conduction, the necessary concepts have already been defined: With \( r \) a point in the conductor, we take the macroscopic electric field \( E(r) \) to measure the cause and the macroscopic current density \( J(r) \) to measure the effect. Conceptually, we then think of \( J \) as some function of \( E \),

\[
J = J(E)
\]  
\hspace{1cm} (9.1)

but we must look to experiment to determine whether Eq. (9.1) applies to a particular sample of conducting material at all and, if it does apply, to determine its specific functional form for that sample. Equation (9.1), sometimes called a constitutive relation, is the point at which the empirical properties of matter find their way into the theoretical framework, and, in the macroscopic approach, the empirically dictated form of Eq. (9.1) is accepted without deeper question. That form, of course, may be extremely complicated. For most materials, however, \( J \) is parallel to \( E \) and Eq. (9.1) assumes the simpler form

\[
J = g(E)E, \quad J \parallel E
\]  
\hspace{1cm} (9.2)

where \( g(E) \) remains to be determined experimentally for each material. Further, for a very large fraction of the materials to which Eq. (9.2) applies, the function \( g(E) \) is actually independent of \( E \), at least for the normal range of field strengths. For these materials, which are called linear or ohmic materials, Eq. (9.2) reduces to

\[
J = gE \quad \text{(ohmic conductors)}
\]  
\hspace{1cm} (9.3)

where the conductivity \( g \) is measured in the mks unit \( A/V \cdot m \), a unit we shall later call the mho/m.\(^2\) For ohmic conductors, the value of \( g \) or equivalently the value of the resistivity \( \eta = 1/g \) fully characterizes the extent to which the material conducts in response to a static

---

\(^2\)The conductivity is defined by Eq. (9.3) in all systems of units with which the author is familiar.
Table 9.1: Resistivity $\eta$ and Temperature Coefficient of Resistivity $d\eta/\eta
dT$ for Selected Materials at Room Temperature. In this table, $T$ represents temperature. The materials divide naturally into three groups: the conductors with resistivities on the order of $10^{-6}$ to $10^{-8}$ ohm-m, the semiconductors with resistivities on the order of $10^2$ to $10^{-4}$ ohm-m, and the insulators with resistivities greater than about $10^8$ ohm-m.

<table>
<thead>
<tr>
<th>Material</th>
<th>Resistivity, $\eta$ (ohm-m)</th>
<th>Temperature Coefficient, $\frac{d\eta}{\eta}dT$ [°C]$^{-1}$</th>
<th>Material</th>
<th>Resistivity, $\eta$ (ohm-m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum$^{(a)}$</td>
<td>$2.824 \times 10^{-8}$</td>
<td>0.0039</td>
<td>Germanium$^{(b)}$</td>
<td>$&lt;10^{13}$ cm$^{-3}$ impurity (0.40)</td>
</tr>
<tr>
<td>Brass$^{(a)}$</td>
<td>$7 \times 10^{-8}$</td>
<td>0.002</td>
<td>Beeswax$^{(c)}$</td>
<td>$\approx 10^{12}$</td>
</tr>
<tr>
<td>Constantan$^{(a)}$</td>
<td>$49 \times 10^{-8}$</td>
<td>$10^{-5}$</td>
<td>Glass$^{(c)}$</td>
<td>$10^9$–$10^{14}$ (d)</td>
</tr>
<tr>
<td>Copper (annealed)$^{(a)}$</td>
<td>$1.724 \times 10^{-8}$</td>
<td>0.0039</td>
<td>Glyptal$^{(c)}$</td>
<td>$10^{14}$</td>
</tr>
<tr>
<td>Gold$^{(a)}$</td>
<td>$2.44 \times 10^{-8}$</td>
<td>0.005</td>
<td>Paraffin (special)$^{(c)}$</td>
<td>$&gt;5 \times 10^{16}$</td>
</tr>
<tr>
<td>Iron (99.98%)$^{(a)}$</td>
<td>$10 \times 10^{-8}$</td>
<td>0.005</td>
<td>Porcelain (unglazed)$^{(c)}$</td>
<td>$3 \times 10^{12}$</td>
</tr>
<tr>
<td>Lead$^{(a)}$</td>
<td>$22 \times 10^{-8}$</td>
<td>0.0039</td>
<td>Quartz (fused)$^{(c)}$</td>
<td>$&gt;5 \times 10^{16}$</td>
</tr>
<tr>
<td>Mercury$^{(a)}$</td>
<td>$95.783 \times 10^{-8}$</td>
<td>$8.9^{-4}$</td>
<td>Shellac$^{(c)}$</td>
<td>$\approx 10^{14}$</td>
</tr>
<tr>
<td>Nichrome$^{(a)}$</td>
<td>$100 \times 10^{-8}$</td>
<td>$4 \times 10^{-4}$</td>
<td>Wood$^{(c)}$</td>
<td>$10^{8}$–$10^{11}$</td>
</tr>
<tr>
<td>Nickel$^{(a)}$</td>
<td>$7.8 \times 10^{-8}$</td>
<td>0.006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Silver$^{(a)}$</td>
<td>$1.59 \times 10^{-8}$</td>
<td>0.0038</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tin$^{(a)}$</td>
<td>$11.5 \times 10^{-8}$</td>
<td>0.0042</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tungsten$^{(a)}$</td>
<td>$5.6 \times 10^{-8}$</td>
<td>0.0045</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


(d) Depending on composition.

According to the electric field.$^3$ Experimentally determined values of $\eta$ for several common materials are shown in Table 9.1.

Metals and other “good” conductors characteristically have very high conductivities (small resistivities) that are only weakly dependent on temperature except at extremes of temperature. Semiconductors typically have much lower conductivities (higher resistivities) that are often more strongly dependent on temperature. Insulators, also called dielectrics, have such small conductivities (high resistivities) that it is not appropriate to think of them as conducting materials at all. The limiting values $g \to \infty$ (\(\eta \to 0\)) and $g \to 0$ (\(\eta \to \infty\)) identify idealized cases of the perfect conductor—sometimes called a superconductor—and the perfect insulator, respectively.

$^3$The symbols $\sigma$ and $\rho$ are often used for conductivity and resistivity, respectively, but that notation is subject to confusion with surface and volume charge densities. We use instead $g$ and $\eta$, which follows the notation used by J. R. Reitz and F. J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1967).
We shall now reexpress the equation $J = gE$ between current density and electric field in a form more suited to the description of currents confined to wires. Assume a wire with cross-sectional area $S$ and length $\ell$ and let a potential difference $\Delta V$ be maintained between its ends (Fig. 9.1). When the potential difference is first applied, transient currents will develop in the wire. Ultimately, however, these transients will establish a charge distribution along the surfaces of the wire, which charges in turn produce the fields necessary to "guide" the final steady-state current around any bends in the wire.\(^4\) Steady-state conduction is established when the surface charges have been so adjusted that the current density $J$ and the electric field $E$ inside the wire are everywhere tangent to the wire. If we now assume that the current is uniformly distributed over the cross section of the wire, then $J = |J|$ is constant throughout the wire. Further, in consequence of Eq. (9.3), $E = |E|$ is also constant throughout the wire and the potential difference between the ends of the wire is

$$\Delta V = -\int_{\text{low end}}^{\text{high end}} E \cdot dl = E\ell \quad (9.4)$$

where the path follows the wire and $dl$ is directed from the low to the high end. Further, the current flowing in the wire is given by

$$I = \int_{\Sigma} J \cdot dS = JS \quad (9.5)$$

where $\Sigma$ is any surface intersecting the wire. Since the material of the wire has been assumed ohmic, however, $J = gE$, and we find from Eqs. (9.4) and (9.5) that

$$\frac{\Delta V}{I} = \frac{E\ell}{gES} \Rightarrow \frac{\Delta V}{I} = \frac{\ell}{gS} = \eta \frac{\ell}{S} = R \quad (9.6)$$

or that

$$\Delta V = IR \quad (9.7)$$

\(^4\)See, for example, S. Parker, *Am. J. Phys.* 38, 720 (1970) and the references given there.
where the resistance $R$ is a constant determined by the dimensions and conductivity of the wire; the mks unit of resistance is the ohm ($\Omega$) and the mks unit of a reciprocal resistance, called a conductance, is the mho. A device whose electrical characteristics are described by Eq. (9.7) is called a (linear) resistor, and Eq. (9.7) itself is the familiar form of Ohm’s Law, which relates the potential difference $\Delta V$ across a (linear) resistor $R$ to the current $I$ through the resistor. Equation (9.3), however, is also referred to as Ohm’s Law. A paragraph discussing the limitations of Ohm’s Law is included in Section 9.2.

Finally, it is of interest to calculate the power dissipated in a conductor carrying a current. Clearly, power must be dissipated, for the external electric field is constantly doing work on the charges, but overall their kinetic energy remains unchanged. The collisions must convert energy to heat, called here Joule heat, at a rate exactly equal to the rate at which the external field supplies energy to the system. Thus, we can calculate the rate of energy dissipation in the conductor by calculating instead the rate $P$ at which the external field does work on the charge carriers. That rate, however, is given by Eq. (3.39), and we have that

$$P = \int J \cdot E \, dv$$  \hspace{1cm} (9.8)

where in mks units $P$ is expressed in watts (W). The integral in Eq. (9.8) extends over the entire volume of the conductor. If the current happens to be in a wire, the replacement $J \, dv \rightarrow I \, dl$ yields the more familiar result

$$P = I \int_{\text{low end}}^{\text{high end}} E \cdot dl = I \Delta V = I^2 R = \frac{(\Delta V)^2}{R}$$  \hspace{1cm} (9.9)

where the last two forms follow in substitution from Eq. (9.7).

### 9.2 Microscopic Description: Carrier Mobility and Collision Times

The basic microscopic response of a conductor to its presence in a static electric field is a steady drift velocity superimposed on the random motion of the individual charged particles making up the conductor. The classical microscopic description of conduction therefore involves relating particle drift velocities, which we view as the effects, to the applied electric field, which we continue to view as the cause. Since several different types of charge carrier may contribute to the total transport of charge, we allow for several different drift velocities. Thus, we expect the drift velocity $v^{(a)}$ of carriers of type $a$ to be some function of the applied electric field,

$$v^{(a)} = v^{(a)}(E)$$  \hspace{1cm} (9.10)

A theoretical prediction of the functional form of Eq. (9.10) for the carriers in a particular conductor can be made only after some further assumptions about the nature of the conductor are made. Since Eq. (9.10) is a statement about the behavior of individual microscopic particles, calculation of its specific form is intrinsically a quantum mechanical
problem, but a brief quantum mechanical treatment would require a background beyond that assumed in this book. (See the references at the end of the chapter.) Instead we present a simple classical theory. Suppose the carriers in a conductor experience forces solely (or at least dominantly) from the external field except during collisions, and assume that the acceleration \( a^{(a)} \) of carriers of type \( a \) is constant between collisions. Then at a particular instant of time, the non-random component of the velocity \( \mathbf{v}^{(a)}_i \) of the \( i \)-th carrier of type \( a \) is

\[
\mathbf{v}^{(a)}_i = a^{(a)} \tau^{(a)}_i = \frac{q_a \tau^{(a)}_i}{m_a} \mathbf{E}
\]

(9.11)

where \( q_a \) and \( m_a \) are the charge and mass of carriers of type \( a \), \( \tau^{(a)}_i \) is the time since the last collision of the \( i \)-th particle of type \( a \), and Newton’s second law has been used to write \( a^{(a)} \) as \( q_a \mathbf{E}/m_a \). The average drift velocity \( \mathbf{v}^{(a)} \) for carriers of type \( a \) is then given by

\[
\mathbf{v}^{(a)} = \langle \mathbf{v}^{(a)}_i \rangle = \frac{q_a \langle \tau^{(a)}_i \rangle}{m_a} \mathbf{E} = \frac{q_a \tau^{(a)}_i}{m_a} \mathbf{E}
\]

(9.12)

where the collision time \( \tau^{(a)} = \langle \tau^{(a)}_i \rangle \) physically is the average time between collisions. (Why? Hint: Collisions do not occur at regular intervals.) The quantity

\[
\mu_a = \frac{q_a \tau^{(a)}_i}{m_a}
\]

(9.13)

is called the mobility of carriers of type \( a \) and will be positive or negative, depending on whether the carrier is positively or negatively charged. In terms of the mobility, Eq. (9.12) becomes

\[
\mathbf{v}^{(a)} = \mu_a \mathbf{E}
\]

(9.14)

from which we see that \( \mu_a \) is the average drift speed per unit field and must have the dimensions of velocity per electric field, \( (\text{m/s})/(\text{V/m}) \) in mks units. On the basis of the simple assumptions in this paragraph, we therefore predict that \( \mathbf{v}^{(a)} \) in Eq. (9.10) should be proportional to \( \mathbf{E} \) and that the constant of proportionality \( \mu_a \) is determined by the intrinsic properties of the carriers and by the nature and frequency of the collisions experienced by the carriers as they migrate through the conductor. Except at extremes of temperature and field strength, the quantum mechanical description of conduction in many materials and particularly in metals leads to the same general conclusions, and the carrier mobilities are important (and frequently measured) parameters describing these materials.

Given Eq. (9.14), we can now derive Ohm’s Law. We need merely introduce the density of carriers of type \( a \), \( n^{(a)} \), and then substitute Eq. (9.14) into Eq. (2.18) to find that the macroscopic current density \( \mathbf{J} \) is given by

\[
\mathbf{J} = \sum_a q_a n^{(a)} \mathbf{v}^{(a)} = \left( \sum_a q_a n^{(a)} \mu_a \right) \mathbf{E}
\]

(9.15)

which is Ohm’s Law, Eq. (9.3), provided we identify the coefficient of \( \mathbf{E} \) with the conductivity. This classical microscopic description of conduction has thus yielded not only a derivation of Ohm’s Law but also the expression

\[
g = \sum_a q_a n^{(a)} \mu_a = \sum_a \left( \frac{q_a^2 n^{(a)} \tau^{(a)}}{m_a} \right)
\]

(9.16)
relating a macroscopic conductivity to microscopic mobilities and collision times. Particularly when only one type of carrier contributes to conduction, measurements of $g$ are therefore effectively measurements of the product $q n \mu$ for the carrier involved. If $q$ is assumed and $n$ is known, say from Hall effect measurements (P9.7), measurements of the macroscopic conductivity yield values for the microscopic mobility. Note that $g$ is necessarily positive even though the mobilities may have either sign.

Ohm’s Law and its microscopic equivalent, Eq. (9.14), are, of course, not universal laws of physics. There are circumstances that these laws do not describe adequately, and the microscopic development leading to Eq. (9.14) suggests areas in which we might expect Ohm’s Law to break down. We have, for example, tacitly assumed that the external field does not significantly influence the collision times, so that $\tau^{(a)}$ is independent of the field. Equivalently, we have required that the drift speeds be small compared to thermal speeds, which requirement in turn limits the strength of the field so that high drift speeds are not produced (P9.6). We also tacitly assumed the density of carriers to be independent of the strength of the applied field. But sufficiently strong fields may ionize more of the neutral particles present (or further ionize multielectron atoms that are already partly ionized), thus increasing the density of carriers and destroying the validity of Ohm’s Law. Third, we have assumed the conducting material to be isotropic (the same in all directions) and the resulting conductivity is a scalar. But some few materials are non-isotropic (or anisotropic) and many materials can be made non-isotropic, for example, by the application of a uniform external magnetic induction field. These materials conduct differently for electric fields applied in different directions and the resulting current density may not even be parallel to the applied field. Salvaging Ohm’s Law for these circumstances then requires allowing the conductivity to become a nine-component entity called the conductivity tensor (P9.8). Still further, we have assumed the conductor to be homogeneous (the same at all interior points). Non-ohmic devices, such as transistors and solid state rectifiers, can be made by assembling a conducting sample whose characteristics (conductivity, carrier concentration, etc.) change slowly or abruptly as an observation point moves through the sample. Fifth, we have assumed that only the external field is significant except during actual collisions, and, further, we have assumed that collisions are well-defined events and that we can tell (in principle at least) when a particle is undergoing a collision and when it is not. One of the primary forces of interaction between charged particles, however, is the Coulomb force, which is a very long-range force. Even when particles are far apart, their Coulomb interaction may be significant. Certainly, this interaction interferes with any clear-cut separation of moments of collision from moments of free motion in an external field. Sixth (and finally), we have assumed that the microscopic carriers experience an electric field defined by a “macroscopic” test charge. Although that assumption is known to be valid for microscopic particles moving in free space (e.g., electrons in a cathode ray tube), it may in fact not be correct for carriers in the interior of some conductor. Having enumerated these sources of possible difficulty, one might even be surprised that Ohm’s Law works as well as it does over such a broad range of materials and applied fields.

This discussion of conduction is far from complete. We have outlined the essential descriptive framework, and we have seen some of how this general framework can be molded to describe specific types of matter more explicitly. What remains is more detailed quantum mechanical calculations of mobilities and collision times, a full examination (both experi-

\footnote{American physicist Edwin Hall, b. 7 November 1855 in Gorham, Maine; d. 20 November 1938 in Cambridge, Massachusetts.}
mentally and theoretically) of departures from Ohm’s Law, and a cataloging of measured properties of specific materials. Further pursuit of these topics, however, would distract us from our main intent of exposing the basic theoretical framework. The interested reader is directed to the references listed below.

PROBLEMS

P9.1. Find the dimensions of conductivity in cgs-esu and ponder the implications of your result.

P9.2. It is desired to use a nichrome wire having a cross-sectional area of 0.2 mm$^2$ to make a heating coil that will produce 600 W when operated at 120 V. How long must the wire in the coil be?

P9.3. A conducting wire of radius $a$ and length $L$ is surrounded by a coaxial conducting sheath of radius $b$. The space between the wire and the sheath is filled with a weakly ionized gas that has a (small) conductivity $g$. Let the wire be maintained at a potential $V$ with respect to the sheath. Ignoring fringing, find the current flowing between the wire and the sheath and show that the resistance of the gas is given by $[\ln(b/a)/2\pi Lg]$.

P9.4. Steady currents in a homogeneous, ohmic medium placed in a static electric field are described by the equations $\nabla \cdot J = 0$, $\nabla \times E = 0$, and $J = gE$. (Why?) Show that the electric field can be derived from a scalar potential that satisfies Laplace’s equation, thus justifying the use of field trays to study electric field patterns. (To make a field tray, take a large flat pan and place about 1/4 “ of salty water in its bottom. Then place a couple of metal disks or other shapes in this water, connecting a battery to maintain a potential difference between them. Because a current will then flow in the salt water, one can use simple techniques to measure the potential difference between one electrode and any point in the water, thus determining equipotential surfaces experimentally. This problem asks for a demonstration that the empirically determinable equipotential surfaces in a field tray are identical to the essentially unmeasurable equipotential surfaces that would exist were the same electrodes simply placed in empty space.)

P9.5. Suppose that at time zero the charge density inside a conductor happens to have the nonzero (non-equilibrium) value $\rho_0(r)$, where $r$ is a point in the conductor. (a) Assume $\rho(r, t) = \rho_0(r) f(t)$ and combine the equation of continuity and Ohm’s Law with Gauss’s Law—the presently available form will do, although strictly you should use the modified form to be developed in Chapter 10—to obtain a differential equation for $f(t)$, show that $f(t) = e^{-t/\tau_r}$, and find an explicit expression for the relaxation time $\tau_r$. (b) Calculate $\tau_r$ numerically for aluminum. In essence, this result demonstrates that aluminum and other good conductors dissipate non-equilibrium charge distributions very rapidly.

P9.6. An aluminum wire at room temperature ($\approx 300$ K) has a cross-sectional area of 0.1 mm$^2$ and carries a current of $5 \times 10^{-4}$ A. (a) If each aluminum atom contributes three conduction electrons and all the current is carried by these electrons, calculate the electronic drift velocity in the wire. *Hints*: Avogadro’s number is $6.02 \times 10^{23}$ atoms/mole; the atomic weight of aluminum is 27; the density of aluminum is 2.70 g/cm$^3$. (b) Estimate the thermal speed $\sqrt{3kT/m_e}$ of the electrons and compare the result with the drift velocity. Here $k$ is Boltzmann’s constant and $m_e$ is the electron mass. (c) Use Eq. (9.16) to estimate the collision time from the measured conductivity. (d) Approximately how far does an electron move between collisions? (e) Use Eq. (9.16) to estimate the electron mobility from the measured conductivity and then estimate the electric field necessary to bring about the given current. Compare this field with the field necessary to produce a drift velocity comparable to the thermal velocity and compare each with typical laboratory fields.
9.2. CARRIER MOBILITY AND COLLISION TIMES

Figure 9.2: Figure for Problem P9.7.

P9.7. A current $I$ is passed through a sample of conducting material in a uniform magnetic induction $\mathbf{B} = B\mathbf{k}$ that is perpendicular to the direction of current flow (Fig. 9.2). Assume the current is carried by a single type of carrier with charge $q$, density $n$, and drift velocity $\mathbf{v}_d$, and let the points $a$ and $b$ be separated by a distance $w$. (a) Argue qualitatively that an electric field $\mathbf{E} = E\mathbf{j}$ is induced, show that $E = vB$, and calculate the potential difference $\Delta V$ between points $a$ and $b$. Which point is at the higher potential? (b) The Hall coefficient for this sample of material is defined by $R_H = E/JB$, where $J$ is the $\mathbf{i}$-component of the current density. Show that $R_H = 1/qn$. (c) Show also that $R_H = \mu/g$, where $\mu$ is the carrier mobility and $g$ is the conductivity of the sample. Since $R_H$ and $g$ are both measurable quantities, this so-called Hall effect can be used to determine the sign of charge carriers, carrier densities, and carrier mobilities, particularly in semiconductors. Further, once $R_H$ is known for a particular sample, the effect can be turned around and measurements of $\Delta V$ can be used to determine magnetic fields; used in this way, the device is called a Hall probe.

P9.8. In the classical model of conduction introduced by Drude (1900), the resistive forces on the carriers in a conducting material are represented by a viscous damping term in the equation of motion, so that the individual carriers of mass $m$ and charge $q$ move in accordance with

$$m\frac{dv}{dt} = q\mathbf{E} - b\mathbf{v}$$

when an external $\mathbf{E}$-field is applied. (a) Justify this view by showing that Eq. (1) predicts a terminal (drift) velocity, relate $b$ to the carrier mobility $\mu$, and, assuming that the carriers are electrons and that $\mu = -1.2 \times 10^{-3} \text{ m}^2/\text{V} \cdot \text{s}$, estimate how quickly the terminal velocity will be reached. Hint: Consider motion only in the $x$-direction. (b) Assume that this same approach applies when a uniform external $\mathbf{B}$-field is added, so that individual carriers now move in accordance with

$$m\frac{dv}{dt} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] - b\mathbf{v}$$

Take $\mathbf{B} = B\mathbf{k}$ and let $\mathbf{E}$ have all three components, $\mathbf{E} = E_1 \mathbf{i} + E_2 \mathbf{j} + E_3 \mathbf{k}$. Determine the terminal drift velocity $\mathbf{v}_d$ in terms of the components of $\mathbf{E}$ and show that

$$(v_{d1}, v_{d2}, v_{d3}) = \begin{pmatrix} \frac{\alpha b}{q} & \alpha B & 0 \\ -\alpha B & \frac{\alpha b}{q} & 0 \\ 0 & 0 & \frac{q}{b} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

8See footnote 2 on page 332.
where \( \alpha = 1/[B^2 + (b^2/q^2)] \). Finally, infer that for this case the mobility \( \mu \) and the conductivity \( g = qn\mu \) are not scalars, and show that \( \mathbf{J} \) is not parallel to \( \mathbf{E} \).
Chapter 10

Properties of Matter II: Dielectric Polarization

In considering the response of matter to applied electric fields, it is useful to distinguish two ideal types of matter. We have already introduced the perfect conductor. Microscopically, it contains large numbers of free charges that move and produce a macroscopic current whenever an external electric field is applied. Real materials, however, conduct only imperfectly and many materials, called insulators or dielectrics, conduct very poorly indeed. A perfect dielectric, again an idealization, contains no free (microscopic) charges, and an external electric field induces not a current but rather a redistribution of the charge in each molecule of the dielectric and perhaps a reorientation of these molecules. This dielectric response, called dielectric polarization, is the subject of this chapter. Although additional charge beyond that intrinsic to its atoms may be placed in or on a dielectric, we shall consider primarily the electrically neutral dielectric. Further, we shall confine our discussion here to static external fields, postponing the treatment of time-dependent polarization to Chapter 12.

In contrast to conduction, which is brought about by one field (the electric field) but itself produces a different field (the magnetic induction field), dielectric polarization is brought about by the same field that it also produces. Thus, when a dielectric is placed in an external electric field and is thereby polarized, the dielectric produces additional electric fields that modify the original fields both inside and outside the dielectric. This modification of the fields in turn alters the response of the dielectric, which again modifies the fields, and so on. Physically, the system ultimately reaches a steady state in which the dielectric is polarized to the extent required by the final field. From a calculational point of view, however, it would seem that we need to know the final field in order to determine the final polarization but we cannot determine that final field until we know the final polarization. The treatment of dielectric polarization is thus more complicated than that of conduction because we cannot ignore the fields produced by the dielectric as we determine the response of the dielectric to an external field.

Because of the difficulty pointed out in the previous paragraph, it is convenient to divide the overall treatment of dielectric polarization into two parts. In the first part, we shall develop a model for the structure of a dielectric and introduce descriptive concepts so that we can
(1) describe quantitatively what might be called a field-causing state of the dielectric, and

(2) determine from this field-causing state the field that it causes, both inside and outside the dielectric.

In this part, we do not ask how a particular field-causing state of the dielectric is brought about or maintained; we simply describe this state by giving values to the appropriate descriptive concepts and then utilize this description to calculate the resulting field. In the second part of the treatment of dielectric polarization, however, the establishment of the field-causing state is of primary concern; we seek methods by which to

(3) determine the response of a dielectric when the final field is assumed known.

Although steps (1) and (2) in this program can be treated quite generally, step (3) involves consideration of specific materials and will of necessity be more complicated, less general, and more empirical than the treatment of steps (1) and (2). As we shall see, for example, in Section 10.5, the results of these three steps taken together will permit us to calculate final fields and polarizations even though neither may be known before the calculation is begun.

As with conduction, there are two different approaches to the treatment of dielectric polarization. The more involved of these is the microscopic approach, which begins by adopting a detailed microscopic model for the composition of matter\(^1\)—matter consists of extremely small molecules, each of which has electric and/or magnetic properties that are presumably understood. These molecules are taken to exist in free space. In this view, there is consequently no intrinsic difference between free space occupied here and there by free charge and/or free currents and free space occupied by matter since (insofar as its electromagnetic properties are concerned) matter is composed ultimately of (microscopic) charges and currents. One describes the field-causing state of matter by specifying the location of every molecule and the (microscopic) charge and current distributions associated with each molecule. With this information as input, the formalism of Chapters 4–6 permits a detailed calculation of the corresponding electromagnetic field, which is called the microscopic electromagnetic field. It is the field experienced by test charges whose dimensions are comparable to molecular dimensions, and it fluctuates rapidly and erratically both in space (because on a molecular scale the charge and current distributions can hardly be regarded as smoothly varying) and in time (because thermal agitation of the molecules introduces an erratic time variation of charge and current distribution on the microscopic scale).

Although in principle the microscopic field can be fully determined by methods already available, in practice the very large number (\(\approx 10^{23}\)) of molecules typically involved precludes its actual evaluation by these means. Fortunately, for macroscopic problems, it is usually not necessary to know the microscopic state of affairs in detail. Only rarely are we interested in the variations of the field over distances comparable to molecular dimensions or on a time scale determined by the thermal fluctuations. In most instances, the presence of fields is detected by test charges that are large compared to molecules (though perhaps small macroscopically) and that move slowly compared to the velocities of thermal vibration (i.e., that stay in one place for a long time if times are measured on an atomic time scale).

\(^1\)We use the word matter rather than dielectric and we speak of electric and magnetic properties so that this discussion can be applied also to the phenomenon of magnetization.
Such test charges are sensitive not to the detailed microscopic field but to the time and space average of the microscopic field, where the averages are taken over temporal and spatial regions large microscopically but small macroscopically. The next step in the development of the full theory in the microscopic approach is therefore to introduce a macroscopic field as an average of the microscopic field. The detailed evaluation of this average turns out naturally to involve making a distinction between free charge and currents placed on matter and the so-called bound charge and current distributions that result from the charges that are an intrinsic part of the matter itself. In evaluating the average, one ultimately (and quite naturally) identifies a pair of auxiliary fields whose sources are essentially the free charges and currents alone. These auxiliary fields differ from the electric field and the magnetic induction field, whose sources are the free and bound distribution combined, by terms that involve directly and explicitly the time and space averages of the microscopic properties of the matter. Finding the macroscopic fields then involves finding expressions for these averages in terms of the fields.

In the formal treatment of the electromagnetic field in matter the microscopic approach is extremely valuable. It provides a deep insight into the structure of the macroscopic fields in the presence of matter, it begins with the vacuum form of Maxwell’s equations, and the auxiliary fields emerge naturally as the averages of the microscopic fields are evaluated. Although we shall use microscopic arguments occasionally and we shall begin with an examination of the microscopic response of molecules to external fields, we shall not adopt the microscopic approach in our basic development. Instead, we adopt the macroscopic approach at the very outset. The price we pay is the inapplicability (at least directly) of the vacuum form of Maxwell’s equations. We must introduce in a rather ad hoc fashion the descriptive terminology that would have arisen naturally had we adopted the microscopic approach. Once we have done so, however, the subject develops easily and with rather less mathematical complexity than from the microscopic view, although the final results are identical.

PROBLEM

P10.1. Explain qualitatively how the macroscopic fields defined as averages of the microscopic fields may be static even though the microscopic fields certainly are time-dependent.

10.1 The Microscopic Description: Electric Polarizability

As a prelude to a more detailed macroscopic development of dielectric theory, we shall consider briefly the response of individual molecules to static electric fields. We assume that the field varies slowly enough in space to be regarded as a constant over the region occupied by a single molecule. Then the molecules, each of which is overall electrically neutral, experience no net force from the field. Even an electrically neutral molecule, however, may have an electric dipole moment and hence may experience torques in an external field. We are thus led to explore the electric dipole moments of molecules as a possible microscopic basis for understanding the dielectric response of macroscopic matter.

2The interested reader is referred, for example, to J. D. Jackson, Classical Electrodynamics (John Wiley & Sons, Inc., New York, 1962), p. 103ff.
Molecules can be divided into two categories by the character of their dipole moments. Some molecules, such as the water molecule, have a naturally asymmetric internal charge distribution and hence possess a permanent dipole moment even though the molecule is overall electrically neutral. When placed in an electric field (whatever its origin), these molecules will tend to align themselves with their dipole moments parallel to the field (P3.10). This tendency of permanent dipoles to align themselves, however, is opposed by the thermal tendency to randomness. Thus, a given permanent dipole assumes an average dipole moment that would be zero except that in the external field the dipole favors a preferred orientation as it undergoes thermal agitation.

To express this discussion more quantitatively, consider the following situation: A dielectric at absolute temperature $T$ composed of permanent dipoles having dipole moments of magnitude $p_0$ is placed in an external electric field. The dipoles adjust to some sort of average equilibrium position, in which state a particular dipole experiences a total field consisting of the applied external field plus the time-averaged microscopic field established by all other dipoles when the external field is present. Let the time average (over the microscopic fluctuations) of this total field be $E_m$. Combination of statistical mechanics with considerations of electrostatic energy (P10.2) leads ultimately to the result that the average dipole moment of a single dipole is given by

$$\langle p_0 \rangle_{\text{perm}} = \alpha E_m; \quad \alpha = \frac{p_0^2}{3kT}$$

where $k$ is Boltzmann’s constant and the condition $p_0 E_m \ll kT$ has been imposed. (This apparent restriction to “high” temperature is not as much of a restriction as it might seem. Typically $p_0$ is quite small and fields approaching the very large value of $10^9$ V/m must be applied before the condition is violated even at room temperature.) The constant $\alpha$ in Eq. (10.1) might be called an orientational molecular polarizability, since it measures the extent to which a molecule possessing a permanent dipole moment orients itself in a polarizing field. In calculating this result, we assumed the dipoles to interact dominantly with the external field and only weakly with one another. The interaction of dipoles with dipoles has been taken into account only through the contribution of the internally generated microscopic field and through recognition that these interactions are responsible for the thermal agitation that each molecule undergoes.

A second category of molecule contains those molecules that do not possess a permanent dipole moment. When placed in an electric field, these molecules acquire an induced dipole moment that depends on the strength of the applied field. In effect, the external field causes a distortion of the molecule because the positive nucleus experiences a force in one direction while the negative electron cloud experiences a force in the opposite direction. Provided that the distortion is not too great, the induced dipole moment is in many cases proportional to the total field experienced by the molecule, i.e.

$$p_{\text{ind}} = \alpha_0 E_m$$

Here $\alpha_0$ might be called a deformational molecular polarizability. A very simple classical model that predicts Eq. (10.2) is explored in P10.3.

---

3 For the rest of this chapter we understand that the electric dipole moment is meant.

4 Actually, the situation is even a bit more complicated. We really want the microscopic field established at a particular molecular site by all other molecules when the other molecules, however, have the orientations that they have when the molecule in question is present.
In a still more general situation, a molecule may exhibit both a permanent dipole moment and a capacity to acquire an additional induced moment when placed in an external field. For such molecules, the total dipole moment is related to the field \( \mathbf{E}_m \) by

\[
\mathbf{p} = \langle \mathbf{p} \rangle_{\text{perm}} + \mathbf{p}_{\text{ind}} = \left( \frac{\mathbf{p}_0^2}{3kT} + \alpha_0 \right) \mathbf{E}_m = \alpha_t \mathbf{E}_m
\]  

(10.3)

where the total molecular polarizability \( \alpha_t \) is given by

\[
\alpha_t = \frac{\mathbf{p}_0^2}{3kT} + \alpha_0
\]  

(10.4)

For some dielectrics, this molecular polarizability can be determined from macroscopically measurable quantities by using the Clausius-Mossotti\(^5,6\) equation (Section 10.6). Sometimes the orientational polarizability and the deformational polarizability can be separated if measurements of the total polarizability over a range of temperatures are made.

In this section, we have been concerned mostly with the qualitative responses of molecules to external electric fields. Our quantitative statements have therefore pertained to the simplest dielectrics. In more general circumstances, we would have to allow for a possible field dependence in both \( \alpha \) and \( \alpha_0 \) and for possible anisotropies that could make \( \alpha \) and/or \( \alpha_0 \) depend on the direction of the external field (P10.25). A detailed treatment of these more complicated cases or of the more satisfactory quantum mechanical approach to determining molecular response would, however, lead us away from our primary examination of the macroscopic dielectric response.

Fortunately, we need not anticipate a need to look at multipole moments of higher order than the dipole moment. A molecule with a permanent moment no lower than the quadrupole moment experiences neither force nor torque in a field whose spatial variation is slow enough that the field can be regarded as constant over the molecule (P10.5). In this field, such a molecule exhibits no response arising from its permanent moment. Instead, the primary response of this molecule and also of molecules with no permanent moments whatever is to acquire an induced dipole moment, and we have returned to a case already considered. In essence, we can ignore moments higher than the dipole moment in treating the response of molecules (and hence of dielectrics) to external fields having a slow spatial variation. Further, once the dipole moment is present, its contribution to the resulting total field dominates the contribution of any possible higher moments as soon as the observation point is more than a few molecular dimensions away from the molecule itself. Thus, at observation points that are at least a few molecular dimensions away from a molecule (which is almost always the case), we can ignore moments higher than the dipole moment in calculating the contribution of the molecule to the total field.

PROBLEMS

**P10.2.** Accepting (1) that the energy of an electric dipole of moment \( \mathbf{p} \) in an electric field \( \mathbf{E} \) is \(- \mathbf{p} \cdot \mathbf{E}\) (P4.18) and (2) that the distribution of dipoles over these energy states is given in statistical thermodynamics by \( e^{-\text{energy}/kT} \), where \( k \) is Boltzmann’s constant and \( T \) is

---

5 German physicist and mathematician Rudolf Julius Emanuel Clausius, b. 2 January 1822 in Köslin, Pomerania, Prussia (now Koszalin, Poland); d. 24 August 1888 in Bonn, Rhine Province, Prussia.

6 Italian physicist Ottaviano-Fabrizio Mossotti, b. 18 April 1791 in Novara, Italy; d. 20 March 1863 in Pisa, Italy.
the absolute temperature, show that the average value $\langle p_0 \rangle$ of the dipole moment of a permanent dipole in an electric field $E_m = E_m \hat{k}$ is given by $p_0 k \mathcal{L}(p_0 E_m / kT)$, where the Langevin function\footnote{French physicist Paul Langevin, b. 23 January 1872 in Paris, France; d. 19 December 1946 in Paris, France.} $\mathcal{L}(\lambda)$ is defined by

$$\mathcal{L}(\lambda) = \coth \lambda - \frac{1}{\lambda}$$

Here $p_0$ is the magnitude of the permanent dipole moment of the dipole. Show also that Eq. (10.1) emerges from this expression when $p_0 E_m \ll kT$. How large can the field $E_m$ be before this condition is violated at room temperature? \textit{Hints:} (1) Let $p = p_0 (\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k})$. (2) Note from statistical mechanics that

$$\langle p_0 \rangle = \frac{\int p e^{p \cdot E_m / kT} \sin \theta d\theta d\phi}{\int e^{p \cdot E_m / kT} \sin \theta d\theta d\phi}$$

where $\theta$ and $\phi$ are the usual spherical angles with $\theta$ being the angle between $p$ and $E_m$. (3) Typical values for permanent molecular dipole moments are on the order of $10^{-30}$ C·m.

\textbf{P10.3.} An extremely crude calculation of the deformational molecular polarizability can be made for a single atom by the following model. Let the atom consist of a positive nucleus of charge $q$ surrounded by an electron charge cloud of radius $r$ containing a total charge $-q$ that is distributed \textit{uniformly} throughout a sphere of radius $r$. In the absence of a polarizing field, the nucleus is located at the center of the spherical charge cloud. When a polarizing field $E_m$ is present, suppose that the electron cloud is displaced without distortion. Thus, in the presence of a polarizing field, the atom (in some crude sense) will have the appearance shown in Fig. 10.1. The induced dipole moment of the atom is therefore $qd \hat{n}$. Since the atom is at rest, the nucleus in particular must experience zero force. (a) Find the electric field established by the electron charge cloud at the position of the nucleus. (b) Determine the total force on the nucleus and then show that $\alpha_0 = 4\pi \epsilon_0 r^3$. (c) Insert a typical value of $r$ into the answer to part (b) and estimate the value of $\alpha_0$. (d) Use this value of $\alpha_0$ to estimate the induced dipole moment in a fairly large field, say $10^6$ V/m. (e) Taking $q$ to be the charge on the proton, estimate the displacement of the electron cloud from the nucleus and compare the result with typical atomic dimensions.

\textbf{P10.4.} Estimate the electric field established by the nucleus of a hydrogen atom at the position of the electron and compare the value with typical laboratory fields. What does this comparison suggest about the possibility of effecting severe distortion of a molecule through the application of external fields?
**P10.5.** Describe the behavior of a linear quadrupole of arbitrary initial orientation in a uniform electric field. Do your conclusions apply also to more general quadrupoles? Defend your answer.

**10.2 The Macroscopic Description: Dielectric Polarization**

As suggested in Section 10.1, we shall now picture a dielectric as a material composed microscopically of electrically neutral molecules that have either a permanent dipole moment or a capability for acquiring an induced dipole moment (or perhaps both). Without asking how to relate a particular field-causing or polarized state of this material to the causes of that state (see Section 10.5), let us consider a dielectric sample that is already polarized. Microscopically the molecular dipoles in this dielectric then possess a nonzero average dipole moment. Now, if $p_i$ is the average dipole moment of the $i$-th molecule, the total dipole moment $\Delta p$ of a selected small volume $\Delta v$ within the dielectric is given by

$$\Delta p = \sum_i p_i \quad (10.5)$$

where the sum extends over all the molecules in $\Delta v$. In our subsequent theory, however, it is more convenient to work instead with a quantity determined by the average properties of the dielectric over a small volume. We therefore introduce the dipole moment per unit volume or dielectric polarization $P(r)$, defined by

$$P(r) = \lim_{\Delta v \to 0} \frac{\Delta p}{\Delta v} \quad (10.6)$$

where the meaning of the limit is the same as always: $\Delta v$ becomes macroscopically small but remains large enough to contain a large number of microscopic dipoles. The polarization—usually the adjective dielectric is suppressed—is a macroscopic concept that plays the same role in specifying the macroscopic state of a polarized dielectric as the charge density plays in specifying the state of a general charge distribution. Of course, the polarization may vary throughout the dielectric and indeed may be time-dependent as well. It does not, however, depend on the origin of the coordinate system chosen for its expression, for the dipole moment of a charge distribution that is overall electrically neutral is invariant to translation of the coordinate system (P3.19).¹⁸

Among other things (Sections 10.3 and 10.4), knowledge of the polarization permits calculation of the macroscopic dipole moment of the dielectric. From Eq. (10.6) we infer that the dipole moment of a small volume element $\Delta v$ centered at $r$ is given by $\Delta p \approx P(r) \Delta v$. Hence, the dipole moment of a macroscopic region of the dielectric is given by

$$p = \int P(r) dv \quad (10.7)$$

where the integral extends over the volume whose dipole moment is sought and in particular may extend over the entire dielectric.

¹⁸The definition of polarization given in this paragraph is adopted in all systems of units with which the author is familiar.
Figure 10.2: Coordinates and vectors for calculating the electric field at a point exterior to a polarized dielectric.

PROBLEM
P10.6. The dipole moment of the water molecule is about $6.2 \times 10^{-30}$ C·m. Find the maximum polarization of water vapor at a temperature of 100°C and a pressure of 1 atmosphere.

10.3 The Macroscopic Scalar Potential and Electric Field at a Point Exterior to a Polarized Dielectric

In Sections 10.3 and 10.4, we shall calculate the contribution made to the macroscopic electric field by a polarized dielectric whose polarization $\mathbf{P}(r')$ at every point $r'$ within the dielectric is known. Consider first an observation point $r$ exterior to the dielectric. In microscopic terms, such an observation point is far away from all of the dipoles composing the dielectric. Thus, a small element of the dielectric having volume $\Delta v'$ and located at $r'$ (Fig. 10.2) can be treated as a dipole of moment $\mathbf{P}(r') \Delta v'$, regardless of where within the dielectric the point $r'$ lies. The contribution of this element of the dielectric to the electrostatic potential at $r$ is then given by

$$\Delta V = \frac{\mathbf{P}(r') \Delta v' \cdot (r - r')}{4\pi \varepsilon_0 |r - r'|^3}$$  \hspace{1cm} (10.8)

[see Eq. (4.51)] and integration of Eq. (10.8) over the volume of the dielectric yields

$$V(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\mathbf{P}(r') \cdot (r - r')}{|r - r'|^3} dv'$$  \hspace{1cm} (10.9)
10.3. POTENTIAL AND FIELD AT AN EXTERIOR POINT

for the contribution of the entire dielectric to the total electrostatic potential at $r$.

Equation (10.9) can be given a more useful physical interpretation if it is rewritten. First, we substitute the identity

$$\nabla' \frac{1}{r - r'} = \frac{r - r'}{|r - r'|^3}$$  (10.10)

[compare Eq. (4.52)] to obtain

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \left[ P(r') \cdot \nabla' \frac{1}{|r - r'|} \right] dv'$$  (10.11)

Then we use the vector identity in Eq. (C.11) in the form

$$Q \cdot \nabla' \Phi = \nabla' \cdot (\Phi Q) - \Phi \nabla' \cdot Q$$  (10.12)

with $\Phi = 1/|r - r'|$ and $Q = P$, to obtain

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \left( \nabla' \cdot \frac{P(r')}{|r - r'|} - \frac{\nabla' \cdot P(r')}{|r - r'|} \right) dv'$$  (10.13)

Continuing, we use the divergence theorem in the first term to find that

$$V(r) = \frac{1}{4\pi\epsilon_0} \oint \frac{P(r') \cdot \hat{n}(r')}{|r - r'|} dS' + \frac{1}{4\pi\epsilon_0} \int \left[ -\nabla' \cdot P(r') \right] dv'$$  (10.14)

where the surface integral extends over the surface bounding the dielectric. Finally, we introduce a unit outward normal $\hat{n}(r')$ to the surface at the point $r'$. Then, with $dS' = |dS'|$, we have that $dS' = \hat{n}(r') dS'$ and further that

$$V(r) = \frac{1}{4\pi\epsilon_0} \oint \frac{P(r') \cdot \hat{n}(r')}{|r - r'|} dS' + \frac{1}{4\pi\epsilon_0} \int \left[ -\nabla' \cdot P(r') \right] dv'$$  (10.15)

This result, however, is immediately recognized as the potential established by a charge distribution described by a volume charge density

$$\rho_b(r) = -\nabla \cdot P(r)$$  (10.16)

and a surface charge density

$$\sigma_b(r) = P(r) \cdot \hat{n}(r)$$  (r on the surface)  (10.17)

[See Eq. (4.57).] Thus, at exterior points, the polarized dielectric produces the same potential as these so-called bound charges, and we can replace our description in terms of the macroscopic polarization with this equivalent description in terms of bound charges if we find it convenient to do so.\(^9\)

We can even see qualitatively how these bound charges arise. Consider the surface distribution first. When a dielectric is polarized, all of the dipoles have a preferred orientation. Thus, at some point on the surface of the polarized dielectric, all of the dipoles tend to point the same way, say with the positive charges out. A net positive charge thus appears on the surface at that point [Fig. 10.3(b)]. In the unpolarized state, some of these

\(^9\)The expressions in Eqs. (10.16) and (10.17) for the bound charge densities apply in all systems of units with which the author is familiar.
CHAPTER 10. PROPERTIES OF MATTER II: DIELECTRIC POLARIZATION

Figure 10.3: An enlarged view of the surface of (a) an unpolarized and (b) a polarized dielectric.

dipoles would point the other way, leaving the surface apparently neutral to a macroscopic observer [Fig. 10.3(a)].

Turning now to a qualitative interpretation of the volume charge density, we note that \( \rho_b = 0 \) unless \( \mathbf{P}(\mathbf{r}) \) is nonuniform. Assume a dielectric that is initially unpolarized. As the material is polarized, the interior charges are redistributed. Some of the charge in a selected but fixed volume element moves out one side while charge moves in from the surrounding volumes at the other side. If the polarization is uniform, the influx balances the outgo and the volume element remains electrically neutral despite the polarized state of the matter. If the polarization is not uniform, however, the two charge motions do not balance and the volume element acquires a net charge of one sign or the other. It is shown in P10.7 that the qualitative mechanisms described here yield Eqs. (10.16) and (10.17) when made quantitative. Further, it is shown in P10.8 that polarization of an initially neutral dielectric does not change that overall neutrality, i.e., that the sum of all bound charges is zero.

Given the potential, we can readily calculate the electric field established at an exterior point by a polarized dielectric; we simply take the negative gradient of the potential, finding that

\[
\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint \frac{\sigma_b(\mathbf{r}')|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|^3} \, dS' + \frac{1}{4\pi\epsilon_0} \int \frac{\rho_b(\mathbf{r}')|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|^3} \, dv'
\]  

(10.18)

Equation (10.18) for the field and Eq. (10.15) for the potential of course give only the contributions of the bound charges on the polarized dielectric; contributions from other sources must be added to those given by these equations.

PROBLEMS

P10.7. Convert the qualitative discussion of the physical origin of bound charges as presented in Section 10.3 into a quantitative derivation of Eqs. (10.16) and (10.17).

P10.8. Show that the total bound charge on a dielectric with arbitrary polarization is zero.

P10.9. A dielectric cube of side 2a has its center at the origin and its side parallel to the coordinate planes, and it has a radial polarization \( \mathbf{P}_{z} = br \), where \( b \) is a constant. Find the charge distribution equivalent to this dielectric and show that the total bound charge is zero.

P10.10. A dielectric cylinder of length \( L \) and radius \( R \) is uniformly polarized with the polarization parallel to the axis of the cylinder. For convenience let the cylinder lie along the \( z \)-axis with its center at the origin and take the polarization to be \( \mathbf{P} = P\hat{k} \). (a) Find the charge distribution equivalent to this dielectric. (b) Find the electrostatic potential at the point \((0, 0, z)\) and then use an available numeric processing program like IDL, MATLAB,
10.4. FIELD AT AN INTERIOR POINT

Figure 10.4: Geometry for calculating the electric field at a point interior to a polarized dielectric. The spherical surface $\Sigma_{II}$ centered at $r$ divides the dielectric into “near” and “far” regions as described in the text.

OCTAVE, or PYTHON to obtain careful graphs of this potential as a function of $z/R$ for various values of $L/R$ over the range $-\infty < z/R < \infty$. Hint: See P4.26. (c) Find the limiting form of the electrostatic potential at the point $(0,0,z)$ in the region $z \gg L$, $z \gg R$ and comment on the multipole character of the distribution. (You may want to use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica’s ability to determine Taylor expansions.) (d) What is the dipole moment of the distribution as inferred from the polarization? Is your result in agreement with the dipole moment inferred from the limiting form of the potential?

**P10.11.** A spherical dielectric with its center at the origin has radius $a$ and uniform polarization $\mathbf{P} = P\hat{k}$. (a) Determine the equivalent charge distribution. (b) Calculate the dipole moment of this dielectric both from the polarization and from the equivalent charge distribution.

10.4 The Macroscopic Electric Field at a Point Interior to a Polarized Dielectric

The calculation of the macroscopic electric field at a point interior to a polarized dielectric is more complicated than the calculation of the field at an exterior point because the interior point is microscopically close to at least some of the molecules in the dielectric. Since these “near” molecules must be treated differently than the more numerous molecules that are microscopically remote from the interior point, we therefore divide the calculation of the interior field into two parts by identifying a sphere of radius $R$ centered at the interior point $r$, choosing $R$ large enough in microscopic terms so that all molecules outside of $\Sigma_{II}$ are microscopically far away from $r$. (See Fig. 10.4.) We then imagine the molecules within $\Sigma_{II}$ to be removed temporarily, without, however, changing the polarization of the remaining
dielectric. Once the near molecules have been removed, the point \( \mathbf{r} \) is an exterior point; therefore, the contribution \( \mathbf{E}_I(\mathbf{r}) \) made to the macroscopic field at \( \mathbf{r} \) by the molecules outside of \( \Sigma_{II} \) can be evaluated by the methods of Section 10.3. We now replace the molecules that were temporarily removed and calculate their contribution \( \mathbf{E}_{II}(\mathbf{r}) \) to the macroscopic field at \( \mathbf{r} \) by an averaging process to be described later in this section. In effect, we express the interior field \( \mathbf{E}_{\text{interior}}(\mathbf{r}) \) as the sum of two parts,

\[
\mathbf{E}_{\text{interior}}(\mathbf{r}) = \mathbf{E}_I(\mathbf{r}) + \mathbf{E}_{II}(\mathbf{r}) \tag{10.19}
\]

where \( \mathbf{E}_I(\mathbf{r}) \) is the contribution of the “distant” molecules and \( \mathbf{E}_{II}(\mathbf{r}) \) is the contribution of the near molecules. We then adopt different methods for evaluating the two contributions. In some respects we are defining what is meant by the electric field at an interior point—macroscopic test charges cannot easily be placed inside a dielectric—and the procedure so far outlined will certainly not be satisfactory unless the final result for \( \mathbf{E}_{\text{interior}}(\mathbf{r}) \) is independent of the radius \( R \) of the (microscopically large but macroscopically small) sphere \( \Sigma_{II} \).

Because the calculations are involved, we present the comparatively simple final results before justifying them. We shall find that the two contributions to the total interior field are given, respectively, by

\[
\mathbf{E}_I(\mathbf{r}) = \mathbf{E}(\mathbf{r}) + \frac{\mathbf{P}(\mathbf{r})}{3\epsilon_0}, \quad \mathbf{E}_{II}(\mathbf{r}) = -\frac{\mathbf{P}(\mathbf{r})}{3\epsilon_0} \tag{10.20}
\]

where \( \mathbf{E} \) is the field contributed by the charge distribution equivalent to the entire dielectric as given by Eq. (10.18) (evaluated, however, at the interior point) and \( \mathbf{P} \) is the polarization of the dielectric. Substituting Eq. (10.20) into Eq. (10.19), we find that

\[
\mathbf{E}_{\text{interior}}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) \tag{10.21}
\]

and the validity of Eq. (10.18) for interior points is therefore established.

We shall now turn to the detailed derivation of Eq. (10.20). To facilitate the discussion, we introduce the notation

\[
\begin{align*}
v & = \text{volume of the original dielectric} \\
v_{II} & = \text{volume of sphere of radius } R \\
v_I & = v - v_{II} \\
\Sigma & = \text{surface of original dielectric} \\
\Sigma_{II} & = \text{surface of sphere of radius } R
\end{align*}
\]

as shown in Fig. 10.4.

Consider first the field \( \mathbf{E}_I(\mathbf{r}) \). Since \( \mathbf{r} \) is exterior to the volume \( v_I \) and further is microscopically remote from every point of that volume, the results of Section 10.3 apply and we find from Eq. (10.18) that

\[
\mathbf{E}_I = \frac{1}{4\pi\epsilon_0} \int_{v_I} \left\{ \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right\} (\mathbf{r} - \mathbf{r}') \, dv' \\
+ \frac{1}{4\pi\epsilon_0} \int_{\Sigma} \frac{\mathbf{P}(\mathbf{r}') \cdot \mathbf{n}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') \, dS' \\
+ \frac{1}{4\pi\epsilon_0} \int_{\Sigma_{II}} \frac{\mathbf{p}(\mathbf{r}') \cdot \mathbf{n}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') \, dS' \tag{10.22}
\]
where on $\Sigma$ and $\Sigma_{II}$ the direction of the unit outward normal is reckoned relative to a point inside the volume $v_I$. The third integral in Eq. (10.22) now has a simple evaluation. Since the volume $v_{II}$ is by hypothesis macroscopically small, the polarization $P(r')$ does not vary macroscopically over its surface $\Sigma_{II}$. Thus, in this third integral, $P(r')$ may be replaced by its value $P(r)$ at the center of the sphere. Furthermore, the vector $\hat{n}(r')$ at a point $r'$ on $\Sigma_{II}$ is given by

$$\hat{n}(r') = \frac{r - r'}{|r - r'|} \quad (10.23)$$

where the outward direction at a point on $\Sigma_{II}$ is a direction *toward* the point $r$. Thus, the third integral $I_3$ in Eq. (10.22) can be written in the form

$$I_3 = \frac{1}{4\pi\varepsilon_0} \int_{\Sigma_{II}} P(r) \cdot \hat{s} \frac{r - r'}{|r - r'|^4} (r - r') dS' \quad (10.24)$$

Now, introduce a spherical polar coordinate system with its center at the point $r$ and its polar axis in the direction of $P(r)$ (Fig. 10.5). Let $\hat{s}$ be a unit vector in the radial direction in this coordinate system. Then, we find that

$$r - r' = -R\hat{s}; \quad |r - r'| = R; \quad dS' = R^2 \sin \theta d\theta d\phi \quad (10.25)$$

and Eq. (10.24) becomes

$$I_3 = \frac{1}{4\pi\varepsilon_0} \int [P(r) \cdot \hat{s}] \sin \theta d\theta d\phi = \frac{P(r)}{4\pi\varepsilon_0} \int \hat{s} \cos \theta \sin \theta d\theta d\phi \quad (10.26)$$

where we have recognized that the angle between $P(r)$ and $\hat{s}$ is $\theta$. Now, in terms of Cartesian unit vectors $\hat{\xi}, \hat{\eta}, \hat{\zeta}$ in the three coordinate directions shown in Fig. 10.5, the unit radial vector $\hat{s}$ has the expression

$$\hat{s} = \sin \theta \cos \phi \hat{\xi} + \sin \theta \sin \phi \hat{\eta} + \cos \theta \hat{\zeta} \quad (10.27)$$
which on substitution into Eq. (10.26) leads to the evaluation

\[ I_3 = \frac{P(r)}{3\varepsilon_0} \]  

(10.28)

We therefore find from Eq. (10.22) that

\[ \mathbf{E}_I(r) = \frac{1}{4\pi\varepsilon_0} \int_{vI} \frac{\left[ -\nabla' \cdot \mathbf{P}(r') \right]}{|\mathbf{r} - \mathbf{r}'|^3} |r - r'| \, dv' + \frac{1}{4\pi\varepsilon_0} \int_{\Sigma} \frac{\mathbf{P}(r') \cdot \mathbf{n}(r')}{|\mathbf{r} - \mathbf{r}'|^3} |r - r'| \, dS' + \frac{\mathbf{P}(r)}{3\varepsilon_0} \]  

(10.29)

Equation (10.29) would be further simplified if the integral over \( v_I \) could be extended to include the volume \( v_{II} \) of the small sphere. To do so would require adding the integral

\[ I_4 = \frac{1}{4\pi\varepsilon_0} \int_{v_{II}} \frac{\left[ -\nabla' \cdot \mathbf{P}(r') \right]}{|\mathbf{r} - \mathbf{r}'|^3} |r - r'| \, dv' \]  

(10.30)

to Eq. (10.29). Over the macroscopically small volume \( v_{II} \), however, we can treat \( -\nabla' \cdot \mathbf{P}(r') \) as a constant. The integral \( I_4 \) then represents the electric field at the center of a uniformly charged sphere of radius \( R \) (Why?), and consequently \( I_4 = 0 \). (See P10.12.) Thus, \( I_4 \) can be added to Eq. (10.29) with no change and we have finally that

\[ \mathbf{E}_I(r) = \frac{1}{4\pi\varepsilon_0} \int_v \frac{\left[ -\nabla' \cdot \mathbf{P}(r') \right]}{|\mathbf{r} - \mathbf{r}'|^3} |r - r'| \, dv' + \frac{1}{4\pi\varepsilon_0} \int_{\Sigma} \frac{\mathbf{P}(r') \cdot \mathbf{n}(r')}{|\mathbf{r} - \mathbf{r}'|^3} |r - r'| \, dS' + \frac{\mathbf{P}(r)}{3\varepsilon_0} \]

(10.31)

where \( \mathbf{E}(r) \) is the field given at the interior point \( r \) by evaluating the exterior expression, Eq. (10.18) at the interior point; i.e., \( \mathbf{E}(r) \) is the field established at the interior point by the bound charge distribution equivalent to the entire dielectric. Equation (10.31) confirms the first member of Eq. (10.20) for the contribution made by the molecules outside \( \Sigma_{II} \).

To evaluate the contribution made by the molecules within \( \Sigma_{II} \) to the macroscopic field at the point \( r \), we note first that this contribution must be insensitive to the precise microscopic location of \( \Sigma_{II} \) within the dielectric. That is, microscopic displacements of the point \( r \) within the dielectric cannot alter the macroscopic field \( \mathbf{E}_{II}(r) \). We therefore define \( \mathbf{E}_{II}(r) \) by the following average. Let \( r \) be located randomly—perhaps within a molecule, perhaps in the space between molecules—and assume that all molecules within \( \Sigma_{II} \) are characterized by the same time-averaged microscopic charge distribution (which is equivalent to assuming that the volume \( v_{II} \) is macroscopically small and that all molecules in \( v_{II} \) are of the same chemical species\(^{10}\)). Finally, let the volume \( v_{II} \) contain \( N \) molecules. Then the probability of finding a molecule centered within the infinitesimal volume element \( dv' \) centered at the point \( r' \) somewhere in \( v_{II} \) is \( N \, dv' / v_{II} \), and the average contribution made by the element \( dv' \) at \( r' \) to the microscopic field at \( r \) is given by

\[ d\mathbf{E}_{II} = \frac{N \, dv'}{v_{II}} \mathbf{E}_{mol}(r, r') \]  

(10.32)

\(^{10}\)If several different types of molecule are present, the argument must be extended to include a sum over these several types.
10.4. FIELD AT AN INTERIOR POINT

Figure 10.6: An exaggerated view of the “near” region in a dielectric. The region occupied by a single molecule is shown.

where \( E_{\text{mol}}(\mathbf{r}, \mathbf{r}') \) is the time average of the microscopic field produced at \( \mathbf{r} \) by a molecule whose center is at \( \mathbf{r}' \). All of the molecules in \( v_{II} \) thus produce an average microscopic field at \( \mathbf{r} \) given by

\[
E_{II} = \frac{N}{v_{II}} \int_{v_{II}} E_{\text{mol}}(\mathbf{r}, \mathbf{r}') \, dv'
\]  

(10.33)

and we take this average microscopic field as the contribution of the molecules in \( v_{II} \) to the macroscopic field at \( \mathbf{r} \).

To obtain a more explicit evaluation of Eq. (10.33), we use the above assumption that all molecules in \( v_{II} \) can be characterized by the same time-averaged molecular charge density, which we take to be given at the point \( \mathbf{r}_1 \) for a molecule whose center is at \( \mathbf{r}_c \) by \( \rho_{\text{mol}}(\mathbf{r}_1 - \mathbf{r}_c) \).

Then, the field \( E_{\text{mol}}(\mathbf{r}, \mathbf{r}') \) is given by Eq. (4.7), viz.,

\[
E_{\text{mol}}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0} \int \rho_{\text{mol}}(\mathbf{r}'' - \mathbf{r}_1) \frac{\mathbf{r} - \mathbf{r}''}{|\mathbf{r} - \mathbf{r}''|^3} \, dv''
\]  

(10.34)

where \( \mathbf{r}'' \) locates a point in the molecule and the proper limits are effectively imposed by the function \( \rho_{\text{mol}} \), which is zero outside the molecule. More conveniently, we can write Eq. (10.34) as an integral on the variable \( \mathbf{r}'' = \mathbf{r}''' - \mathbf{r}' \), finding

\[
E_{\text{mol}}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0} \int \rho_{\text{mol}}(\mathbf{r}'') \frac{\mathbf{r} - \mathbf{r}' - \mathbf{r}''}{|\mathbf{r} - \mathbf{r}' - \mathbf{r}''|^3} \, dv''
\]  

(10.35)

The relationships among \( \mathbf{r} \), \( \mathbf{r}' \), \( \mathbf{r}'' \), and \( \mathbf{r}''' \) are illustrated in Fig. 10.6. In terms of the (unknown) molecular charge density, Eq. (10.33) now becomes

\[
E_{\text{mol}}(\mathbf{r}, \mathbf{r}') = \frac{N}{4\pi\epsilon_0 v_{II}} \int_{v_{II}} \rho_{\text{mol}}(\mathbf{r}'') \left[ \int \frac{\mathbf{r} - \mathbf{r}' - \mathbf{r}''}{|\mathbf{r} - \mathbf{r}' - \mathbf{r}''|^3} \, dv' \right] \, dv''
\]  

(10.36)
where we have chosen to do the integral on the single primed variables first, since the full dependence of the integrand on that variable is known. The integral $I_1$ in square brackets, however, is more conveniently evaluated if we write it as an integral on the variable $r_{iv} = r' - r$ which locates the point $r'$ relative to the center of $v_{II}$ (Fig. 10.6); we thus have that

$$I_1 = \int_{v_{II}} \frac{(-r'') - r_{iv}}{|(-r'') - r_{iv}|^3} dv_{iv} \quad (10.37)$$

which is written to facilitate the recognition that $I_1$ is numerically the same as the electric field at the point $(-r'')$ inside a spherical charge distribution having the uniform charge density $\rho(r_{iv}) = 4\pi\epsilon_0$.$^{11}$ We can therefore evaluate $I_1$ by exploiting Gauss’s law to calculate the corresponding electric field. Because this source distribution is invariant to rotation about any diameter, the field itself can be a function only of the distance $R$ from the center to the point of observation and must be radially directed, i.e.,

$$E(R) = E(R)\hat{R} \quad (10.38)$$

where $\hat{R}$ is the position vector of the observation point relative to the center of the sphere. Since the flux out of a spherical surface $\Sigma'$ having radius $R$ and centered at the center of the distribution is therefore given by

$$\oint_{\Sigma'} E \cdot dS = 4\pi R^2 E(R) \quad (10.39)$$

and the total charge inside the volume $V$ bounded by $\Sigma'$ is given by

$$\int_V \rho dv = 4\pi\epsilon_0 \left(\frac{4}{3}\pi R^3\right) \quad (10.40)$$

Gauss’s law, which requires the flux to be $(1/\epsilon_0)$ times the total charge, yields that

$$4\pi R^2 E(R) = 4\pi \left(\frac{4}{3}\pi R^3\right) \implies E(R) = \frac{4}{3}\pi R \implies E(R) = \frac{4}{3}\pi R\hat{R} = \frac{4}{3}\pi\hat{R} \quad (10.41)$$

Thus, we obtain the value of $I_1$ as

$$I_1 = E(-r'') = -\frac{4}{3}\pi r'' \quad (10.42)$$

and, finally, on substitution of Eq. (10.42) for the term in square brackets in Eq. (10.36), we find that

$$E_{II}(r) = \frac{N}{4\pi\epsilon_0 v_{II}} \left(-\frac{4\pi}{3}\right) \int r''\rho_{mol}(r'') dv'' \quad (10.43)$$

The remaining integral, however, is the dipole moment of a single molecule located near the point $r$ in the dielectric. Thus, the integral times $N$ is the total dipole moment in $v_{II}$ and that product divided by $v_{II}$ is the macroscopic polarization $P(r)$ of the dielectric at $r$. Equation (10.43) therefore has the macroscopic evaluation

$$E_{II}(r) = -\frac{P(r)}{3\epsilon_0} \quad (10.44)$$

thus verifying the second member of Eq. (10.20) and also confirming the conclusion that the macroscopic interior field is given by the same expression that gives the exterior field, i.e., by Eq. (10.18).

$^{11}$Compare Eq. (4.7) with $dq' \rightarrow \rho(r_{iv}) dv_{iv}$, $r' \rightarrow r_{iv}$, and $r \rightarrow -r''$.  

PROBLEM

P10.12. Assuming that \( v_{II} \) in Eq. (10.30) is small enough so that \( \nabla' \cdot P(r') \) can be regarded as constant over the volume show that the integral in Eq. (10.30) is zero (a) by relating the integral to the field at the center of a uniformly charged sphere and invoking symmetry, (b) by direct evaluation of the integral. Hint: Pick a spherical coordinate system with its origin at the center of \( v_{II} \), and (c) by noting Eqs. (10.10) and (C.21) and evaluating the resulting surface integral.

10.5 The Basic Equations of Electrostatics When Dielectrics Are Present

In this section, we shall translate the basic equations of electrostatics into a convenient form for treating problems involving dielectrics. We have already established that the static electric field produced at any point in space by a polarized dielectric can be viewed as originating in a suitable distribution of static charges in free space. Thus, the static electric field \( E \) established jointly by a polarized dielectric and by any simultaneously present distributions of free charge still satisfies both the restricted Faraday Law

\[
\oint E \cdot dl = 0 \quad ; \quad \nabla \times E = 0 \quad (10.45)
\]

and Gauss's Law

\[
\oint E \cdot dS = \frac{1}{\epsilon_0} \int \rho_t \, dv \quad ; \quad \nabla \cdot E = \frac{\rho_t}{\epsilon_0} \quad (10.46)
\]

provided we now interpret the charge density \( \rho_t \) in Gauss’s Law as the total charge density, which includes any “free” charges placed in space and any bound charges present on polarized dielectrics. Since we know how to relate the bound charge to the polarization \( P \), however, we can reexpress Eq. (10.46) in a better form. Let \( \rho \) now denote only the free charge density. Then

\[
\rho_t = \rho + \rho_b = \rho - \nabla \cdot P \quad (10.47)
\]

Hence, with some rearrangement, Eq. (10.46) yields

\[
\oint (\epsilon_0 E + P) \cdot dS = \int \rho \, dv \quad ; \quad \nabla \cdot (\epsilon_0 E + P) = \rho \quad (10.48)
\]

(The integral form is obtained after using the divergence theorem.) We stress again that the charge density appearing on the right-hand side is the free charge density; the bound charges have been explicitly introduced and appear now on the left-hand side in somewhat disguised form. We can now push the polarization, which we usually do not know initially, into the background altogether by introducing the displacement field \( D \) defined in mks units by

\[
D = \epsilon_0 E + P \quad (10.49)
\]

\[\text{The vector } D \text{ is defined differently in different systems of units but is always some linear combination of } E \text{ and } P. \text{ In cgs-esu, cgs-emu, Gaussian units, and Heaviside-Lorentz units } D \text{ is defined so that } \nabla \cdot D = 4\pi \rho, 4\pi \rho, 4\pi \rho, \text{ and } \rho, \text{ respectively.} \quad (\text{See P10.24.})\]
At least the flux of $D$ over closed surfaces or equivalently the divergence of $D$ is determined solely by the free charge, as evidenced by the new form of Gauss’s Law,

$$\oint D \cdot dS = \int \rho \, dv \quad \nabla \cdot D = \rho$$

(10.50)

obtained by substituting Eq. (10.49) into Eq. (10.48). The displacement field $D$ is the first of the two auxiliary fields mentioned in the introductory paragraphs of Chapter 10. Although we shall continue to regard $E$ as the basic field, arguments supporting the opposite view can be presented. Whichever field is viewed as basic, the field $E$ remains the field that determines the force on a charged particle.

We have now determined that the static electric field in the presence of dielectrics satisfies the basic equations

$$\oint E \cdot dl = 0 \quad ; \quad \nabla \times E = 0 \quad \text{(restricted Faraday Law)}$$

(10.51)

$$\oint D \cdot dS = \int \rho \, dv \quad ; \quad \nabla \cdot D = \rho \quad \text{(Gauss’s Law for $D$)}$$

(10.52)

$$D = \epsilon_0 E + P$$

(10.53)

This system, of course, reduces to the vacuum form when there are no dielectrics and $P$ is therefore zero. In that case, $D$ and $E$ are trivially different and Eqs. (10.51) and (10.52) give sufficient information about the field in vacuum to determine that field. When polarizable dielectrics are present, however, $D$ may differ nontrivially from $E$, the two differential equations involve two essentially different vectors, and we can make little progress toward a solution of these equations until we know a relationship between $D$ and $E$. Equation (10.53) is a step toward specifying that relationship, but it is not a complete specification because the vector $P$ is still not known. The specific properties of the dielectric involved enter at this point because the extent to which a dielectric acquires a macroscopic polarization depends not only on the polarizing field but on the intrinsic nature of the material as well. Since the polarization depends on the forces that the charges experience when they are immersed in an electric field, it is reasonable to expect some constitutive relation of the general form

$$P = P(E)$$

(10.54)

to describe the response of the dielectric, even though the individual molecules in fact polarize in response to the microscopic rather than the macroscopic field. A more explicit form for Eq. (10.54) can be determined only by an empirical study of specific dielectrics or, in some cases, by a detailed quantum mechanical calculation. Once the form of Eq. (10.54) is known for a particular material, Eqs. (10.51)–(10.53) provide sufficient information to determine the two fields $E$ and $D$ if the free charge distribution and/or suitable boundary conditions—are known.

The constitutive relation appropriate to a particular material may in fact be quite complicated. A material composed of molecules with a permanent dipole moment may be characterized by a different constitutive relation than a material composed of non-polar molecules; some materials develop polarizations that are not in the same direction as the polarizing field; especially when the applied fields are strong, the polarization may not depend in any simple way on $E$; some materials do not respond in the same way to fields
applied in different directions; some few materials even exhibit polarization in the absence of a polarizing field (P10.22). Even if materials displaying these and other complications are excluded, however, there remain many materials that develop a polarization in the same direction as the polarizing field. For these dielectrics, the constitutive relation has the form

$$P = \chi_e(E)E$$  \hfill (10.55)

where the static dielectric susceptibility $\chi_e(E)$ may still depend on the field strength. For many materials, however, $\chi_e$ happens to be constant, at least if $E$ is not too large. These simplest of all dielectrics are called linear dielectrics and the displacement vector in such a dielectric is proportional to the electric field,

$$D = \epsilon_0 E + P = (\epsilon_0 + \chi_e)E = \epsilon E$$  \hfill (10.56)

where the static permittivity of the dielectric $\epsilon$ is defined by

$$\epsilon = \epsilon_0 + \chi_e$$  \hfill (10.57)

Since $\chi_e$ is necessarily positive (Why?), the permittivity $\epsilon$ always exceeds $\epsilon_0$ and the static dielectric constant (or the relative permittivity) $K_e$, defined by

$$K_e = \frac{\epsilon}{\epsilon_0} = 1 + \frac{\chi_e}{\epsilon_0}$$  \hfill (10.58)

always exceeds 1. Values of $K_e$ for a selection of common dielectric materials are shown in Table 10.1.

We shall conclude this section with an example that not only illustrates the use of the new form of Gauss’s Law but also provides the basis of one method for measuring dielectric constants. Consider a parallel plate capacitor (P4.16) having plates of area $A$ and separation $d$, let the space between the plates be filled with a linear dielectric of permittivity $\epsilon$ (Fig. 10.7), and ignore fringing so that the translational and rotational invariances of the whole arrangement require the fields to be perpendicular to the plates. We seek the polarization $P$, the electric field $E$, and the displacement $D$ in the region between the plates.

If we knew $P$, we could calculate the bound charge densities from $P$ and then calculate $E$ from the total charge. But we do not know $P$. Let us therefore begin by calculating $D,$
Table 10.1: Static Dielectric Constants for Selected Materials. The values in this table are quoted from the *Handbook of Chemistry and Physics* (Chemical Rubber Publishing Company, Cleveland, 1965), Forty-sixth Edition, pp. (E-49)–(E-55), and are used by permission of the Chemical Rubber Company.

<table>
<thead>
<tr>
<th>Material</th>
<th>Temperature</th>
<th>$K_e$</th>
<th>Material</th>
<th>Temperature</th>
<th>$K_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air (1 atm)</td>
<td>Room</td>
<td>1.000590</td>
<td>Oxygen (liquid)</td>
<td>80.0°K</td>
<td>1.507</td>
</tr>
<tr>
<td>Diamond</td>
<td>Room</td>
<td>1.500940</td>
<td>Oxygen (1 atm)</td>
<td>100°C</td>
<td>2.3</td>
</tr>
<tr>
<td>Glass: Corning 8870</td>
<td>Room</td>
<td>2.375-4.1 (a)</td>
<td>Polyethylene</td>
<td>Room</td>
<td>6.75</td>
</tr>
<tr>
<td>Glass: Pyrex 7070</td>
<td>Room</td>
<td>4.00</td>
<td>Porcelain</td>
<td>Room</td>
<td>2.8</td>
</tr>
<tr>
<td>Glass: Fused quartz</td>
<td>Room</td>
<td>3.75</td>
<td>Paraffin</td>
<td>Room</td>
<td>2.8</td>
</tr>
<tr>
<td>Hydrogen (liquid)</td>
<td>4.10°K</td>
<td>1.048</td>
<td>Water (hard)</td>
<td>Room</td>
<td>88.0</td>
</tr>
<tr>
<td>Helium (1 atm)</td>
<td>1.40°C</td>
<td>1.000084</td>
<td>Water (liquid)</td>
<td>Room</td>
<td>78.54</td>
</tr>
<tr>
<td>Helium (1 atm)</td>
<td>20.4°K</td>
<td>1.228</td>
<td>Water (liquid)</td>
<td>Room</td>
<td>55.33</td>
</tr>
<tr>
<td>Hydrogen (1 atm)</td>
<td>100°C</td>
<td>1.000264</td>
<td>Water (steam; 1 atm)</td>
<td>Room</td>
<td>1.0126</td>
</tr>
<tr>
<td>Nylon (F-M. 3001)</td>
<td>Room</td>
<td>3.5</td>
<td>(a) Depending on composition</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10.5. BASIC EQUATIONS WHEN DIELECTRICS ARE PRESENT

the flux of which is determined solely by the (known) free charge distribution. Applying Gauss’s Law, Eq. (10.52), in integral form to the pillbox shown in Fig. 10.7, we find that

\[ \oint D \cdot dS = DS = \int \rho \, dv = \sigma S = \frac{qS}{A} \implies D = \frac{\sigma}{A} \implies D = \frac{q}{A} \hat{x} \]  

(10.59)

where \( S \) is the cross-sectional area of the pillbox, \( \sigma \) is the surface density of free charge, and \( q \) is the total free charge on the upper plate of the capacitor. [The surface integral in Eq. (10.59) has a contribution only from the lower face of the pillbox because \( D \) is parallel to the vertical face and the top face is in a conductor where \( D = 0 \).] Thus

\[ E = \frac{D}{\epsilon} = \frac{q}{\epsilon A} \hat{x} \]  

(10.60)

and

\[ P = \chi_e E = \frac{\epsilon - \epsilon_0}{\epsilon} \frac{q}{A} \hat{x} \]  

(10.61)

Equations (10.16) and (10.17) then give

\[ \rho_b = 0 \]

\[ \sigma_b (\text{upper}) = \mathbf{P} \cdot (-\hat{x}) = -\frac{\epsilon - \epsilon_0}{\epsilon} \frac{q}{A} \]

\[ \sigma_b (\text{lower}) = \mathbf{P} \cdot (\hat{x}) = \frac{\epsilon - \epsilon_0}{\epsilon} \frac{q}{A} \]  

(10.62)

for the bound charge distribution induced on the dielectric. Knowing \( E \) as in Eq. (10.60), we find that the potential difference \( \Delta V \) between the plates is given by

\[ \Delta V = -\int E \cdot dl = \frac{qd}{\epsilon A} \]  

(10.63)

and further the capacitance of the arrangement as defined by Eq. (4.55) is given by

\[ C = \frac{q}{\Delta V} = \frac{A}{d} \]  

(10.64)

In particular the capacitance of the arrangement when \( \epsilon = \epsilon_0 \) (no dielectric) is \( C_0 = \epsilon_0 A/d \) and

\[ \frac{C}{C_0} = \frac{\epsilon}{\epsilon_0} = K_e \]  

(10.65)

Since \( K_e > 1 \), placing a dielectric between the plates of a capacitor increases its capacitance. Further, Eq. (10.65) shows that careful measurements of capacitance with and without a dielectric filler can yield a numerical value for the dielectric constant of the filler.

PROBLEMS

P10.13. Apply Gauss’s Law, Eq. (10.52), to an infinite linear dielectric and derive Coulomb’s Law for the force between two point charges in such a medium. Is this force larger or smaller than the force between the same two charges at the same separation in vacuum? By what factor is the force changed?

P10.14. Suppose a dielectric is linear but inhomogeneous (\( \epsilon \) is independent of field strength but depends on position in the dielectric). Starting with Eqs. (10.51)–(10.53), argue the existence of an electrostatic potential \( V \) and show that it satisfies the equation \( \nabla \cdot (\epsilon \nabla V) = -\rho \), which reduces to Poisson’s equation \( \nabla^2 V = -\rho/\epsilon \) if the dielectric is homogeneous (\( \epsilon \) independent of position). Hint: See item (3) in Section 2.5.
P10.15. Replace the dielectric in Fig. 10.7 with the equivalent charge distribution in Eq. (10.62) and determine the electric field between the plates by applying Gauss’s Law in free space. Show that the result agrees with Eq. (10.60).

P10.16. The dielectric constant of the material filling the space between the two plates of a parallel plate capacitor varies linearly from the value \( K_1 \) at one plate to the value \( K_2 \) at the other. If the plates have separation \( d \) and area \( A \), determine the capacitance and verify that your result gives the correct value in the limiting case \( K_1 = K_2 \). \textit{Hint}: Ignore fringing.

P10.17. Two parallel conducting plates of area \( A \) are separated by a gap of thickness \( d \) in which is placed a dielectric slab of uniform thickness \( t \) (< \( d \)) and of dielectric constant \( K \). Let the potential difference between the plates be \( \Delta V \). (a) Determine \( E \) and \( D \) at all points between the plates? \textit{Hint}: Ignore fringing. (b) What is the free charge density on the conducting plates? (c) Determine the capacitance of the arrangement and compare it with the capacitance that the arrangement would have if the dielectric were removed.

P10.18. A conducting sphere of radius \( a \) carries a total charge \( q \) and is embedded in a concentric spherical shell of inner radius \( a \) and outer radius \( b \) made of a linear dielectric of permittivity \( \varepsilon \). (a) Determine \( D \), \( E \), and \( P \) at points within the dielectric. \textit{Hint}: Note the symmetry and apply Gauss’s Law for the displacement vector. (b) What is the equivalent charge distribution for this dielectric? (c) Determine \( D \) and \( E \) at points in the free space outside the dielectric. (d) Calculate \( D \) and \( E \) at a point infinitesimally beyond the outer surface of the dielectric and compare with \( E \) and \( D \) at a point infinitesimally inside that surface. Show that \( E \) has a discontinuity and relate that discontinuity to the bound surface charge density. (e) A conducting spherical shell of radius \( b \) is now placed about the arrangement and carries a free charge \(-q\). Determine the capacitance of this new arrangement and compare the result with the capacitance that the device would have if the dielectric were removed.

10.6 Connecting the Microscopic Polarizability and the Macroscopic Dielectric Constant: The Clausius-Mossotti Relation

We shall now derive the Clausius-Mossotti relation, which permits a determination of the microscopic molecular polarizabilities of some (but not all) materials from measured values of the macroscopic dielectric constants. We shall begin by finding the field \( \mathbf{E}_m \) that a single molecule in a dielectric experiences, i.e., by finding the microscopic field established at the position of a molecule by all molecules except the one located at that site. Unfortunately, the field \( \mathbf{E}_m \) is not the same field as the interior macroscopic field calculated in Section 10.4, because the macroscopic field involves an average of the microscopic field over points within some volume and hence includes contributions from the microscopic field not only at molecular sites but also at other points in the dielectric. The procedure used in Section 10.4, however, can be modified to apply to the present situation. Again, we remove the molecules from a macroscopically small but microscopically large spherical region centered this time on a selected molecular site but we do not permit the part of the dielectric remaining to readjust to the absence of the molecules removed. Then we use macroscopic...
10.6. THE CLAUSIUS-MOSSOTTI RELATION

methods to calculate the contributions $E_I$ made by the “far” molecules and by any free charges to the field $E_m$. Finally, we replace the molecules removed (except for the one at the selected molecular site) and add the field $E'$ contributed by these near molecules. In brief, we express $E_m$ as the sum of two parts

$$E_m = E_I + E'$$  \hspace{1cm} (10.66)

and evaluate each part separately. In fact, we have already calculated $E_I$; its definition here is essentially identical to its definition in Section 10.4 and it is therefore given by Eq. (10.31) except that $E$ must now be interpreted to include contributions not only from the bound charge on the dielectric but also from any free charges present; i.e., $E$ is the total macroscopic field at the molecular site. Thus,

$$E_m = E + \frac{P}{3\epsilon_0} + E'$$  \hspace{1cm} (10.67)

The difference between the present calculation and the calculation in Section 10.4 arises because the field $E'$ here is very different from the field $E_{II}$ there, for we are now interested in the field at a more specific point, namely at the site of a molecule. In fact, $E'$ is very difficult to evaluate for any particular case. Its value depends on the properties of the specific molecule, on the relative placement of the molecules in the structure of the dielectric, and on the interrelations among these molecules. For gases and liquids (in which the near molecules are randomly positioned) and for some crystals (in which very strong translational symmetries exist), however, $E'$ turns out to be zero (P10.23). For those materials, the so-called Lorentz form of the field $E_m$, where

$$E_m = E + \frac{P}{3\epsilon_0} \quad \text{(if $E' = 0$)}$$  \hspace{1cm} (10.68)

applies and $E_m$ differs from $E$ by a term proportional to the polarization.

With an expression for $E_m$ now available, we can develop an equation relating the molecular polarizability $\alpha_t$ of Eq. (10.4) to the dielectric constant $K_e$. We simply note from Eq. (10.3) that the dipole moment $p$ of a single molecule is given by

$$p = \alpha_t E_m = \alpha_t \left( E + \frac{P}{3\epsilon_0} \right)$$  \hspace{1cm} (10.69)

Then, the polarization of the dielectric is given by

$$P = Np = N\alpha_t \left( E + \frac{P}{3\epsilon_0} \right)$$  \hspace{1cm} (10.70)

where $N$ is the number of molecules per unit volume in the dielectric. Equation (10.70), however, has the solution

$$P = \frac{N\alpha_t}{1 - (N\alpha_t/3\epsilon_0)} E$$  \hspace{1cm} (10.71)

for $P$. Finally, on comparing Eq. (10.71) with Eq. (10.55), we find that

$$\chi_e = \frac{N\alpha_t}{1 - (N\alpha_t/3\epsilon_0)} = \epsilon_0(K_e - 1)$$  \hspace{1cm} (10.72)
which can be solved either for $\alpha_t$ in terms of $K_e$ or vice versa, giving that

$$\alpha_t = \frac{3\varepsilon_0 K_e - 1}{N K_e + 2}, \quad K_e - 1 = \frac{N\alpha_t/\varepsilon_0}{1 - (N\alpha_t/3\varepsilon_0)} \quad (10.73)$$

Equation (10.72) is known as the Clausius-Mossotti relation when applied to non-polar molecules and the Debye equation\(^{14}\) when applied to polar molecules. For materials conforming to the Lorentz form of $E_m$ in Eq. (10.68), the (microscopic) molecular polarizability can therefore be determined from measurements of the (macroscopic) dielectric constant. More complicated relations than Eq. (10.73) can be derived for materials that do not conform to Eq. (10.68), but both a fuller theoretical study of the connection between macroscopic and microscopic properties for all dielectrics and a complete cataloging of known experimental properties of dielectrics lie outside the scope of this book.

**PROBLEMS**

P10.19. Derive Eq. (10.73) from Eq. (10.72).

P10.20. Combine the Clausius-Mossotti relation with the result in P10.3(b) to obtain an expression for the radius of a non-polar atom. Given that the dielectric constant of air at 1 atmosphere is 1.00059, estimate the radius of an atom in an air molecule.

P10.21. Nitrogen gas at room temperature ($\approx 27^\circ$C) and 1 atmosphere pressure has a dielectric constant $K_e = 1.000580$. The density of liquid nitrogen at its boiling point ($-195.8^\circ$C) is 0.808 g/cm$^3$. (a) Show from Eq. (10.73) that $K_e - 1 \approx N\alpha_t/\varepsilon_0$ when $K_e \approx 1$ and find $N\alpha_t/\varepsilon_0$ for nitrogen gas. (b) Use the ideal gas laws to determine the density of molecules in a gas at room temperature and then use part (a) to find $\alpha_t$ for the nitrogen molecule. (c) Assuming that $\alpha_t$ as a property of individual molecules is approximately the same for molecules in the liquid as for those in the gas, find $N\alpha_t/\varepsilon_0$ for liquid nitrogen and then use Eq. (10.73) without approximation to predict the dielectric constant of liquid nitrogen at its boiling point. The measured value is $K_e = 1.474$. Optional: Write a computer program to predict the dielectric constants of liquids by the method of this problem, look up data on other substances (e.g., A, C$_6$H$_6$, CS$_2$, O$_2$, He, Br$_2$, CCl$_4$) in the Handbook of Chemistry and Physics, and compare your predictions with measured values.

P10.22. Consider a dielectric crystal of the type to which Eq. (10.71) applies and let the individual molecules have a polarizability dominated by the orientational part, $\alpha_t = p_0^2/3kT$. Sketch a graph of $\chi_e$ versus $T$. What happens when $kT \leq Np_0^2/9\varepsilon_0$? Do negative values of $\chi_e$ make physical sense? Hint: Some few materials, called ferroelectrics, exhibit a spontaneous permanent polarization at temperatures below a critical temperature whose value depends on the material.

P10.23. Because the distortion of a molecule brought about even by fairly strong fields is very small compared to molecular dimensions [P10.3(e) and P10.4], the contribution made by a near molecule to the field $E'$ at a molecular site can be computed by considering the near molecule to be a dipole whose field is given by Eq. (4.10). Introduce a coordinate system with its origin at the molecular site and its $z$-axis parallel to the macroscopic polarization at the molecular site. Then all molecules contributing to $E'$ have the same dipole moment $p = p\hat{k}$. Finally, let the $i$-th contributing molecule be at $r_i$. (a) Show that the field $E'$ produced at the origin by all contributing molecules is given by

$$E' = \frac{p}{4\pi\varepsilon_0} \sum_i \frac{3z_ir_i\hat{i} + 3z_iy_i\hat{j} + (3z_i^2 - r_i^2)\hat{k}}{r_i^5}$$

\(^{14}\)See footnote 18 on page 239.
(b) Show that $E' = 0$ if the near molecules are located randomly, as, for example, in a liquid or a gas. *Hint:* With random location $\sum (z_i x_i / r_i^5) = \sum (z_i y_i / r_i^5) = 0$ and $\sum (z_i^2 / r_i^5) = \frac{1}{5} \sum (r_i^2 / r_i^5)$. (Why?)

(c) Show that $E' = 0$ if the molecules are located at the sites of a cubic lattice of side $a$ and the $z$-axis is defined by $p$, which coincides with a crystal axis; i.e., show that $E' = 0$ if all coordinates $x_i$, $y_i$, and $z_i$ are (positive or negative) integer multiples of $a$. *Hint:* Consider the contributions from nearest neighbors, next nearest neighbors, etc., separately. *Optional:* Generalize part (c) to a proof when the dipole moment is not parallel to one of the crystal axes.

**SUPPLEMENTARY PROBLEMS**

**P10.24.** Given that $\rho_b = -\nabla \cdot P$ in all systems of units and that $\nabla \cdot E = 4\pi \rho_t$, $4\pi c^2 \rho_t$, $4\pi \rho_t$, and $\rho_t$ in cgs-esu, cgs-emu, Gaussian units, and Heaviside-Lorentz units, respectively, find the relationship among $D$, $E$, and $P$ in these systems of units. *Hint:* See Footnote 4.

**P10.25.** Consider an anisotropic crystal made up of molecules consisting of a fixed nucleus carrying charge $q$ and an electron cloud (charge $-q$) that is attracted to its nucleus by a harmonic force $F = -k_x \hat{x} - k_y \hat{y} - k_z \hat{z}$, where $k_x$, $k_y$, and $k_z$ may be different. Introduce a $3 \times 3$ matrix $K$ whose diagonal elements are $k_x$, $k_y$, and $k_z$ and whose off-diagonal elements are zero. (a) Show that $F = -Kr$. (b) Show that a polarizing field $Em$ induces a dipole moment given by $p = q^2 K^{-1} Em$. (c) Assuming the crystal is one for which $E' = 0$ in Eq. (10.67), show that the dielectric constant of this material is given by the matrix

$$K_e = \frac{\epsilon}{\epsilon_0} = I + \frac{Nq^2}{\epsilon_0} \left( K - \frac{Nq^2}{3\epsilon_0} I \right)^{-1}$$

where $I$ is the $3 \times 3$ unit matrix and $N$ is the number of molecules per unit volume in the crystal, and $\epsilon_0$ is called the permittivity tensor. (d) For the sake of a more concrete example, consider a probably unrealistic material for which $Nq^2/\epsilon_0 = 1$, $k_x = k_y = 1$, $k_z = 2$, and $E$ lies in the $yz$-plane making an angle $\beta$ with the $y$-axis, i.e., $E = E(\cos \beta \hat{j} + \sin \beta \hat{k})$. Using an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON, obtain a careful graph of the angle between $D$ and $E$ as a function of $\beta$. *Optional:* Assuming that $Nq^2/\epsilon_0 \ll k_x$, $k_y$, and $k_z$, find an expression for the angle between $D$ and $E$. *Hint:* Write $D = \epsilon_0 (I + K')E$, where $K'$ is a matrix with small elements, and expand in powers of $K'$. 

Chapter 11

Properties of Matter III: Magnetization

In this chapter, we shall consider the response of matter to externally applied static magnetic fields. Because magnetic monopoles are not found in the physical world, there is no magnetic response corresponding to electric conduction and we might therefore conclude that magnetic materials could be considered in a single category. Such a hasty conclusion, however, is not supported by real magnetic materials. Just as we found two different microscopic mechanisms—molecular orientation and molecular deformation—that contributed to the response of a dielectric in an electric field, we must here distinguish two mechanisms contributing to the response of matter in a magnetic induction field. In sharp contrast to the dielectric mechanisms (both of which gave rise to essentially the same macroscopic effect), the two magnetic mechanisms produce two qualitatively different effects. Thus, even in the absence of magnetic “conduction”, two basic types of magnetic material (paramagnetic and diamagnetic) can be differentiated.

As with conduction and dielectric polarization, the magnetic response of matter can also be examined from either a microscopic or a macroscopic viewpoint. The microscopic approach is essentially identical to that described in the introduction to Chapter 10, and that discussion will not be repeated. We shall again adopt the macroscopic approach, but not before exploring briefly the microscopic response of individual molecules to externally applied magnetic induction fields. The development will once again be divided into a descriptive part, concerned solely with describing field-causing states of matter and with determining the fields caused by matter in a given state, and a causal part, concerned with the ways in which field-causing states are brought into being and maintained. Together, the descriptive and causal parts constitute the theoretical framework for finding magnetic induction fields in the presence of magnetizable matter, even though the contribution of the matter to the total field is not in general known until after the total field has been found.

11.1 The Microscopic Description: Magnetic Polarizability

From a microscopic point of view, a sample of magnetically responsive material is composed of molecules residing in free space. A discussion similar to that in the opening and closing
paragraphs of Section 10.1 supports the adequacy of characterizing each molecule solely by a magnetic dipole moment. Magnetically responsive matter is then viewed microscopically as a dense assemblage of molecular dipoles located in free space. In this section, the response of individual molecules to their presence in an external magnetic induction field will be explored.

The magnetic moment of a single atom or molecule arises from two contributions. On the one hand, each electron circulating the nucleus constitutes a current loop and hence contributes to the magnetic moment of the atom of molecule. On the other hand, the electrons and the nuclei in the molecule possess intrinsic angular momenta or spins. With each intrinsic spin is associated a further magnetic moment that adds to the moment produced by the orbiting electrons. For some molecules, these contributions may add up to produce a net permanent magnetic moment; for others, the contributions may cancel, leaving an atom or molecule with no permanent magnetic moment but with the potential for developing an induced magnetic moment when an external field is applied.

Consider first the response of a molecule possessing a permanent dipole moment. (For the rest of this chapter, we understand that the magnetic dipole moment is meant.) We shall confine our discussion to a classical treatment, recognizing, however, that the only fully correct approach is a quantum mechanical approach (P11.29). When the molecule of concern is placed in a magnetic induction field, it experiences a torque tending to align its magnetic moment with the direction of the field (P3.13). In an assembly of permanent dipoles, however, this tendency to alignment is opposed by the thermal tendency to randomness. Thus, a given permanent dipole assumes an average dipole moment that would be zero except that in the external field each dipole favors a preferred orientation as it undergoes thermal agitation. To express this discussion more quantitatively, consider the following situation: A magnetic material at absolute temperature $T$ composed of permanent dipoles having dipole moments of magnitude $m_0$ is placed in an external magnetic induction field. The dipoles adjust to some sort of average equilibrium position, in which state a particular dipole experiences a total field consisting of the applied external field plus the time-averaged microscopic field established by all other dipoles when the external field is present.\(^1\) Let the time average (over the microscopic fluctuations) of this total field be $B_m$. Combination of statistical mechanics with considerations of magnetostatic energy (P11.1) leads ultimately to the result that the average dipole moment of a single dipole is given by

$$\langle m_0 \rangle_{\text{perm}} = \beta B_m, \quad \beta = \frac{m_0^2}{3kT}$$

(11.1)

where $k$ is Boltzmann’s constant and the condition $m_0B_m \ll kT$ has been imposed. [Compare Eq. (10.1).] The constant $\beta$ might be called an orientational (magnetic) molecular polarizability, since it measures the extent to which a molecule possessing a permanent dipole moment is oriented in a polarizing field. The orientational molecular polarizability is positive, and materials composed of permanent dipoles are said to be paramagnetic. The paramagnetic response of molecules to an external field results in an average dipole moment having the same direction as the field, and the strength of this response decreases with increasing temperature.

Molecules that possess no permanent dipole moment exhibit a different response. When placed in a magnetic induction field, these molecules develop an induced dipole moment that

\(^1\)See footnote 4, Chapter 10.
11.1. THE MICROSCOPIC DESCRIPTION: MAGNETIC POLARIZABILITY

is opposite in direction to the polarizing field. Materials exhibiting this property are said to be diamagnetic. We shall see that all materials should exhibit a diamagnetic response. In materials composed of polar molecules, however, the stronger paramagnetic response usually masks the weaker diamagnetic response of the material. Some separation of the two responses can be made because the diamagnetic response is not temperature-dependent, but experiments to distinguish the two responses on this basis are difficult because of the weakness of the diamagnetic compared to the paramagnetic response.

Diamagnetism arises from the response of individual electronic orbits as these orbits adjust to new equilibria in the presence of an external field. Although this response is extremely complicated and can be fully treated only by using quantum mechanics, a classical model that leads to a crude microscopic understanding of the phenomenon can be constructed. Consider first the qualitative features involved when an electron in a circular orbit is placed in an external magnetic induction field at right angles to the plane of the orbit. Some interaction—Coulomb or otherwise—provides the centripetal force that keeps the electron in its orbit when no external fields are present. When the field is turned on, the circulating electron experiences an additional radially directed force that may tend to expand or contract the orbit, depending on the direction in which the electron is circulating. Simultaneously, as the field is turned on, the electron experiences tangential forces from the induced electric field (Faraday law), and these forces may speed up or slow down the electron, again depending on the direction in which the electron is circulating. Table 11.1 summarizes qualitatively the direction of these two effects for various directions of the field and of the circulation of the electron.

Now consider the adjustments that the orbit might be expected to undergo as a result of these two effects. From Table 11.1, we conclude that a final magnetic induction that reduces the centripetal force is accompanied by an induced electric effect that slows down the electron and vice versa. But a slower electron can stay at the same radius if the centripetal force is suitably reduced and similarly a faster electron can stay at the same radius if the centripetal force is appropriately increased. Classically, the change in radius as the $B$-field is turned on is therefore likely to be less significant than the change in electron speed, since the radius is affected by two phenomena that tend to compensate. Let us then assume that the radius does not change at all. (It is shown in P11.2 that in fact this assumption is correct provided that the change in angular velocity is small compared to the initial angular velocity.)

To make our discussion more quantitative, consider an electron with charge $-q_e$ and mass $m_e$ circulating in the $xy$-plane with angular velocity $\omega$ in an orbit of radius $\tau$ (Fig. 11.1). The circulating electron constitutes a current of strength $I = q_e \omega / 2\pi$ in the counterclockwise direction as seen from a point on the positive $z$-axis. Hence, the magnetic moment of the resulting current loop is given by

$$\mathbf{m} = (\text{current})(\text{area}) = \frac{q_e \omega}{2\pi} \left(\pi \tau^2 \mathbf{k}\right) = \frac{q_e}{2} \omega \tau^2 \mathbf{k} = -\frac{q_e}{2} \omega \tau^2$$  \hspace{1cm} (11.2)

where $\omega = -\omega \mathbf{k}$ is the angular velocity of the electron and $\omega$ is positive for an electron circulating clockwise and negative for an electron circulating counterclockwise.

We shall now calculate the change induced in this magnetic moment when an external magnetic induction field perpendicular to the plane of the orbit is turned on. Let the field
change from zero to a final value \( B_m \) during the time interval \( 0 < t < t_f \) in accordance with

\[
B = B(t) \hat{k}
\]  

(11.3)

where \( B(0) = 0 \) and \( B(t_f) = B_m \). Remembering Faraday’s Law, we infer that, as this field is turned on, an induced electric field tangential to the orbit of the electron is produced. If we assume that the electron is circulating very rapidly compared to the time scale associated with the changing field, the electron makes many revolutions before the field has changed appreciably. It is then appropriate to calculate the electric field induced at an instant of time and use it to determine the work done on the electron over short intervals of time.

Applied to the orbit of the electron, Faraday’s Law,

\[
\oint E \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}
\]

(11.4)

gives

\[
2\pi E_\phi = -\frac{d}{dt}(\pi \tau^2 B)
\]

(11.5)

where we have assumed that \( E = E_\phi(\tau) \hat{\phi} \), \( \hat{\phi} \) being a unit vector in the direction of increasing \( \phi \). Hence, we find first that

\[
E = -\left[ \frac{1}{2\pi} \frac{d}{dt}(\pi \tau^2 B) \right] \hat{\phi}
\]

(11.6)

and then that the rate \( P \) at which this field does work on the electron is given by

\[
P = \mathbf{F} \cdot \mathbf{v} = (-q_e E) \cdot (\omega \tau \hat{\phi}) = -\frac{q_e \omega}{2\pi} \frac{d}{dt}(\pi \tau^2 B)
\]

(11.7)

The rate at which the energy of the electron increases, however, is also given by the time rate of change of the kinetic energy of the electron, i.e.,

\[
P = \frac{d}{dt} \left( \frac{1}{2} m_e \tau^2 \omega^2 \right)
\]

(11.8)

Thus, on combining Eqs. (11.7) and (11.8), we find that the motion of the electron in this (slowly) changing field must satisfy the equation

\[
\frac{d}{dt}(\tau^2 \omega^2) = -\frac{q_e \omega}{m_e} \frac{d}{dt}(\pi \tau^2 B)
\]

(11.9)

Assuming now that \( \tau \) is constant, we find from Eq. (11.9) that

\[
\frac{d\omega}{dt} = -\frac{q_e}{2m_e} \frac{dB}{dt}
\]

(11.10)

Direct integration then yields

\[
\Delta \omega = \omega_f - \omega_0 = -\frac{q_e B_m}{2m_e} = \pm \omega_L
\]

(11.11)

where \( \omega_f \) is the final angular velocity of the electron, \( \omega_0 \) is its initial angular velocity, and \( \omega_L = \frac{q_e |B_m|}{2m_e} \) is the so-called Larmor frequency; in Eq. (11.11), the upper (lower) sign applies when \( B_m < 0 \) (\( B_m > 0 \)) and the condition of slowly changing fields is more explicitly stated by the requirement \( |\Delta \omega| \ll \omega_0 \). (Compare P11.2.) Finally, we find from Eq. (11.2)
Table 11.1: Effects on the Orbit of an Electron when a Polarizing Field is Turned On. In this table, the viewer is assumed to be located at a point on the positive z-axis in Fig. 11.1. Since the electron has a negative charge, the force that it experiences in an electric field is opposite in direction to that of the field.

<table>
<thead>
<tr>
<th>Direction of Final B</th>
<th>Direction of Electron Circulation</th>
<th>Direction of Resulting Magnetic Force</th>
<th>Direction of Induced Electric Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>+z</td>
<td>Radially out</td>
<td>Slows electron down</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(smaller centripetal force)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>+z</td>
<td>Radially in</td>
<td>Speeds electron up</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(greater centripetal force)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>−z</td>
<td>Radially in</td>
<td>Speeds electron up</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(greater centripetal force)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>−z</td>
<td>Radially out</td>
<td>Slows electron down</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(smaller centripetal force)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 11.1: Path of an electron circulating in the $xy$-plane with angular velocity $\omega$. 
that the change in magnetic moment induced by the turning on of this external field is given by
\[
\Delta m = q_e \frac{r^2}{2} \Delta \omega \hat{k} = q_e \frac{r^2}{2} (\omega_f - \omega_0) \hat{k} = q_e \frac{r^2}{2} \left( -\frac{q_e B_m}{2m_e} \right) \hat{k} = -\frac{q_e^2 r^2}{4m_e} B_m
\] (11.12)

Thus, when an external B-field perpendicular to the orbit is turned on, the induced change in the magnetic moment of a circulating charge is independent of \(\omega\) (and hence does not depend on the direction of circulation), is independent of the sign of the charge, and finally is opposite in direction to the inducing field [as evidenced by the minus sign in Eq. (11.12)]. This minus sign should be contrasted with the plus sign in Eq. (11.1); it is this difference in sign that distinguishes diamagnetic from paramagnetic response.

Despite its classical origin, Eq. (11.12) can be converted into an expression that happens to be quantum mechanically correct, at least for molecules with a single nucleus (i.e., atoms) in which the electron distribution about the nucleus is spherically symmetric. Quantum mechanically, an electron orbit does not have a well-defined radius. We would expect then that \(r^2\) in Eq. (11.12) should be replaced by \(\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle\), where \(\langle \cdots \rangle\) denotes an average over the electron wave function \(\psi\), e.g., \(\langle x^2 \rangle = \int \psi^* x^2 \psi \, dv\). But for a spherically symmetric wave function, \(\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \frac{1}{3} \langle r^2 \rangle\), where \(r\) is the distance of the point \((x, y, z)\) from the center of the atom. Thus, \(\langle r^2 \rangle = \frac{2}{3} \langle r^2 \rangle\) and Eq. (11.12) is replaced with
\[
\Delta m = -\frac{q_e}{6m_e} \langle r^2 \rangle B_m
\] (11.13)

The task of quantum mechanics is then to calculate \(\langle r^2 \rangle\) for each electron in the atom.

Whether Eq. (11.13) or a more complicated equation applies, the total change in the magnetic moment of the molecule when the field is turned on is obtained by summing the contributions of each electron. Since we have been discussing non-polar molecules, which have no initial magnetic moment, this change is equal to the final induced moment. Thus, if Eq. (11.13) applies, we have that
\[
m_{\text{ind}} = \left( -\frac{q_e}{6m_e} \sum_i \langle r^2 \rangle_i \right) B_m = \beta_0 B_m
\] (11.14)

where \(\langle r^2 \rangle_i\) is the average value of \(r^2\) for the \(i\)-th electron. Since all electrons contribute to changing \(m\) in the same direction, a net change results. The coefficient \(\beta_0\) in Eq. (11.14) multiplying \(B_m\) might be called a deformational (magnetic) molecular polarizability because it relates to a magnetic moment produced through deformation of the internal structure of the molecule.

If both paramagnetic and diamagnetic effects are present, the total molecular polarizability is obtained by combining Eqs. (11.1) and (11.14) to find that
\[
m = \langle m_0 \rangle_{\text{perm}} + m_{\text{ind}} = \beta_t B_m
\] (11.15)

where the total (magnetic) molecular polarizability \(\beta_t\) is given by
\[
\beta_t = \frac{m_0^2}{3kT} - \frac{q_e^2}{6m_e} \sum_i \langle r^2 \rangle_i
\] (11.16)
As we have already remarked, more precise and more general expressions for these polarizabilities can be calculated only by using quantum mechanics; we leave those calculations to other authors.

PROBLEMS

P11.1. Accepting (1) that the energy of a magnetic dipole of moment \( \mathbf{m} \) in a magnetic induction field \( \mathbf{B} \) is \( -\mathbf{m} \cdot \mathbf{B} \) (P3.21) and (2) that the distribution of dipoles over these energy states is given in statistical thermodynamics by \( \exp(-\text{energy}/kT) \), where \( k \) is Boltzmann’s constant and \( T \) is the absolute temperature, determine the average value of the dipole moment \( \langle \mathbf{m}_0 \rangle \) of a dipole in a magnetic induction field \( \mathbf{B}_m \). Show also that Eq. (11.1) emerges when \( m_0B_m \ll kT \). How large can the field \( B_m \) be before this condition is violated at room temperature? 

Hints: (1) See P10.2 for the introduction of the Langevin function.

(2) Typical values for permanent molecular dipole moments are on the order of \( 10^{-23} \) J/T (mks units).

P11.2. The appropriateness of our assumption of a constant orbital radius at Eq. (11.10) can be shown as follows. A particle of mass \( m_e \) moving in a circle of radius \( r \) with angular velocity \( \omega_0 \) must have acting on it a centripetal force \( F_c = -m_e\omega_0^2 r \hat{r} \). Suppose \( \omega_0 \) is changed to \( \omega_0 + \Delta \Omega \). (a) What must be the change \( \Delta F_c \) in \( F_c \) if the particle is to continue to move in a circular orbit of the same radius? Assume \( \Delta \Omega \ll \omega_0 \). (b) Show that turning on the magnetic induction \( B_m \hat{k} \) in Fig. 11.1 changes \( F_c \) by \( \Delta F_c = q_e\omega_r B_m \hat{r} \). (c) Combine parts (a) and (b) to show that, if \( \Delta \Omega = -q_e B_m/2m_e \), then adding the magnetic induction will not change the radius. (d) Compare this particular \( \Delta \Omega \) with \( \Delta \omega \) as given in Eq. (11.11) to conclude that \( r \) is constant provided \( |\Delta \omega| \ll \omega_0 \). (e) Estimate \( \omega_0 \) numerically for the electron in the hydrogen atom and then find how large \( B_m \) can become before \( |\Delta \omega| \) exceeds 0.01\( \omega_0 \). Compare this value with typical laboratory fields, say 10,000 G.

P11.3. (a) Show from Eq. (11.2) that \( \mathbf{m} = -q_e \mathbf{L}/2m_e \), where \( \mathbf{L} \) is the orbital angular momentum of the electron. (b) Quantum mechanically, the component of \( \mathbf{L} \) parallel to a magnetic induction field can assume only the values \( m_l \hbar \), where \( m_l \) is an integer and \( \hbar \) is Planck’s constant divided by \( 2\pi \). The Bohr magneton \( M_B \), which is a convenient unit for expressing atomic magnetic moments, is defined so that the component of the magnetic moment parallel to the field is given by \( -m_l M_B \). Show that \( M_B = q_e\hbar/2m_e \). (c) Suppose an electron has \( m_l = 1 \). What (numerically) is its magnetic moment? Use this result in Eq. (11.2) to estimate its angular velocity. (d) How large can \( B_m \) become in Eq. (11.11) before \( \Delta \omega \) ceases to be small compared to \( \omega_0 \), thereby invalidating the assumption that the field changes slowly?

P11.4. Given that \( \mathbf{m} = -q_e \mathbf{L}/2m_e \), where \( \mathbf{L} \) is the orbital angular momentum of the electron (see P11.3 and P3.22), evaluate the torque \( \mathbf{N} \) exerted on the electron by the field in Eq. (11.6) and integrate Newton’s second law \( d\mathbf{L}/dt = \mathbf{N} \) to show ultimately that the magnetic dipole moment of the electron is changed by an amount given by Eq. (11.12) when the magnetic induction field is turned on. Assume \( r \) is constant.

11.2 The Macroscopic Description: Magnetization

The macroscopic approach to magnetic properties is founded on a concept called magnetization, which is analogous to the concept of polarization introduced in the discussion of
dielectrics. Each elementary magnetic dipole in a given volume element $\Delta v$ makes an additive contribution to the total dipole moment of that volume element. If the $i$-th molecular dipole has \textit{average} dipole moment $m_i$, then the \textit{magnetization} (magnetic dipole moment per unit volume) is defined by

\[
M(r) = \lim_{\Delta v \to 0} \left( \frac{1}{\Delta v} \sum_i m_i \right) \tag{11.17}
\]

where the sum extends over all the molecules in the volume element and the meaning of the limit is the same as always—$\Delta v$ becomes macroscopically small but remains large enough to contain a large number of microscopic dipoles.\(^3\) In terms of the magnetization, each small volume element $\Delta v$ therefore has associated with it a magnetic dipole moment $\Delta m \approx M(r) \Delta v$ and the total magnetic dipole moment of the sample will be

\[
m = \int M(r) \, dv \tag{11.18}
\]

\(\textbf{11.3 \ The Macroscopic Vector Potential and Magnetic Induction Field at a Point Exterior to a Magnetized Object; Bound Currents}\)

In Sections 11.3, 11.4, and 11.5, we shall calculate the contribution made to the macroscopic magnetic induction field by a magnetized object whose magnetization $M(r')$ at every point $r'$ within the object is known. Consider first an observation point $r$ exterior to the object. As in the analogous dielectric case, such an observation point is microscopically far from \textit{all} of the dipoles in the object. Thus, a small element of the object having volume $\Delta v'$ and located at $r'$ (Fig. 10.1) can be treated as a dipole of moment $M(r') \, dv'$, regardless of where within the object $r'$ lies. The contribution of this element of the object to the magnetic vector potential at $r$ is then given by

\[
\Delta A = \frac{\mu_0}{4\pi} \frac{M(r') \, dv' \times (r - r')}{|r - r'|^3} \tag{11.19}
\]

[see Eq. (5.56)] and integration of Eq. (11.19) over the volume of the object yields

\[
A(r) = \frac{\mu_0}{4\pi} \int \frac{M(r') \times (r - r')}{|r - r'|^3} \, dv' \tag{11.20}
\]

for the contribution of the entire object to the total vector potential at $r$.

Equation (11.20) can be rewritten in two different ways, each of which leads to a useful physical interpretation. We shall postpone treating the reformulation that leads to an identification of equivalent magnetic monopoles to the next section and shall treat here

\(^3\)The definition of magnetization given in this paragraph is adopted in all systems of units with which the author is familiar.
11.3. FIELD AT AN EXTERIOR POINT

the reformulation leading to the identification of bound currents. First, we substitute the identity in Eq. (10.10) into Eq. (11.20) to obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \left[ \mathbf{M}(\mathbf{r}') \times \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] d\mathbf{v}'$$  \hspace{1cm} (11.21)

Then we use the vector identity in Eq. (C.8) in the form

$$\mathbf{Q} \times \nabla' \Phi = \Phi \nabla' \times \mathbf{Q} - \nabla' \times (\Phi \mathbf{Q})$$  \hspace{1cm} (11.22)

with $\Phi = 1/|\mathbf{r} - \mathbf{r}'|$ and $\mathbf{Q} = \mathbf{M}$, to obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{v}' - \frac{\mu_0}{4\pi} \int \nabla' \times \left( \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) d\mathbf{v}'$$  \hspace{1cm} (11.23)

Finally, we use the identity in Eq. (C.25) in the second term to find that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{v}' + \frac{\mu_0}{4\pi} \oint \frac{\mathbf{M}(\mathbf{r}') \times \hat{n}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{S}'$$  \hspace{1cm} (11.24)

where the surface integral extends over the surface bounding the magnetized object, $\hat{n}(\mathbf{r}')$ is a unit outward normal to that surface at $\mathbf{r}'$, and $d\mathbf{S}' = |d\mathbf{S}'|$. This result, however, is the vector potential established by a current distribution described by a volume current density

$$\mathbf{J}_m(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r})$$  \hspace{1cm} (11.25)

and a surface current density (in rationalized mks units)

$$\mathbf{j}_m(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{n}(\mathbf{r})$$  \hspace{1cm} (r on the surface) \hspace{1cm} (11.26)

[See Eq. (5.48) and P5.34.] Thus, at exterior points, the magnetized object produces the same vector potential as these so-called bound (or Amperian) currents and we can replace our description in terms of the macroscopic magnetization with this equivalent description in terms of bound currents if we find it convenient to do so.

These bound currents can also be obtained by a longer but less formal derivation that makes their physical origin more clear. A suitable model for a magnetized object is shown in Fig. 11.2. Each volume element $\Delta v$ into which the object is divided has a magnetic dipole moment $\Delta \mathbf{m} \approx \mathbf{M} \Delta v$ associated with it, and we can view this moment as arising from a current circulating about the boundary of the volume element. The proper current to associate with a volume element at which the magnetization is $\mathbf{M}$ is found by computing the magnetic dipole moment of the element in two ways. On the one hand, for example, the $y$-component of the magnetic moment is given by $M_y \Delta v = M_y \Delta x \Delta y \Delta z$. On the other hand, if we let $i_{xz}$ be the current circulating about the volume element in orbits whose plane is parallel to the $xz$-plane and adopt the usual sign convention that $i_{xz} > 0$ when the current circulates in the counterclockwise direction as viewed from a point far away on the positive $y$-axis, then the $y$-component of the magnetic moment of the volume element is also given by the current $i_{xz}$ times the area $\Delta x \Delta z$ of the face of the element in the $yz$-plane. Equating these two evaluations of the $y$-component of the magnetic moment gives

$$M_y \Delta x \Delta y \Delta z = i_{xz} \Delta x \Delta z \implies i_{xz} = M_y \Delta y$$  \hspace{1cm} (11.27)
By applying similar arguments to the other two coordinate directions, we find that

\[ i_{xy} = M_z \Delta z, \quad i_{yz} = M_x \Delta x \]  

(11.28)

An enlarged view of these currents at a single volume element is shown in Fig. 11.3.

In this model, bound currents arise because it is possible for these circulating currents to combine in such a way that there appears to be a macroscopic current even though there is no net macroscopic transport of charge. Consider first the bound surface current. A volume element at the surface has at least one face—the face toward the outside—on which the supposed circulating current is apparent from the outside. If, for example, the outside face parallel to the \( yz \)-plane in Fig. 11.2 is viewed, one might see currents flowing.

---

4 Equations (11.25) and (11.26) retain the same form in cgs-esu and cgs-emu. In Gaussian and Heaviside-Lorentz units, we would find that \( \mathbf{J}_m = c \nabla \times \mathbf{M} \), and \( \mathbf{j}_m = c \mathbf{M} \times \mathbf{n} \), where \( c \) is the speed of light in cm/s.
11.3. FIELD AT AN EXTERIOR POINT

Figure 11.4: Currents in a surface of a magnetized material. The surface shown is in a plane parallel to the $yz$-plane. The current $i_{yz}$, which circulates in this plane, is not shown because it makes no contribution to the bound surface currents in this plane.

Figure 11.5: Vectors for defining the surface current density. Both vectors shown lie in the plane of the paper.

as shown in Fig. 11.4. Both the currents $i_{xy}$ and the currents $i_{xz}$ have the appearance of currents flowing in the surface. Even though no charges are transported physically from one “cell” to the next, the net macroscopic appearance of all of the contributions from $i_{xy}$ is a current flowing to the right in the figure and the net macroscopic appearance of all of the contributions from $i_{xz}$ is a current flowing toward the bottom of the picture. Now, surface currents have a direction and are described by a density similar to volume current densities (P2.26). With reference to Fig. 11.5, let us define a surface current density $j$ so that the current $i$ flowing across a (short) line in the surface can be represented by the equation

$$i = j \cdot l$$  \hspace{1cm} (11.29)

where $l$ is a vector whose magnitude is the length of the line and whose direction is perpendicular to the line in the direction of the current desired. In effect, $j$ measures the rate at which charge is transported across a line of unit length perpendicular to $j$. We now look at the surface of one of the blocks illustrated in Fig. 11.4. A more detailed diagram is shown in Fig. 11.6. Only the currents that contribute to an apparent macroscopic charge transport in the surface are shown. The current $i$ flowing across the slanted line in Fig. 11.6
Figure 11.6: Currents in one of the surface elements shown in Fig. 11.4.

is composed of two parts, and we find that

\[
i = i_{xy} \frac{a}{\Delta z} - i_{xz} \frac{b}{\Delta y} \]

[Compare Eqs. (11.27) and (11.28)]

\[
= M_x a - M_y b
\]

\[
= (M_x \hat{j} - M_y \hat{k}) \cdot (a \hat{j} + b \hat{k})
\]

\[
= (\mathbf{M} \times \hat{i}) \cdot \mathbf{l}
\] (11.30)

where the final form follows because \( \mathbf{l} = a \hat{j} + b \hat{k} \). (Why?) Direct comparison of Eq. (11.30) with Eq. (11.29) now leads us to identify a surface current density

\[
\mathbf{j}_m = \mathbf{M} \times \hat{i}
\] (11.31)

in the surface parallel to the \( yz \)-plane. In this result, of course, \( \hat{i} \) is the unit vector normal to the surface used in the example. More generally, we would expect that the unit normal \( \hat{\mathbf{n}} \) would appear instead of \( \hat{i} \) and we have obtained Eq. (11.26) by arguments based on the model in Fig. 11.2.

Bound volume currents arise when the magnetization is not uniform. Under such conditions, the current associated with a particular volume element is different from that associated with an adjacent volume element and the currents flowing in opposite directions along the boundary between the two elements do not cancel. The result is an apparent macroscopic transport of charge in the interior of the magnetized object. To obtain a quantitative expression for this volume current density, consider an area having sides \( \Delta x \) and \( \Delta y \) and lying in a plane parallel to the \( xy \)-plane. Let this area be positioned with its center at the point \( (x, y, z) \). Now, divide the magnetized object into elements having sides \( \Delta x \), \( \Delta y \), and \( \Delta z \) in such a way that the selected area lies in two of these elements as shown in Fig. 11.7. The circulating currents that contribute to charge flow across the selected surface are also shown. In balance, the net current flow in the positive \( z \)-direction is

\[
i_1 = i_{xz}(x + \frac{1}{2} \Delta x) - i_{xz}(x - \frac{1}{2} \Delta x) = \frac{\partial i_{xz}}{\partial x} \Delta x = \frac{\partial M_y}{\partial x} \Delta x \Delta y
\] (11.32)

where Eq. (11.27) has been used to relate \( i_{xz} \) to \( M_y \). An additional contribution to the net current flow across this surface in the positive \( z \)-direction is made by currents circulating in the \( yz \)-plane (Fig. 11.8) and is given by
11.3. FIELD AT AN EXTERIOR POINT

Figure 11.7: A selected area and two volume elements for determining part of the bound volume current across that area. Part (a) shows the two elements and part (b) shows the view as seen from the point marked with an eye symbol in part (a).

Figure 11.8: Two volume elements for determining a second contribution to the bound volume current across the same area shown in Fig. 11.7. Part (a) shows the two elements and part (b) shows the view as seen from the point marked with an eye symbol in part (a).

\[ i_2 = i_{yz}(y + \frac{1}{2}\Delta y) - i_{yz}(y - \frac{1}{2}\Delta y) = -\frac{\partial i_{yz}}{\partial y}\Delta y = -\frac{\partial M_x}{\partial y}\Delta x \Delta y \]  

\[ (11.33) \]

where Eq. (11.28) has been used to relate \( i_{yz} \) to \( M_x \). Combining Eqs. (11.32) and (11.33), we find that the net current across the surface in the positive \( z \)-direction is given by

\[ i_1 + i_2 = \left( \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \right) \Delta x \Delta y = (\nabla \times \mathbf{M}) \cdot \mathbf{k} \Delta x \Delta y \]  

\[ (11.34) \]

Since the current across a small surface \( \Delta S \) placed in a current flow described by the current density \( \mathbf{J} \) is given by \( \mathbf{J} \cdot \Delta S \) and since in the present example \( \Delta S = \mathbf{k} \Delta x \Delta y \), we deduce from Eq. (11.34) that the circulating currents give rise to a bound volume current density given by \( \mathbf{J}_m = \nabla \times \mathbf{M} \). Strictly, of course, our development supports this conclusion only
for the \( z \)-component of \( J_m \), but the other components can be similarly derived (P11.9). We have therefore obtained Eq. (11.25) by arguments based on the model in Fig. 11.2.

Given the vector potential, we can readily calculate the field established at an exterior point by a magnetized object; we take the curl of the potential, finding that

\[
B = \frac{\mu_0}{4\pi} \int \frac{J_m(r') \times (r - r')}{|r - r'|^3} \, dv' + \frac{\mu_0}{4\pi} \oint \frac{J_m(r') \times (r - r')}{|r - r'|^3} \, dS'
\]

(P11.10). Equation (11.35) for the field and Eq. (11.24) for the vector potential of course give only the contributions of the bound currents on the magnetized object; contributions from other sources (if any) must be added to those given by these equations.

**PROBLEMS**

**P11.5.** A cylindrical rod of length \( L \) and radius \( R \) is uniformly magnetized with magnetization \( \mathbf{M} \) directed along its axis. Let the axis coincide with the \( z \)-axis. Find the bound current densities and then argue that the exterior field produced by this bar magnet is identical to the field produced by a solenoid (P5.14) of the same dimensions. If the solenoid has \( n \) turns per unit length, what should be the current in the wire to duplicate the field of the magnet exactly?

**P11.6.** A sphere of radius \( R \) has a uniform magnetization \( \mathbf{M} \). Choosing a coordinate system with its origin at the center of the sphere and its polar axis parallel to \( \mathbf{M} \) so that \( \mathbf{M} = M\mathbf{k} \), (a) find the equivalent bound currents and (b) calculate the dipole moment of the sphere both from the magnetization and from the bound currents.

**P11.7.** Show that the total bound current crossing an open surface within a magnetized object is given by the line integral of the magnetization about the path bounding the surface.

**P11.8.** Derive the expression in Eq. (11.28) for \( i_{xy} \).

**P11.9.** Following the pattern illustrated in the text, start from the model in Fig. 11.2 and derive the expression \( J_{mx} = (\nabla \times \mathbf{M})_x \) for the \( x \)-component of the bound current density.

**P11.10.** Evaluate the curl of Eq. (11.24) to derive Eq. (11.35).

**P11.11.** A spherical cavity of radius \( R \) is cut in the interior of an object with uniform magnetization \( \mathbf{M} \). Find the equivalent currents on the surface of that cavity and show that these currents contribute an amount \( -\frac{2}{3}\mu_0\mathbf{M} \) to the field at the center of the cavity.

**11.4 An Alternative Approach to the Exterior Field: Equivalent Poles**

Although the expression for the exterior field in terms of bound currents is at least in one way more general than the expression we now derive (see Section 11.5), the integrals contain cross products and their evaluation is often very complicated. A formalism that expresses \( \mathbf{B} \) as the gradient of a scalar is simpler and more useful in some cases than the expression in Eq. (11.35). We therefore describe an alternative writing of Eq. (11.20). First, we replace \((r - r')|r - r'|^3\) by its equivalent \(-\nabla(1/|r - r'|)\) as in Eq. (4.52) and then we calculate \( \mathbf{B} \)
11.4. AN ALTERNATIVE APPROACH TO THE EXTERIOR FIELD

where, to obtain the second form, we used the vector identity in Eq. (C.16) with \( Q = M(r') \) and \( R = \nabla (1/|r - r'|) \), and we noted that such combinations as \( \nabla \cdot M(r') \) are zero because \( \nabla \) differentiates with respect to the components of \( r \). Now, for points exterior to the magnetized object, \( r' \) is never equal to \( r \). Hence, \( \nabla^2 (1/|r - r'|) = 0 \) throughout the domain of integration and Eq. (11.36) therefore reduces to

\[
\mathbf{B}(r) = \frac{\mu_0}{4\pi} \int (M(r') \cdot \nabla) \nabla \left( \frac{1}{|r - r'|} \right) dv'
\]

Continuing, we use the vector identity in Eq. (C.15) with \( Q \) and \( R \) as before to rewrite Eq. (11.37) in the form

\[
\mathbf{B}(r) = \mu_0 \nabla \left[ \frac{1}{4\pi} \int M(r') \cdot \nabla' \left( \frac{1}{|r - r'|} \right) dv' \right]
\]

Equation (11.38), however, expresses the field at an exterior point as the gradient of a scalar. Thus, if we introduce the magnetic scalar potential \( V^{(m)} \) in such a way that

\[
\mathbf{B}(r) = -\mu_0 \nabla V^{(m)}(r)
\]

we find from Eq. (11.38) that, apart from an arbitrary additive constant,

\[
V^{(m)}(r) = \frac{1}{4\pi} \int M(r') \cdot \nabla' \left( \frac{1}{|r - r'|} \right) dv'
\]

where the operator \( \nabla \) acting on the function \( 1/|r - r'| \) has been replaced by its equivalent \( -\nabla' \). Since Eq. (11.40) is now identical in form with Eq. (10.11), further rewriting of Eq. (11.40) follows the same path as was described for Eq. (10.11). Analogous to Eq. (10.15) we ultimately find that

\[
V^{(m)}(r) = \frac{1}{4\pi} \int \frac{M(r') \cdot \hat{n}(r')}{|r - r'|} dS' + \frac{1}{4\pi} \int \frac{[-\nabla' \cdot M(r')]}{|r - r'|} dv'
\]

Equation (11.41) suggests that, at least for purposes of calculation if not as a physically real description, we introduce a volume density of magnetic monopoles given by

\[
\rho_m(r) = -\nabla \cdot M(r)
\]

and a surface density of magnetic monopoles given by

\[
\sigma_m(r) = M(r) \cdot \hat{n}(r)
\]
Thus, insofar as the exterior field is concerned, the magnetized object can be replaced either by an equivalent distribution of currents [Eqs. (11.25) and (11.26)] or by an equivalent distribution of (fictitious) magnetic (mono)poles [Eqs. (11.42) and (11.43)]. If the poles are adopted, the field, calculated by taking the gradient of the magnetic scalar potential, is given by

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint \frac{\sigma_m(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dS' + \frac{\mu_0}{4\pi} \int \frac{\rho_m(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' \]  

(11.44)

[Compare Eq. (10.18).] Because of the similarity of Eqs. (11.41) and (11.44) to expressions for the electrostatic potential and field, many problems in magnetostatics correspond to analogous problems in electrostatics and solutions in one area can be directly translated to the other area. (See P11.13.) Of course, Eqs. (11.41) and (11.44) give only the contribution of the magnetized object to the \( \mathbf{B} \)-field; the contributions of any other sources must be added to these equations to obtain the total field.

### PROBLEMS

**P11.12.** Patterning your argument after the development associated with 10.11, derive Eq. (11.41) from Eq. (11.40).

**P11.13.** Find the distribution of poles equivalent to the magnetized cylinder in P11.5 and then find the magnetic scalar potential at the point \((0,0,z)\) by direct transcription of the solution to the analogous electrostatic problem in P10.10.

**P11.14.** Find the distribution of poles equivalent to the uniformly magnetized sphere of P11.6 and calculate the dipole moment of the sphere from these poles. Hint: By analogy with electrostatics, \( \mathbf{m} = \int \mathbf{r} \rho_m(\mathbf{r}) dV \) for a volume distribution of poles.

**P11.15.** Show that the magnetic moment \( \mathbf{m} \) of an object with magnetization \( \mathbf{M} \) can be calculated from the equivalent magnetic poles, i.e.,

\[ \mathbf{m} = \int \mathbf{r} (-\nabla \cdot \mathbf{M}) dV + \oint \mathbf{r} (\mathbf{M} \cdot \hat{n}) dS \]

Hint: Integrate Eq. (C.11) over the volume of the object, setting \( Q = \mathbf{M} \) and \( \Phi = x, y, \) and \( z \) successively. Then, use the divergence theorem and note Eq. (11.18).

**P11.16.** Derive Eq. (11.44) for the \( \mathbf{B} \)-field by evaluating \(-\mu_0 \nabla V^{(m)}\), with \( V^{(m)} \) given by Eq. (11.41).

**P11.17.** A spherical cavity of radius \( R \) is cut in the interior of an object with uniform magnetization \( \mathbf{M} \). Find the equivalent poles on the surface of that cavity and show that these poles contribute an amount \( \mathbf{B} = \frac{1}{3} \mu_0 \mathbf{M} \) to the field at the center of the cavity.

### 11.5 The Macroscopic Magnetic Induction Field at a Point Interior to a Magnetized Object

With two alternative expressions—Eqs (11.35) and (11.44)—for the magnetic induction field at a point exterior to a magnetized object, we have two starting points for the calculation of the field at an interior point. In both cases, we proceed as we did for the dielectric by separating the material into near and far regions, treating the far regions macroscopically by the methods described in one or the other of the preceding two sections and considering
the near region by methods similar to those used in Section 10.4. In this section we shall outline the starting points for the two calculations of the interior field and then state the results. Detailed derivation of these results is left to the problems.

If we elect to express the field in terms of equivalent currents, then we find from Eq. (11.35) that

\[ \mathbf{B}_{\text{interior}} = \frac{\mu_0}{4\pi} \int_{\mathcal{V}_1} \mathbf{J}_m(r') \times \frac{(r - r')}{|r - r'|^3} dV' + \frac{\mu_0}{4\pi} \int_{\Sigma} \mathbf{j}_m(r') \times \frac{(r - r')}{|r - r'|^3} dS' \]

\[ + \frac{\mu_0}{4\pi} \int_{\Sigma_{\text{II}}} \mathbf{j}_m(r') \times \frac{(r - r')}{|r - r'|^3} dS' + \mathbf{B}_{\text{II}}(r) \quad (11.45) \]

where \( \mathcal{V}_1 \) is the volume of the magnetized material \textit{excluding} a spherical volume of radius \( R \) that contains the near molecules, \( \Sigma \) is the surface of the original object, \( \Sigma_{\text{II}} \) is the surface of the cavity that remains in the magnetized object when the near molecules are removed, and \( \mathbf{B}_{\text{II}} \) is the contribution to the macroscopic \( \mathbf{B} \)-field made by the near molecules. (The notation is the same as that used in Section 10.4; see Fig. 10.3.) Now, since \( \mathbf{j}_m(r') = \mathbf{M}(r') \times \mathbf{n}(r') \) and \( \mathbf{M}(r') \) can be considered constant over the macroscopically small surface \( \Sigma_{\text{II}} \), the third integral in Eq. (11.45) gives the contribution made by the equivalent currents on the surface of a spherical cavity in a uniformly magnetized object to the \( \mathbf{B} \)-field at the center of the cavity, and it has the value \( -\frac{2}{3} \mu_0 \mathbf{M} \) (P11.11). Further, as in the dielectric case, extending the integral over \( \mathcal{V}_1 \) to include the volume \( \mathcal{V}_{\text{II}} \) enclosed by the sphere \( \Sigma_{\text{II}} \) adds zero to the overall expression (P11.18). Thus, Eq. (11.45) reduces to the simpler expression

\[ \mathbf{B}_{\text{interior}}(r) = \mathbf{B}(r) - \frac{2}{3} \mu_0 \mathbf{M}(r) + \mathbf{B}_{\text{II}}(r) \quad (11.46) \]

where \( \mathbf{B}(r) \) is the field given at the \textit{interior} point \( r \) by evaluating the exterior expression, Eq. (11.35), at the interior point; i.e., \( \mathbf{B}(r) \) is the field established at the interior point by the bound current distribution equivalent to the entire magnetized object.

If, on the other hand, we express the field in terms of equivalent poles, then we find from Eq. (11.44) that

\[ \mathbf{B}_{\text{interior}} = \frac{\mu_0}{4\pi} \int_{\mathcal{V}_1} \mathbf{\rho}_m(r') \frac{r - r'}{|r - r'|^3} dV' + \frac{\mu_0}{4\pi} \int_{\Sigma} \mathbf{\sigma}_m(r') \frac{r - r'}{|r - r'|^3} dS' \]

\[ + \frac{\mu_0}{4\pi} \int_{\Sigma_{\text{II}}} \mathbf{\sigma}_m(r') \frac{r - r'}{|r - r'|^3} dS' + \mathbf{B}_{\text{II}}(r) \quad (11.47) \]

where the volumes and surfaces are the same as those described in connection with Eq. (11.45). Now, since \( \mathbf{\sigma}_m(r') = \mathbf{M}(r') \cdot \mathbf{n}(r') \) and \( \mathbf{M}(r') \) can be considered constant over the macroscopically small surface \( \Sigma_{\text{II}} \), the third integral in Eq. (11.47) gives the contribution made by the equivalent poles on the surface of a spherical cavity in a uniformly magnetized object to the \( \mathbf{B} \)-field at the center of the cavity, and it has the value \( \frac{1}{3} \mu_0 \mathbf{M} \) (P11.17). Further, the integral over \( \mathcal{V}_1 \) can again be extended to include the volume \( \mathcal{V}_{\text{II}} \) without changing the value of the integral (P11.19). Thus, Eq. (11.47) reduces to the simpler expression

\[ \mathbf{B}_{\text{interior}}(r) = -\mu_0 \nabla V(m)(r) + \frac{1}{3} \mu_0 \mathbf{M}(r) + \mathbf{B}_{\text{II}}(r) \quad (11.48) \]
where \( V^{(m)}(r) \) is the magnetic scalar potential given at the interior point \( r \) by evaluating the exterior expression, Eq. (11.41), at the interior point; i.e., \( V^{(m)}(r) \) is the potential established at the interior point by the distribution of poles equivalent to the entire magnetized object.

The contribution \( B_{II}(r) \) made by the near molecules is the same, whichever approach to calculating the far field is adopted. Following the pattern in Section 10.4, we define

\[
B_{II}(r) = \frac{N}{v_{II}} \int_{v_{II}} B_{mol}(r, r') \, dv'
\]  

(11.49)

where \( N \) is the number of molecules in \( v_{II} \) and \( B_{mol}(r, r') \) is the time average of the microscopic field produced at \( r \) by a molecule at \( r' \). [Compare Eq. (10.33).]

Now, we introduce the time-averaged molecular current density which we take to be given at the point \( r_1 \) for a molecule whose center is at \( r_c \) by \( J_{mol}(r_1 - r_c) \). Then the field \( B_{mol}(r, r') \) is given by Eq. (5.10), viz.,

\[
B_{mol}(r, r') = \frac{\mu_0}{4\pi} \int J_{mol}(r'' - r') \times \frac{r - r''}{|r - r''|^3} \, dv''
\]  

(11.50)

[Compare Eq. (10.34).] Now, we rewrite Eq. (11.50) as an integral over the variable \( r'' = r'' - r' \) [compare Eq. (10.35)] and then substitute the result into Eq. (11.49) to find that

\[
B_{II}(r) = \frac{N\mu_0}{4\pi v_{II}} \int J_{mol}(r'') \times \left[ \int_{v_{II}} \frac{r - r' - r''}{|r - r' - r''|^3} \, dv' \right] \, dv''
\]  

(11.51)

[Compare Eq. (10.36).] The integral in square brackets, however, is the integral \( I_1 \), whose value is \( I_1 = -\frac{4\pi}{3}r'' \) [Eq. (10.42)]. Thus, Eq. (11.51) reduces to

\[
B_{II}(r) = \frac{N\mu_0}{4\pi v_{II}} \left( -\frac{4\pi}{3} \right) \int J_{mol}(r'') \times r'' \, dv''
\]  

(11.52)

We note finally that the remaining integral is nothing but the negative of twice the dipole moment of a single molecule located near the point \( r \) in the magnetized object. [Replace \( I \, dr \) by \( J \, dv \) in Eq. (3.36).] Thus, \( N \) times the integral is the negative of twice the total dipole moment in \( v_{II} \) and that product divided by \( v_{II} \) is the negative of twice the macroscopic magnetization \( M(r) \) of the object at \( r \). Equation (11.52) therefore becomes

\[
B_{II}(r) = \frac{2}{3}\mu_0 M
\]  

(11.53)

The results for the macroscopic interior field now have different expressions depending on whether we adopt bound currents or equivalent poles. Substituting Eq. (11.53) into Eq. (11.46), we find that

\[
B_{\text{interior}}(r) = B(r)
\]  

(11.54)

and the interior field and the exterior field are both given correctly by the bound currents alone. In contrast, if we substitute Eq. (11.53) into Eq. (11.48), we find that

\[
B_{\text{interior}}(r) = -\mu_0 \nabla V^{(m)}(r) + \mu_0 M(r)
\]  

(11.55)

Although this result is valid outside the object as well as inside (the magnetization is zero outside the material), we nevertheless find that the equivalent poles by themselves are not sufficient to account for the entire field of a magnetized object. The object can for all
purposes be replaced by a set of bound currents, but it cannot be replaced by a set of equivalent poles without recognizing the additional term in Eq. (11.55).

**PROBLEMS**

**P11.18.** Show that the integral over \( v_I \) in Eq. (11.45) can be extended to include the volume \( v_{II} \) with no change in value.

**P11.19.** Show that the integral over \( v_I \) in Eq. (11.47) can be extended to include the volume \( v_{II} \) with no change in value.

**P11.20.** In the text, we represented the molecule by a current distribution in order to calculate \( B_{II}(r) \). Suppose instead we represented the molecules by a volume distribution of magnetic poles, say \( \rho_{mol}(r_1 - r_c) \). Then, in accordance with Eq. (11.44), we would write

\[
B_{mol}(r, r') = \frac{\mu_0}{4\pi} \int \rho_{mol}(r'' - r') \frac{r - r''}{|r - r''|^3} dv''
\]

instead of Eq. (11.50). The resulting equation for \( B_{II}(r) \) is identical in form with Eq. (10.36) and hence has the value \( B_{II}(r) = -\frac{1}{3} \mu_0 M(r) \), as inferred from Eq. (10.44). This result differs significantly from the result in Eq. (11.53). Explain why Eq. (11.53) is correct and the result obtained by representing the molecule in terms of equivalent poles is incorrect.

### 11.6 The Basic Equations of Magnetostatics When Magnetically Responsive Matter is Present

In this section, we shall translate the basic equations of magnetostatics into a convenient form for treating problems involving magnetically responsive matter. We have already established that the static magnetic induction field produced at any point in space by a magnetized object can be viewed as originating in a suitable distribution of (steady) currents in free space. Thus, the static magnetic induction field \( B \) established jointly by a magnetized object and by any simultaneously present free currents still satisfies both the magnetic flux law,

\[
\oint B \cdot dS = 0 ; \quad \nabla \cdot B = 0 \quad (11.56)
\]

and Ampere’s Circuital Law

\[
\oint B \cdot dl = \mu_0 \oint J_t \cdot dS ; \quad \nabla \times B = \mu_0 J_t \quad (11.57)
\]

provided we now interpret the current density \( J_t \) as the total current density, which includes any free currents placed in space and any bound currents present on magnetized objects. Since we know how to relate the bound currents to the magnetization \( M \), however, we can reexpress Eq. (11.57) in a better form. Let \( J \) now denote only the free current density. Then

\[
J_t = J + J_m = J + \nabla \times M \quad (11.58)
\]

Hence, with some rearrangement, Eq. (11.57) yields

\[
\oint \left( \frac{B}{\mu_0} - M \right) \cdot dl = \oint J \cdot dS ; \quad \nabla \times \left( \frac{B}{\mu_0} - M \right) = J \quad (11.59)
\]
(The integral form is obtained after using Stokes' theorem.) We stress again that the current density appearing on the right-hand side is the \textit{free} current density; the bound currents have been explicitly introduced and appear now on the left-hand side in somewhat disguised form. We can now push the magnetization, which we usually do not know initially, into the background altogether by introducing the \textit{magnetic field intensity} $\mathbf{H}$ defined in (rationalized) mks units by\textsuperscript{5,6}

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (11.60)$$

The mks unit of $\mathbf{H}$ is the ampere/meter, A/m.

At least the line integral of $\mathbf{H}$ about a closed path or equivalently its curl is determined solely by the free currents, as evidenced by the new form of Ampere’s circuital law,

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{S} \quad ; \quad \nabla \times \mathbf{H} = \mathbf{J} \quad (11.61)$$

obtained by substituting Eq. (11.60) into Eq. (11.59). The magnetic intensity $\mathbf{H}$ is the second of the two auxiliary fields mentioned in the introductory paragraphs of Chapter 10. Although we shall continue to regard $\mathbf{B}$ as the basic field, arguments supporting the opposite view can be presented. Whichever field is viewed as basic, the field $\mathbf{B}$ remains the field that determines the force on a charged particle.

We have now determined that the static magnetic induction field in the presence of magnetically responsive matter satisfies the basic equations

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{S} \quad ; \quad \nabla \times \mathbf{H} = \mathbf{J} \quad (11.62)$$

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0 \quad ; \quad \nabla \cdot \mathbf{B} = 0 \quad (11.63)$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad ; \quad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (11.64)$$

As with the analogous electrostatic equations, this system reduces to the vacuum form when no magnetically responsive matter is present and $\mathbf{M}$ is therefore zero. In that case $\mathbf{B}$ and $\mathbf{H}$ are trivially different and Eqs. (11.62) and (11.63) are sufficient to determine the field.

When magnetizable matter is present, however, $\mathbf{B}$ may differ nontrivially from $\mathbf{H}$, the two differential equations involve essentially different fields, and we can make little progress toward a solution of these equations until we know a relationship between $\mathbf{B}$ and $\mathbf{H}$. Equation (11.64) is a step toward specifying that relationship, but it is not a complete specification because the vector $\mathbf{M}$ is still not known. As with dielectrics, the specific properties of matter enter the theory at this point. Following established custom we write the necessary constitutive relation in the general form

$$\mathbf{M} = \mathbf{M}(\mathbf{H}) \quad (11.65)$$

---

\textsuperscript{5} The magnetic field intensity is frequently referred to as the magnetic intensity or the magnetic field, which is not to be confused with the magnetic induction $\mathbf{B}$.

\textsuperscript{6} The vector $\mathbf{H}$ is defined differently in different systems of units but is always some linear combination of $\mathbf{B}$ and $\mathbf{M}$. In cgs-esu, cgs-emu, Gaussian units, and Heaviside-Lorentz units, $\mathbf{H}$ is defined so that $\nabla \times \mathbf{H} = 4\pi \mathbf{J}$, $4\pi \mathbf{J}$, $4\pi \mathbf{J}/c$, and $\mathbf{J}/c$, respectively, where $c$ is the speed of light in cm/s. In particular, the Gaussian unit of $\mathbf{H}$, which is both dimensionally and numerically the same thing as the gauss, is nevertheless commonly called the oersted; an $\mathbf{H}$-field of 1 ampere/m in (rationalized) mks units is the same field as an $\mathbf{H}$-field of $4\pi \times 10^{-3}$ oersted in Gaussian units.
thinking of $M$ more directly as a function of $H$ rather than of $B$. A more specific form for Eq. (11.65) can be determined only by an empirical study of specific magnetizable objects or, in some cases, by a detailed quantum mechanical calculation. Once the form of Eq. (11.65) is known for a specific material, Eqs. (11.62)–(11.64) provide sufficient information to determine the two fields $B$ and $H$ if the free currents and/or suitable boundary conditions—see Section 12.7—are known.

As in the analogous dielectric case, the specific form of the constitutive relation appropriate to a given material may be quite complicated. Magnetically responsive materials may be anisotropic (different in different directions), they may exhibit a permanent or spontaneous magnetization even in the absence of a magnetizing field, they may show properties determined in part by their past history (hysteresis), etc. Even though magnetic materials displaying these complications are more common and more important than materials displaying the analogous dielectric complications, there are nonetheless many magnetic materials that develop a magnetization that is parallel to and also simply related to the magnetizing field. For these materials, the constitutive relation has the form

$$M = \chi_m(H)H$$  \hspace{1cm} (11.66)

where the static magnetic susceptibility $\chi_m(H)$ may still depend on the field strength. For many materials, however, $\chi_m$ is constant, at least if $H$ is not too large. These simplest of all magnetic materials are called linear materials. They are diamagnetic when $\chi_m < 0$ and paramagnetic when $\chi_m > 0$. (Why?) Further, the magnetic induction field in such a material is proportional to the magnetic field intensity,

$$B = \mu_0(H + M) = \mu_0(1 + \chi_m)H = \mu H$$  \hspace{1cm} (11.67)

where the static magnetic permeability $\mu$ of the material is defined by

$$\mu = \mu_0(1 + \chi_m)$$  \hspace{1cm} (11.68)

Since $\chi_m$ may be positive or negative, $\mu$ may be larger or smaller than $\mu_0$ and the relative permeability defined by

$$K_m = \frac{\mu}{\mu_0} = 1 + \chi_m$$  \hspace{1cm} (11.69)

may be either larger or smaller than unity. Values of $\chi_m$ for a selection of common materials are shown in Table 11.2.

We shall conclude this section with an example that not only illustrates the new form of the circuital law but also provides the basis of one method for measuring magnetic permeabilities. Consider the so-called Rowland ring,\(^7\) which consists of a coil wound around a toroidal core of some magnetic material (Fig. 11.9). Using symmetries by now familiar for these geometries, we apply the new form of the circuital law, Eq. (11.62), to a circular path of radius $r$ in the material of the core, shown dashed in Fig. 11.9, and find that

$$\oint H \cdot dl = 2\pi r H \phi = NI \implies H = \frac{NI}{2\pi r} \phi$$  \hspace{1cm} (11.70)

---

\(^7\)American physicist Henry Augustus Rowland, b. 27 November 1848 in Honesdale, Pennsylvania; d. 16 April 1901 in Baltimore, Maryland.
Table 11.2: Static Magnetic Susceptibilities for Selected Materials. The values in this table are quoted from the Handbook of Chemistry and Physics (Chemical Rubber Publishing Company, Cleveland, 1955), Thirty-seventh Edition, pp. 2390–2400, and are used by permission of the Chemical Rubber Company. Values from the handbook (in cgs units) must be multiplied by \(4\pi\) to obtain the values tabulated here (in mks units). Note that paramagnetic materials (\(\chi_m > 0\)) typically exhibit stronger response than diamagnetic materials (\(\chi_m < 0\)) and that both paramagnetic and diamagnetic metals may be found.

<table>
<thead>
<tr>
<th>Material</th>
<th>Temperature</th>
<th>(\chi_m = K_m - 1)</th>
<th>Material</th>
<th>Temperature</th>
<th>(\chi_m = K_m - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air (1 atm)</td>
<td>20°C</td>
<td>+30.36 (\times 10^{-5})</td>
<td>Hydrogen</td>
<td>20°C</td>
<td>-2.48 (\times 10^{-5})</td>
</tr>
<tr>
<td>Aluminum</td>
<td>18°C</td>
<td>+0.82 (\times 10^{-5})</td>
<td>Magnesium</td>
<td>18°C</td>
<td>+0.69 (\times 10^{-5})</td>
</tr>
<tr>
<td>Argon (1 atm)</td>
<td>20°C</td>
<td>-0.56 (\times 10^{-5})</td>
<td>Neon (1 atm)</td>
<td>20°C</td>
<td>-0.41 (\times 10^{-5})</td>
</tr>
<tr>
<td>Bismuth</td>
<td>18°C</td>
<td>-1.70 (\times 10^{-5})</td>
<td>Nitrogen (1 atm)</td>
<td>20°C</td>
<td>-0.430 (\times 10^{-5})</td>
</tr>
<tr>
<td>Cobaltous chloride</td>
<td>25°C</td>
<td>+114 (\times 10^{-5})</td>
<td>Oxygen (liquid)</td>
<td>-219°C</td>
<td>+390 (\times 10^{-5})</td>
</tr>
<tr>
<td>Copper</td>
<td>18°C</td>
<td>-0.11 (\times 10^{-5})</td>
<td>Oxygen (1 atm)</td>
<td>20°C</td>
<td>+133 (\times 10^{-5})</td>
</tr>
<tr>
<td>Diamond</td>
<td>20°C</td>
<td>-0.62 (\times 10^{-5})</td>
<td>Paraffin</td>
<td>Room</td>
<td>-0.75 (\times 10^{-5})</td>
</tr>
<tr>
<td>Dysprosium oxide</td>
<td>16°C</td>
<td>+288 (\times 10^{-5})</td>
<td>Silicon</td>
<td>18°C</td>
<td>-0.16 (\times 10^{-5})</td>
</tr>
<tr>
<td>Ferric chloride</td>
<td>20°C</td>
<td>+108 (\times 10^{-5})</td>
<td>Silver</td>
<td>18°C</td>
<td>-0.25 (\times 10^{-5})</td>
</tr>
<tr>
<td>Germanium</td>
<td>18°C</td>
<td>-0.15 (\times 10^{-5})</td>
<td>Sodium</td>
<td>18°C</td>
<td>+0.64 (\times 10^{-5})</td>
</tr>
<tr>
<td>Glass (crown)</td>
<td>Room</td>
<td>-1.1 (\times 10^{-5})</td>
<td>Titanium</td>
<td>20°C</td>
<td>+1.57 (\times 10^{-5})</td>
</tr>
<tr>
<td>Helium (1 atm)</td>
<td>20°C</td>
<td>-0.59 (\times 10^{-5})</td>
<td>Tungsten</td>
<td>18°C</td>
<td>+0.35 (\times 10^{-5})</td>
</tr>
</tbody>
</table>

Figure 11.9: A Rowland ring.

where \(I\) is the current in the coil, \(N\) is the total number of turns, and \(\hat{\phi}\) is a unit vector tangent to the path. From Eqs. (11.67) and (11.69), we then have that

\[
B = \mu H = \frac{\mu N I}{2\pi r} \hat{\phi} = K_m \frac{\mu_0 N I}{2\pi r} \hat{\phi} = K_m B_0
\]  

(11.71)

where \(\mu\) is the permeability of the core and \(B_0\) is the magnetic induction field that would be present if the core were removed. The self-inductance \(L\) of this coil, which is defined in
Eq. (6.28), is now given by

\[ L = \frac{d}{dI} \left[ N \int_{\Sigma} \mathbf{B} \cdot d\mathbf{S} \right] = K_m \frac{d}{dI} \left[ N \int_{\Sigma} \mathbf{B}_0 \cdot d\mathbf{S} \right] = K_m L_0 \]  

(11.72)

where \( \Sigma \) is a surface bounded by a single turn of the coil and \( L_0 \) is the self-inductance that the coil would have if the core were removed. Thus, measurement of two self-inductances, one with the core present and the second with the core removed, leads to a determination of the relative permeability of the material composing the core. Unlike the corresponding dielectric case, it is here possible for \( L \) to be either less than \( L_0 \) (diamagnetic core) or greater than \( L_0 \) (paramagnetic core). We should realize, however, that for many materials \( K_m \) does not differ from unity by very much, and the change in self-inductance may be small enough to require sophisticated measuring techniques.

PROBLEMS

P11.21. (a) Add to Eq. (11.55) a term giving the contribution of free currents with current density \( \mathbf{J}(\mathbf{r}) \) and then show that

\[ \mathbf{H} = -\nabla V^{(m)} + \frac{1}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}' = -\nabla V^{(m)} + \frac{1}{\mu_0} \nabla \times \mathbf{A}_{\text{free}} \]

where \( \mathbf{A}_{\text{free}} \) is the vector potential established by the free currents. (Note that these relationships are valid both inside and outside magnetized matter.) (b) Show that \( \nabla \times \mathbf{H} = \mathbf{J} \) from this result.

P11.22. Show from Eqs. (11.62) and (11.63) that the magnetic field intensity in a region containing no free currents (\( \mathbf{J} = 0 \)) but filled with a linear material having permeability \( \mu \) can be derived from a scalar potential \( V^{(m)} \) by \( \mathbf{H} = -\nabla V^{(m)} \) and that \( V^{(m)} \) satisfies \( \nabla \cdot [\mu \nabla V^{(m)}] = 0 \), which reduces to Laplace’s equation if \( \mu \) is independent of position. Hint: See item (3) in Section 2.5. Optional: Reconcile \( \mathbf{H} = -\nabla V^{(m)} \) with the result in P11.21, in which the integral in general makes a contribution to \( \mathbf{H} \) even at points where \( \mathbf{J} \) is zero. Hint: Imagine the free currents to be the currents equivalent to a suitably magnetized object and then replace the object instead with a set of equivalent poles. At what points in space is this replacement valid?

P11.23. An infinite solenoid (P5.14) of radius \( R \) has \( N \) turns per unit length, carries current \( I \), and is wound on a core of linear material that has permeability \( \mu \). Let the axis of the solenoid coincide with the z-axis. (a) Calculate \( \mathbf{H} \), \( \mathbf{B} \), and \( \mathbf{M} \) in the core. (b) Show that \( \mathbf{H} \) has the same value that it would have if the core were vacuum but that \( \mathbf{B} \) is \( K_m \) times the vacuum value. (c) Calculate the bound surface current on the core, add this current to the free current, and then calculate \( \mathbf{B} \) using Eq. (11.57).

11.7 Connecting the Microscopic Polarizability with the Macroscopic Relative Permeability

A relation similar to the Clausius-Mossotti\(^8\) relation in Section 10.6 can be derived to determine the microscopic (magnetic) molecular polarizability of some (but not all) materials

---

\(^8\)Rudolf Clausius and Ottaviano Mossotti; see footnote 5 in Chapter 10, page 257, and footnote 6 in Chapter 10, page 257.
from measured values of the macroscopic relative permeability. Following the pattern of Section 10.6, we begin by finding the field $B_m$ that a single molecule in a magnetic material experiences, i.e., by finding the (time-averaged) microscopic field established at the position of a molecule by all molecules except the one located at that site. The same separation of the material into near and far molecules that led to Eq. (10.66) here leads to the expression

$$B_m = B_I + B'$$

(11.73)

where $B_I$ is the contribution of the far molecules and $B'$ the contribution of the near molecules. In fact, we have already calculated $B_I$ by two different methods; its definition here is essentially identical with the definition of $B_{\text{interior}} = B_I$ in Section 11.5 except that we must include any contributions $B_{\text{free}}$ from free currents. Thus, we have from Eq. (11.46) that

$$B_m = B - \frac{2}{3}\mu_0 M + B'$$

(11.74)

where $B$ now includes $B_{\text{free}}$, or we have from Eq. (11.48) that

$$B_m = -\mu_0 \nabla V^{(m)} + \frac{1}{3}\mu_0 M + B_{\text{free}} + B'$$

(11.75)

Continuing as in Section 10.6, we confine our attention to those materials for which $B' = 0$ (see P10.23). Further, since the fields $B_m$ and $B_{\text{free}}$ are regarded to exist in free space, $B_m = \mu_0 H_m$ and $B_{\text{free}} = \mu_0 H_{\text{free}}$. Thus, Eqs. (11.74) and (11.75) both give the expressions

$$H_m = H + \frac{1}{3} M ; \quad B_m = \mu_0 \left( H + \frac{1}{3} M \right)$$

(11.76)

for the microscopic field at a molecular site; here, $H$ is the total $H$-field, including contributions of the free currents and of the magnetized matter. Finally, again as in Section 10.6, we combine Eq. (11.76) with Eq. (11.15) to find that

$$M = N m = N\beta_t B_m = N\beta_t \mu_0 \left( H + \frac{1}{3} M \right) \implies M = \frac{N\beta_t \mu_0}{1 - \frac{1}{3} N\beta_t \mu_0} H$$

(11.77)

where $N$ is the number of molecules per unit volume. Thus, we have by comparison with Eq. (11.66) that

$$\chi_m = \frac{N\beta_t \mu_0}{1 - \frac{1}{3} N\beta_t \mu_0}$$

(11.78)

and, further, by combining Eq. (11.78) with Eq. (11.69), we have that

$$K_m = \frac{1 + \frac{2}{3} N\beta_t \mu_0}{1 - \frac{1}{3} N\beta_t \mu_0}, \quad \beta_t = \frac{3}{N\mu_0} \frac{K_m - 1}{K_m + 2}$$

(11.79)

In contrast to the dielectric case, however, the materials for which Eqs. (11.78) and (11.79) work well are the diamagnetic and paramagnetic materials for which $|\chi_m| \ll 1$ or, equivalently, $K_m \approx 1$. For these materials $N\beta_t \mu_0$ is small compared to one (Why?), and Eqs. (11.78) and (11.79) can be approximated to give

$$\chi_m \approx N\beta_t \mu_0, \quad \beta_t \approx \frac{1}{N\mu_0} (K_m - 1)$$

(11.80)
which we would have obtained directly by ignoring the difference between \( H \) and \( H_m \) altogether. When they apply, Eqs. (11.79) and (11.80) express the (microscopic) molecular polarizability in terms of the (macroscopic) relative permeability and are therefore analogous to the Clausius-Mossotti relation of Section 10.6. Unfortunately, magnetic materials that fail to conform with the assumptions of this section are more common than dielectrics that fail to conform with the analogous assumptions of Section 10.6, and the above equations are therefore less generally useful than the Clausius-Mossotti relation.

**PROBLEMS**

**P11.24.** Derive Eq. (11.76) for \( H_m \) and \( B_m \) (a) from Eq. (11.74) and (b) from Eq. (11.75).

**P11.25.** For a paramagnetic material, with \( \beta_t = m_0^2 / 3kT \) [Eq. (11.1)], Eq. (11.77) suggests that at the right temperature \( T_c \) (the Curie temperature\(^9\)), \( M \) can be nonzero even if \( H \) is vanishingly small, and at \( T \leq T_c \) we might therefore expect the material to exhibit a spontaneous magnetization. (a) Show that \( T_c = \frac{N\mu_0 m_0^2}{9k} \). (b) Very small volumes of some materials, called ferromagnetic materials, in fact do exhibit spontaneous magnetization at temperatures below some critical temperature. Calculate \( T_c \) for pure iron \( (m_0 = 2.05 \times 10^{-23} \text{ J/T}, \text{ specific gravity at } 20^\circ\text{C} = 7.874) \) and compare your result with the measured value 1043 K. (c) In the Weiss theory of ferromagnetism,\(^10\) the factor of \( \frac{1}{3} \) in Eq. (11.76) is replaced by a different factor \( \gamma \) so that \( H_m = H + \gamma M \). What value must \( \gamma \) have in order to predict the critical temperature of iron correctly?

### 11.8 Ferromagnetism

Some materials, known as *ferromagnetic materials*, exhibit a very extreme nonlinear paramagnetic response that arises from a very strong inclination of the constituent (microscopic) permanent dipoles to orient themselves parallel to an externally applied magnetizing field. The extreme strength of this ferromagnetic response can be understood only by quantum mechanical arguments and the phenomenon itself is therefore a macroscopic manifestation of a purely quantum mechanical behavior; ferromagnetism cannot even be approximately understood by classical arguments. We cannot digress here to present the full quantum mechanical treatment. In essence, however, it turns out that, under the conditions existing in ferromagnetic materials, parallel orientation of all of the molecular dipoles within macroscopically small but still microscopically large volumes of the material is energetically preferred to a more random orientation of these dipoles. Thus, a macroscopic sample of ferromagnetic material breaks up *spontaneously* into macroscopically small but microscopically large *domains*, each of which contains a very large number of molecular dipoles all oriented in the same way. Now, in a macroscopically *unmagnetized* sample, these domains are randomly oriented within the constraints imposed by the crystal structure of the sample [Fig. 11.10(a)]. When placed in an external field, however, the sample magnetizes by preferential orientation of domains or by the motion of domain walls so as to enlarge some domains at the expense of others [Fig. 11.10(b)]. The resulting *collective* response of huge

\(^9\)French physicist Pierre Curie, b. 15 May 1859 in Paris, France; d. 19 April 1906 in Paris, France.

\(^{10}\)French physicist Pierre-Ernest Weiss, b. 25 March 1865 in Mulhouse, France; d. 24 October 1940 in Lyon, France.
numbers of molecules can produce an extremely large magnetization; the value of $K_m$—to the extent that it is meaningful at all—may be as large as $10^5$ or $10^6$.

Although the relationship $B = \mu_0(H + M)$ is always correct, for ferromagnetic materials the relationship between $M$ and $H$ and consequently also the relationship between $B$ and $H$ is very complicated. Even the qualitative properties of ferromagnetic materials are therefore somewhat involved. Ferromagnetic response in a particular ferromagnetic material, for example, is not observed at all unless the temperature $T$ is below the critical Curie temperature $T_c$ for that material; at temperatures above the Curie temperature ferromagnetic materials exhibit much weaker paramagnetic response. Even when $T \leq T_c$, $M$ and hence $B$ may both be nonzero even if $H$ is zero, and we have a permanent magnet. Further, whether $H$ is zero or not (but $T$ is still below $T_c$), the values of $M$ and $B$ in general are not unique, for the ferromagnetic response is usually not even approximately a single-valued function of $H$; it depends both on how $H$ is built up from zero and on the initial state of magnetization of the sample. Finally, ferromagnetic materials may be anisotropic, so that $B$, $H$, and $M$ need not be parallel except possibly when $H$ is applied in one of a few very special directions in the material. Clearly, we need to know quite a bit about the properties of a particular ferromagnetic sample, about its present environment, and about its past history before we can determine $B$ and $M$ from knowledge of $H$.

A common method for presenting information about the relationship between $B$ and $H$ in a particular ferromagnetic material, at least when $B$ and $H$ are parallel, is to display a graph of $B$ versus $H$, where $B$ and $H$ here mean not the magnitudes of $B$ and $H$ but the components of $B$ and $H$ along a line parallel to $B$ and $H$. (In mks units, it is perhaps better to plot $B$ versus $\mu_0 H$ so that the axes are labeled with quantities having the same dimensions.) Even a graph of $B$ versus $H$, however, depends on the past history of the sample to which it applies. For the sake of standardization, let us therefore start with a macroscopically unmagnetized sample, which we place, say, as the core of a Rowland ring (Fig. 11.9). Now, let $H$ be monotonically increased by gradually increasing the current. If we measure $B$ for various values of $H$, we obtain a relationship having the general form of Fig. 11.11, which is called the magnetization curve for the particular material involved. The quantity $B$ increases non-linearly with $H$, and ultimately a point is reached

\[11\] One way to determine $B$ in the core is to use, for example, a Hall probe (P9.7) to measure $B$ in a narrow gap in the core. $B$ in the gap is the same as $B$ in the core (P12.19).
above which further increases in $H$ do not alter $B$ significantly. Above this point, the material has magnetized to saturation, all of the domains are fully aligned, $M$ cannot be further increased, and the effect of increasing $H$ is no longer amplified by the response of the material; $B = \mu_0 (H + M)$ increases only slowly with further increases in $H$. Two permeabilities are sometimes used to describe the material to which Fig. 11.11 applies. The permeability analogous to the permeability used for diamagnetic and paramagnetic materials is geometrically the slope of a line from the origin to some point on the graph; it is defined by

$$B = \mu(H)H \quad \text{or} \quad \mu(H) = \frac{B(H)}{H} \quad (11.81)$$

and is a strong function of $H$. The differential permeability $\mu_d$ is geometrically the slope of the graph and is also a strong function of $H$. Which (if either) of these permeabilities is best suited to a particular problem involving ferromagnetic materials must be determined by examining the problem.

We can calculate the amount of energy required to move from one point on the magnetization curve to another by using the Faraday Law, which in fact is valid even in the presence of matter (see Section 12.1). Thus, the emf $\mathcal{E}$ induced in the Rowland ring of Fig. 11.9 is given by $\mathcal{E} = -N d\Phi_m/dt$, where $N$ is the number of turns and $\Phi_m$ is the flux of $B$ across a single turn of the coil. The power input $dW/dt$ as given by Eq. (6.23) then is

$$\frac{dW}{dt} = NI \frac{d\Phi_m}{dt} \quad (11.83)$$

To simplify the argument, suppose the ring is large enough so that the fields are essentially uniform within its core. Then $\Phi_m = BS$, where $S$ is the area of the cross section of the core and $B$ is the tangential component of $B$. Further, from Eq. (11.70), the tangential component of $H$, which we now denote merely $H$, is given by $H = NI/2\pi \ell = NI/\ell$, where $\ell$ is the (median) length of the core. Thus, Eq. (11.83) can be written in the form

$$\frac{dW}{dt} = (\ell H) \frac{d}{dt}(BS) = VH \frac{dB}{dt} \quad (11.84)$$
where $V = \ell S$ is the volume of the core. Consequently, the energy input \textit{per unit volume} required to increase the $B$-field from $B$ to $B + dB$ when the $H$-field is $H$ is given by

$$d \left( \frac{W}{V} \right) = H dB$$

(11.85)

and the total energy required \textit{per unit volume} to change the $B$-field from $B_{initial} = B_i$ to $B_{final} = B_f$ is given by

$$\frac{W}{V} = \int_{B_i}^{B_f} H(B) dB$$

(11.86)

which can be interpreted geometrically as the area bounded by the magnetization curve, the vertical axis, and the lines $B = B_i$ and $B = B_f$. If $B_f > B_i$ in Fig. 11.11, work must be done on the fields; if $B_f < B_i$, energy will be taken out of the fields.

The magnetization curve, of course, does not convey all properties of a ferromagnetic material. Suppose now that we gradually reduce $H$ from the maximum value reached in Fig. 11.11. We find that $B$ does not trace back to zero along the curve already obtained. Instead, $B$ follows a curve such as that shown from $a$ to $b$ in Fig. 11.12. When $H$ has been reduced to zero, $B$ is still nonzero, and a permanent magnetization has resulted. Application of an $H$-field in the opposite direction brings about further reduction of $B$, and finally $M$ saturates in the other direction at some point $c$. If $H$ is now increased from this point, $B$ again rises but still does not go to zero when $H$ has again become zero. The curve passes through some point $d$ as $H$ passes through zero. Further increase in $H$ causes $B$ ultimately to rise again to the point $a$ where $M$ saturates. If this cycle is now repeated many times in succession, after awhile the curve traced out in the $BH$-plane stabilizes with a shape and an extent determined by the material and by the range through which $H$ varies. The resulting stable curve is known as a \textit{hysteresis loop}, the word \textit{hysteresis} itself referring to the lag between the magnetization $M$ and the field $H$ that produces $M$. The loop is a (actually the) major hysteresis loop if $H$ becomes large enough to produce saturation and a minor hysteresis loop otherwise. Two parameters of the major hysteresis loop are often given to specify at least a part of the properties of a ferromagnetic material: The \textit{remanence} is the (positive) value of $B$ when $H$ is zero, and the \textit{coercive force} is the magnitude of $H$ when $B$
is zero. Together the remanence and the coercive force of a particular material outline the approximate region occupied by the major hysteresis loop of the material.

We now note from Eq. (11.86) that the amount of energy input to the fields as a ferromagnetic sample is moved along \( cda \) in Fig. 11.12 is greater than the amount recovered as the sample is returned to its initial point along \( abc \). Thus, some energy is converted to heat in the sample as it is carried around its hysteresis loop. It is shown in P11.27 that in cycling a ferromagnetic material once around its hysteresis loop, an amount of energy given by the integral

\[
\text{energy dissipated per unit volume per cycle} = \oint H \, dB
\]

about the hysteresis loop is dissipated per unit volume in the material. Geometrically, this energy loss is equal to the area of the hysteresis loop.

**PROBLEMS**

**P11.26.** (a) Reading data from the magnetization curve in Fig. 11.11, obtain a graph of \( \mu/\mu_0 \) versus \( \mu_0 H \) for this material. (b) Suppose this material now composes the core of an infinitely long solenoid with 200 turns per unit length. Find the magnetic induction in the core when the current in the coil is 0.5, 2, and 4 A.

**P11.27.** Show that the energy dissipated per unit volume in a ferromagnetic material cycled once around its hysteresis loop is given by \( \oint H \, dB \) and verify that this integral expresses the area of the hysteresis loop.

**P11.28.** Describe the general characteristics of the hysteresis loop of ferromagnetic materials that are particularly good for making (a) permanent magnets and (b) transformer cores. Defend your answers.

**SUPPLEMENTARY PROBLEMS**

**P11.29.** Quantum mechanically, the magnetic moment \( \mathbf{m} \) of an atomic system can be written in terms of the total angular momentum \( \mathbf{J} \) of the system in the form \( \mathbf{m} = -g(q_e/2m_e)\mathbf{J} \), where \( q_e \) is the magnitude of the electronic charge, \( m_e \) is the mass of the electron, and \( g \) is a dimensionless factor—called the Landé \( g \)-factor—that characterizes the specific system.\(^{12}\) (In particular, \( g \) has the value 1 if the angular momentum is purely orbital—compare P11.3—the value 2 if the angular momentum originates entirely in spin, and some intermediate value if the angular momentum has both orbital and spin components.) Further, when an external magnetic induction \( \mathbf{B}_m = B_m \mathbf{k} \) is applied to the system, its energy is changed by an amount

\[
E = -\mathbf{m} \cdot \mathbf{B}_m = -m_z B_m = g q_e \hbar / 2m_e B_m \frac{J_z}{\hbar}
\]

where \( \hbar \) is Planck’s constant divided by \( 2\pi \). Now, for a system with total angular momentum quantum number \( j \) (which must have one of the values 0, 1, \( \frac{1}{2} \), ..., \( \frac{3}{2} \), ...), \( J_z/\hbar \) can assume only the \( 2j + 1 \) values \( s = -j, -j + 1, \ldots, j - 1, j \). Thus, only the energies \( E_s = g M_B s B_m \), where \( M_B = q_e \hbar / 2m_e \) is the Bohr magneton, are allowed. Further, the \( z \)-component of the magnetic moment can assume only the values \( m_z = -g M_B s \). Finally, statistically,

\(^{12}\) German-American physicist Alfred Landé, b. 13 December 1888 in Elberfeld, Germany; d. 30 October 1976 in Columbus, Ohio.
(m_z) = \frac{\sum_s (-gM_B s)e^{-E_s/kt}}{\sum_s e^{-E_s/kt}} 

(2)

where \( k \) is Boltzmann’s constant and \( T \) is the absolute temperature. (a) Show that

\[ \langle m_z \rangle = m_0 \tanh \left( \frac{m_0 B_m}{kT} \right) \] 

(3)

when \( j = \frac{1}{2} \). Here, \( m_0 = \frac{1}{2} gM_B \), \( \tanh x = \sinh x / \cosh x \) and \( \sinh x \) and \( \cosh x \) are defined in footnote 5, Chapter 8. (b) If there are \( n \) atoms per unit volume, then the magnetization \( M \) is given by \( n \langle m_r \rangle \). Sketch a graph of \( M \) versus \( m_0 B_m / kT \), and show that the saturation value of the magnetization of this sample is determined by the solution to the transcendental system

\[ \frac{M}{M_s} = \tanh x ; \quad M = \frac{kT}{\gamma \mu_0 n m_0^2} x - \frac{H}{\gamma n m_0} \] 

(4)

(d) On the same axes, sketch graphs of \( M / M_s \) versus \( x \) as given by both of these equations and then, by allowing the straight line to move, obtain qualitative data and sketch graphs of \( M / M_s \) versus \( H \) for various fixed temperatures. (e) Now let \( H = 0 \) and obtain a sketch of \( M / M_s \) versus \( T \). Show in particular that \( M / M_s = 0 \) unless \( T < T_c = \gamma \mu_0 n m_0^2 / k \). (f) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to find \( M / M_s \) versus \( T / T_c \) for \( H = 0 \) to three significant figures and plot a careful graph. Optional: Find and explore an expression analogous to Eq. (3) for some other value for \( j \).

P11.30. Write Eq. (11.41) in the form

\[ V^{(m)}(r) = \frac{1}{4\pi} \int \frac{dq_m'}{|r - r'|} \]

where \( dq_m' = \rho_m dv' \) or \( \sigma_m dS' \) as appropriate to the distribution, follow the development in Section 4.6, and derive a multipole expansion for the magnetic scalar potential. Make sure to present a convincing mathematical argument for the absence of the monopole term.

P11.31. Consider a cylindrical bar magnet of length \( \ell \) and cross-sectional area \( S \) uniformly magnetized to magnetization \( \mathbf{M} \). Let this magnet be placed in a uniform external magnetic induction \( \mathbf{B} \), with \( \mathbf{B} \) perpendicular to \( \mathbf{M} \). (a) Calculate the torque on the magnet from the expression \( \mathbf{m} \times \mathbf{B} \), where \( \mathbf{m} = \mathbf{M} \ell S \) is the dipole moment of the magnet. (b) Calculate the torque also by replacing the magnet with equivalent poles and assuming a magnetic force \( \mathbf{F}_m \) parallel to \( \mathbf{B} \) on the positive pole and an equal but opposite force on the negative pole. (c) Equate these two evaluations of the torque to show that a magnetic charge \( q_m \) in the field \( \mathbf{B} \) experiences a magnetic force given by \( q_m \mathbf{B} \). (d) Show from Eq. (11.44) that the \( \mathbf{B} \)-field established at \( \mathbf{r} \) by a magnetic pole of strength \( q_m \) at the origin is given by \( \mathbf{B} = \mu_0 q_m \mathbf{R} / 4\pi r^3 \) and infer “Coulomb’s” Law \( \mathbf{F}(q_m, q_m') = \mu_0 q_m q_m' \mathbf{r} / 4\pi r^3 \) for the force between two magnetic poles. (e) Under what conditions would you expect to observe such an inverse square force experimentally?

P11.32. If Eq. (11.36) is applied to a point interior to the magnetized object, \( \mathbf{r}' \) assumes the value \( \mathbf{r} \) somewhere in the domain of integration and the first term no longer contributes zero. However, \( \nabla^2 \left( 1 / |\mathbf{r} - \mathbf{r}'| \right) \) is still zero except at \( \mathbf{r}' = \mathbf{r} \), so the first term can be reduced to an integral over a small volume \( \Delta \mathbf{v} \) centered on the point \( \mathbf{r} \). If \( \Delta \mathbf{v} \) is small enough, \( \mathbf{M}(\mathbf{r}') \) is essentially constant throughout \( \Delta \mathbf{v} \) and can be removed from the integral with the value \( \mathbf{M}(\mathbf{r}) \). Evaluate the remaining integral formally and combine your result with Eq. (11.39), which expresses the second term in Eq. (11.36), to derive Eq. (11.55) for the interior
field. Criticize this method of deriving the interior field. 

*Hints:* Write $\nabla^2 \left(1/|\mathbf{r} - \mathbf{r}'|\right)$ as $\nabla' \cdot \left(\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\right)$. (2) Use the divergence theorem. (3) Use Eq. (2.37).

**P11.33.** Given that $\mathbf{J}_m = \nabla \times \mathbf{M}$, $\nabla \times \mathbf{M}$, $c\nabla \times \mathbf{M}$, and $c\nabla \times \mathbf{M}$ and that $\nabla \times \mathbf{B} = \frac{4\pi \mathbf{J}_t}{c^2}$, $4\pi \mathbf{J}_t$, $4\pi \mathbf{J}_t/c$, and $\mathbf{J}_t/c$ in cgs-esu, cgs-emu, Gaussian units, and Heaviside-Lorentz units, respectively, find the equation relating $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{M}$ in each of these systems of units. 

*Hint:* See Footnote 6.
Chapter 12

Time-Dependent Fields When Matter is Present: Maxwell’s Equations Revised

In Chapters 10 and 11, we introduced the displacement field $D$ and the magnetic intensity $H$, which supplement the electric intensity $E$ and the magnetic induction $B$ when matter is present. When the fields are static (which requires also that the polarization $P$ and magnetization $M$ be static), the four fields satisfy the equations

$$\oint D \cdot dS = \int \rho \, dv ; \quad \nabla \cdot D = \rho$$

(12.1)

$$\oint E \cdot dl = 0 ; \quad \nabla \times E = 0$$

(12.2)

$$\oint H \cdot dl = \int J \cdot dS ; \quad \nabla \times H = J$$

(12.3)

$$\oint B \cdot dS = 0 ; \quad \nabla \cdot B = 0$$

(12.4)

where $\rho$ is the free charge density and $J$ is the free current density. Supplemented with definitions of $D$ and $H$ and with the constitutive relations determining $P$ and $M$ from the fields, these equations constitute the formal framework of electricity and magnetism when matter is present, but this framework will not be fully general until we have extended it to include time-dependent fields. In some ways, this generalization is simple: All of the quantities in Eqs. (12.1)-(12.4)—including $P$ and $M$, whose explicit appearance has been suppressed by introducing the vectors $D$ and $H$—merely acquire a time dependence. In other ways, this generalization is far less simple, for additional terms appear in the equations. We have already seen the form of these additional terms in vacuum. We shall now determine their form when matter is present. In addition, we shall examine some of the general consequences of the revised equations.

12.1 Maxwell’s Equations in Matter

Equations (12.2) and (12.3) acquire additional terms when the full set of equations is generalized to include time-dependent fields. Consider first the generalization of Eq. (12.2). As
with the analogous generalization in Section 6.1 for the fields in vacuum, we begin with the concept of emf, which we still define as the work done on a unit charge moved once around the path described in the first subsection of Section 6.1. Throughout our development of properties of matter, however, the fields \( E \) and \( B \) have retained their connection with forces experienced by charged particles. Thus, even in the presence of matter, the electromagnetic part of the emf is still given by

\[
\varepsilon_{\text{em}}(t) = \oint F_{\text{em}} \cdot dl = \oint \left( E(r,t) + v(r,t) \times B(r,t) \right) \cdot dl
\]

(compare Eq. (6.9)) where \( F_{\text{em}} \) is the force on a particle of charge \( q \) in the electromagnetic field \( E, B \) and \( v(r,t) \) is the velocity of the path \( \Gamma \) at the point \( r \) at the time \( t \). In particular, this definition includes a motional emf that we can still write in the form

\[
\varepsilon_{\text{em}}^{\text{mot}} = \oint (v \times B) \cdot dl = -\frac{d}{dt} \int_{\Sigma} B \cdot dS = -\frac{d\Phi_m}{dt}
\]

(12.6)

where \( \Sigma \) is a surface bounded by \( \Gamma \). [See Eqs. (6.13)–(6.17).] Thus, at least the motional emf remains unchanged by the presence of matter. The more general validity of the analogous expression in Eq. (6.12) suggests that Eq. (12.6) may also apply to emf’s originating in time-dependent fields as well as to motional emf’s. Assuming that generality, we would then have that

\[
\varepsilon_{\text{em}} = -\frac{d\Phi_m}{dt}
\]

(12.7)

or, equivalently, that

\[
\oint E \cdot dl = -\oint \frac{\partial B}{\partial t} \cdot dS \quad ; \quad \nabla \times E = -\frac{\partial B}{\partial t}
\]

(12.8)

[Compare Eqs. (6.18)–(6.21) and Eq. (6.49).] Equations (12.7) and (12.8), of course, contain new elements that are not present in Eq. (12.6), and the support for Eqs. (12.7) and (12.8) lies not in the argument that led to them—that argument is a plausibility argument, not a derivation—but rather in an experimental confirmation of the predictions based on these equations. To date, no experimental contradictions have been found, and we therefore conclude that Faraday’s Law assumes the same form whether matter is present or not.

Consider now the generalization of Eq. (12.3) to time-dependent fields. The necessary modification is determined most directly if we return temporarily to the equation relating the curl of \( B \) to the total current density, whatever its origin, viz., to

\[
\nabla \times B = \mu_0 (J + J_m)
\]

(12.9)

This equation is applicable only to static fields. When the fields (and the macroscopic polarization and magnetization vectors) become time-dependent, additional currents appear. We already know (Section 6.2) that a time-dependent electric field has associated with it a current density \( J_e \) given by \( \epsilon_0 \partial E/\partial t \). In addition, a time-dependent polarization, which involves a time-dependent re-adjustment of charge within the dielectric, gives rise to an apparent macroscopic current, the displacement current. To obtain a quantitative expression for the corresponding polarization current density, we recognize first that the total bound charge \( Q_b \) contained within an arbitrary (but fixed) volume of a polarized dielectric is given by

\[
Q_b = \int \rho_b \, dv = -\int \nabla \cdot P \, dv = -\oint P \cdot dS
\]

(12.10)
where the final form is obtained by using the divergence theorem. Consequently, the net rate at which bound charge flows out of this volume is given by

\[
-\frac{\partial Q_b}{\partial t} = \oint \frac{\partial \mathbf{P}}{\partial t} \cdot d\mathbf{S}
\]  

(12.11)

and we infer that a time-dependent polarization gives rise to a current density \(\mathbf{J}_p\) given by \(\partial \mathbf{P}/\partial t\). In extending Eq. (12.9) to time-dependent fields, both \(\mathbf{J}_e\) and \(\mathbf{J}_p\) must be included and we find that

\[
\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \right)
\]

\[
\nabla \times \mathbf{B} - \mu_0 \frac{\partial \mathbf{M}}{\partial t} = \mathbf{J} + \frac{\partial}{\partial t} (\epsilon_0 \mathbf{E} + \mathbf{P})
\]  

(12.12)

Finally, returning to the fields \(\mathbf{H} = \mu_0^{-1} \mathbf{B} - \mathbf{M}\) and \(\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}\), we have that

\[
\oint \mathbf{H} \cdot d\mathbf{l} = \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} ; \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\]  

(12.13)

As with Faraday’s Law, the real test of this equation lies much more in the experimental confirmation of its predictions than in the above “derivation”.

On the basis of our present knowledge, the only other terms that we might expect in Maxwell’s equations would be a free magnetic charge density in Eq. (12.4) and a free magnetic current density in Eq. (12.2). Both of these terms are ruled out by the experimental absence of free magnetic charge. Thus, Maxwell’s equations for time-dependent fields in matter are

\[
\oint \mathbf{D} \cdot d\mathbf{S} = \int \rho \, dv ; \quad \nabla \cdot \mathbf{D} = \rho
\]  

(12.14)

\[
\oint \mathbf{E} \cdot d\mathbf{l} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} ; \quad \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}
\]  

(12.15)

\[
\oint \mathbf{H} \cdot d\mathbf{l} = - \int \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} ; \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\]  

(12.16)

\[
\oint \mathbf{B} \cdot d\mathbf{S} = 0 ; \quad \nabla \cdot \mathbf{B} = 0
\]  

(12.17)

These equations are to be supplemented with the definitions

\[
\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}
\]  

(12.18)

\[
\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}
\]  

(12.19)

and with the Lorentz force

\[
\mathbf{F} = q [\mathbf{E} + \mathbf{v} \times \mathbf{B}]
\]  

(12.20)

on a particle with charge \(q\) moving with velocity \(\mathbf{v}\) in the fields. Equations (12.14)–(12.20) are the fundamental equations of electricity and magnetism when matter is present. They are valid regardless of the properties of whatever matter may be present, although they are restricted to use in a coordinate system in which all matter is at rest. Because these equations, as the mathematical expression of particular experimental results, have a very
broad range of applicability, they have come to be regarded as a very basic expression of
the properties of the electromagnetic field. From expressions of experimental results, these
equations have been elevated to the status of basic principles and are regarded essentially
as the starting point for all theoretical considerations involving the electromagnetic field.
Indeed, in some developments of the subject, Maxwell’s equations are taken as postulates
and the entire subject is developed from these equations as an assumed starting point.

Of course, it is not possible to obtain solutions to these equations in the absence of
more specific information about $\rho$, $J$, $P$, and $M$. For some problems these quantities will
be part of the given information. In other circumstances, they may be determined by the
fields and by the response of whatever matter is present to those fields. If the latter is
the case, empirical information about the response of that matter is usually needed. The
simplest materials are linear. For these materials, the constitutive relations have the form

$$P = \chi_e E \quad ; \quad M = \chi_m H \quad ; \quad J = \gamma E$$

and Eqs. (12.18) and (12.19) become

$$D = \varepsilon E \quad ; \quad B = \mu H$$

In contrast to Eqs. (12.14)–(12.20), Eqs. (12.21) and (12.22) are valid only for specific
materials. More complicated constitutive relations may be needed to describe the empirical
properties of other materials adequately.

**PROBLEMS**

**P12.1.** Show that Maxwell’s equations admit a principle of superposition; that is, show that if
$E_i$, $H_i$, $D_i$, and $B_i$ satisfy the equations with sources $J_i$ and $\rho_i$, then $\sum E_i$, $\sum H_i$, $\sum D_i$, and $\sum B_i$ satisfy the equations with sources $\sum J_i$ and $\sum \rho_i$.

**P12.2.** Show that the fields $E' = \beta \mu H$, $H' = -\beta \varepsilon E$ satisfy Maxwell’s equations in a linear
medium with $J = 0$ and $\rho = 0$ if $E$ and $H$ satisfy the equations. Here, $\beta$ is a constant.
Apart from a few constants and a minus sign, the fields $E$ and $H$ can therefore be
exchanged.

**P12.3.** Reexpress Eqs. (12.14)–(12.17) by eliminating $D$ and $H$ in favor of $E$ and $B$, thereby
displaying explicitly where $P$ and $M$ appear in the equations.

**P12.4.** Starting with Maxwell’s equations, derive a law for the force between two point charges
in a linear dielectric. *Hint:* Use Gauss’s Law in integral form.

**P12.5.** A conductor of arbitrary shape is surrounded by a linear dielectric. Show that, provided
the fields are static, the free surface charge density $\sigma(P)$ at a point $P$ on the surface of
the conductor is given by $\sigma(P) = D(P) \cdot \hat{n}(P)$, where $\hat{n}(P)$ is a unit vector normal to
the conductor at $P$.

**P12.6.** (a) Show that in linear matter with no free charge or currents all Cartesian components
of the electric field and all Cartesian components of the magnetic induction satisfy the
homogeneous wave equation. *Hint:* Take the curl of Eqs. (12.15) and (12.16) and use
Eq. (C.19). (b) What is the speed of propagation of an electromagnetic wave in linear
matter characterized by a permittivity $\varepsilon$ and a permeability $\mu$? Predict the index of
refraction $n$ for this matter, where $n$ is defined as the ratio of the speed of light in
vacuum to the speed of light in matter.
12.2  The Equation of Continuity

In the next several sections, we shall consider how the present form of Maxwell’s equations alters various expressions with which we are already familiar in the form applicable when matter is not present. Consider first the equation of continuity. Just as in the absence of matter, this equation expressing conservation of charge can be derived from Maxwell’s equations. Since the divergence of a curl is automatically zero, the divergence of Eq. (12.16) yields the result
\[ \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) = 0 \]  

(12.23)

Substitution from Eq. (12.14) then gives
\[ \oint_{\Sigma} \mathbf{J} \cdot d\mathbf{S} = -\int_{V} \frac{\partial \rho}{\partial t} \, dv \; ; \; \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \]  

(12.24)

which has the same form as the equation of continuity in vacuum except that \( \rho \) and \( \mathbf{J} \) now refer to the free charge density and the free current density. Because of their origin, the bound charge and current distributions automatically satisfy a conservation law and consequently Eq. (12.24) does not include that component of the charge distribution (P12.7).

PROBLEM

P12.7. Explain why the equation of continuity involves only free charges and currents. Hint: Write equations of continuity for the polarization and magnetization currents, and show that these equations are automatically satisfied.

12.3  The Energy Theorem

An equation that can be interpreted as an energy theorem can be derived from Maxwell’s equations even when matter is present. Following the pattern of Section 6.4, we subtract the dot product of Eq. (12.16) with \( \mathbf{E} \) from the dot product of Eq. (12.15) with \( \mathbf{H} \) and use Eq. (C.12) to find that
\[ \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{J} \cdot \mathbf{E} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \]  

(12.25)

We then integrate Eq. (12.25) over some volume and use the divergence theorem to obtain
\[ -\oint_{\Sigma} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_{V} \mathbf{J} \cdot \mathbf{E} \, dv + \int_{V} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) \, dv \]  

(12.26)

Each of the terms in this result can be given an interpretation similar to that of the corresponding term in Eq. (6.53). Since, for example, presence of matter does not alter the relationship between the fields \( \mathbf{E} \) and \( \mathbf{B} \) and the forces experienced by charged particles, the arguments associated with Eq. (3.39) apply also to the present case and \( \int \mathbf{J} \cdot \mathbf{E} \, dv \) still represents the rate at which work is done by the fields on the particles in the volume over which the integral extends.
The second integral on the right in Eq. (12.26) is more difficult to interpret. If the medium is linear, so that \( \mathbf{D} = \epsilon \mathbf{E} \) and \( \mathbf{B} = \mu \mathbf{H} \), with \( \epsilon \) and \( \mu \) constants, then

\[
\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \epsilon \mathbf{E}) = \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \right) \tag{12.27}
\]

and similarly

\[
\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \right) \tag{12.28}
\]

For these media, the final term in Eq. (12.26) assumes the form

\[
\frac{\partial}{\partial t} \int_V \left( \frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \right) \, dv \tag{12.29}
\]

from which by analogy with the similar term in Eq. (6.53) we are led to associate the energy density

\[
u_{EM} = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \tag{12.30}
\]

with the electromagnetic field. Further, we are led to interpret the final term in Eq. (12.26) as the rate at which energy stored in the fields is increasing. Equations (12.29) and (12.30) remain valid even for those anisotropic media in which the components of \( \mathbf{D} \) are related to those of \( \mathbf{E} \) by a symmetric permittivity tensor \( \epsilon_{ij} \) through the equation \( D_i = \sum_j \epsilon_{ij} E_j \), where \( \epsilon_{ij} \) is time-independent and field-independent but may depend on the spatial coordinates (P12.8). Even in more general situations for which an energy density cannot be identified, the final term in Eq. (12.26) is still interpreted as the rate at which energy stored in the fields in a particular volume is increasing.

Accepting these interpretations, we conclude that the right-hand side of Eq. (12.26) is the total energy per unit time that must be coming into the volume over which the integrals extend. Since we have (tacitly) assumed that there are no sources or sinks of energy within any volume, the power input represented by the right-hand side of Eq. (12.26) must arise from a flow of energy across the boundary of the volume. Conveniently, the left-hand side of Eq. (12.26) involves a surface integral, and we therefore identify that integral with the rate at which energy is transported into the volume by virtue of energy flow across the bounding surface. Replacing the minus sign with a plus sign, we conclude that

\[
\left( \text{rate at which energy flows out of } V \text{ across the bounding surface } \Sigma \right) = \oint_{\Sigma} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \tag{12.31}
\]

which in turn leads to the identification of the vector \( \mathbf{S} \)

\[
\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{12.32}
\]

—still called the Poynting vector—as a vector representing energy transport by the electromagnetic fields. It is common to interpret the vector \( \mathbf{S} \) as representing a density of energy flow across surfaces, but there has been considerable discussion about the appropriateness and correctness of this interpretation. [See the remarks following Eq. (6.57) and the references given there.] Note, however, that this interpretation of the Poynting vector has not been specific to any particular medium; it applies whatever form the constitutive relations may happen to assume. Note also that energy transport in an electromagnetic field is most directly discussed by using the vectors \( \mathbf{E} \) and \( \mathbf{H} \).
12.4. THE MOMENTUM THEOREM

In Section 6.5, the momentum to be associated with an electromagnetic field in free space was inferred by considering the forces experienced by a charge and current distribution by virtue of the interaction of that distribution with the fields that the distribution itself establishes. Although in the presence of matter, the force $F$ experienced by a charge and current distribution in an electromagnetic field is still given from the force density $\rho E + J \times B$ by

$$F = \int_V \left( \rho E + J \times B \right) \, dv$$

and this force is still the rate of change of the linear momentum $P$ of the matter within the volume over which the integral extends, Maxwell’s equations with no restrictions on the constitutive properties of whatever matter is present lead now to the result

$$\frac{d}{dt} \left( P + \int_V (D \times B) \, dv \right) = \int_V \left( E(\nabla \cdot D) - D \times (\nabla \times E) - B \times (\nabla \times H) \right) \, dv$$

instead of to Eq. (6.61). Because the integral under the time derivative appears on the same footing as the momentum $P$, Eq. (12.34) suggests that the fields in the volume $V$ should be assigned a momentum density $G$ given by

$$G = D \times B$$

Since no restricting assumptions have been made about the properties of whatever matter is present, this identification is independent of how $D$ and $B$ are related to $E$ and $H$. Thus, the momentum density $G$ (determined from $D$ and $B$) and the Poynting vector $S$
(determined from $E$ and $H$) are in general not simply related. If, however, the medium involved is linear, then

$$G = \epsilon E \times \mu H = \epsilon \mu E \times H = \epsilon \mu S$$

which reduces to Eq. (6.64) in vacuum where $\epsilon = \epsilon_0$ and $\mu = \mu_0$.

An alternative expression for the quantity appearing on the right-hand side in Eq. (12.34) is explored in P12.28.

### 12.5 On Which Fields are Basic

Four fields—$E$, $D$, $B$, $H$—have been introduced but only two of these four are independent. The other two are determined by Eqs. (12.18) and (12.19). We have so far regarded $E$ and $B$ as basic and $D$ and $H$ as auxiliary. Now that we have seen how these fields enter into various aspects of the general formalism, we point out that the choice of which fields to regard as basic is in part arbitrary, this choice being primarily determined by the circumstances of the problem at hand. If, for example, we are interested in

- forces on charged particles, we take $E$ and $B$ as basic.
- response of matter, we take $E$ and $H$ as basic.
- energy transport, we take $E$ and $H$ as basic.
- momentum in the fields, we take $D$ and $B$ as basic.
- field problems when bound charge and current distributions are initially unknown, we take $D$ and $H$ as basic.
- symmetric discussion of Maxwell’s equations, we take either $D$ and $B$ or $E$ and $H$ as basic.

Perhaps other situations could be added to this list. These few observations should, however, be sufficient to point out that the choice of basic fields is as much a matter of taste and convenience as of the intrinsic theory.

### 12.6 The Potentials

Even when matter (of whatever type) is present, it is possible to express the fields $E$ and $B$ in terms of scalar and vector potentials $V$ and $A$. The arguments are identical with those presented to support Eqs. (6.65)–6.67). In essence,

$$\nabla \cdot B = 0 \implies B = \nabla \times A$$

and then

$$\nabla \times E = -\frac{\partial B}{\partial t} \implies \nabla \times \left( E + \frac{\partial A}{\partial t} \right) = 0$$

$$\implies E + \frac{\partial A}{\partial t} = -\nabla V$$

$$\implies E = -\nabla V - \frac{\partial A}{\partial t}$$

(12.38)
12.6. THE POTENTIALS

[See items (3) and (4) in Section 2.5.] Further, the fields continue to be invariant to the gauge transformation

\[ A_2 = A_1 + \nabla \Lambda \quad ; \quad V_2 = V_1 - \frac{\partial \Lambda}{\partial t} \]  

(12.39)

[Compare Eqs. (6.68) and (6.70).] and the divergence of \( \mathbf{A} \) remains arbitrary.

As in Section 6.6, two of Maxwell’s equations—Eqs. (12.15) and (12.17)—are automatically satisfied when the fields are expressed in terms of the potentials \( \mathbf{A} \) and \( V \); the remaining two generate differential equations whose solutions determine the potentials. When matter is present, however, this second pair of equations—Eqs. (12.14) and (12.16)—involves the fields \( \mathbf{D} \) and \( \mathbf{H} \), which are not given directly by the potentials. Thus, the form of the equations for \( \mathbf{A} \) and \( V \) depends on the form of the relationships between \( \mathbf{D} \) and \( \mathbf{E} \) and between \( \mathbf{H} \) and \( \mathbf{B} \). If \( \mathbf{D} \) and \( \mathbf{H} \) are linearly related to \( \mathbf{E} \) and \( \mathbf{H} \), as in Eq. (12.22), then combining the potentials with Eqs. (12.14) and (12.16) ultimately yields that

\[ \left( \nabla^2 - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) A = -\mu \mathbf{J} \]  

(12.40)

and that

\[ \left( \nabla^2 - \mu \epsilon \frac{\partial^2}{\partial t^2} \right) V = -\frac{\rho}{\epsilon} \]  

(12.41)

provided the potentials satisfy the Lorentz condition

\[ \nabla \cdot \mathbf{A} + \mu \epsilon \frac{\partial V}{\partial t} = 0 \]  

(12.42)

[Compare Eqs. (6.74), (6.75, and (6.73).] For more general relationships between \( \mathbf{D} \) and \( \mathbf{E} \) and between \( \mathbf{H} \) and \( \mathbf{B} \), the resulting equations for \( \mathbf{A} \) and \( V \) may be more complicated, but they can always be obtained by substituting Eqs. (12.37) and (12.38) into Eqs. (12.18) and (12.19) and then substituting those results into Eqs. (12.14) and (12.16).

Although the vector and scalar potentials \( \mathbf{A} \) and \( V \) are the only potentials that can always be identified without imposing special conditions on the fields, the sources, the polarization, or the magnetization, other potentials with more limited applicability can be identified. We have, for example, already introduced the magnetic scalar potential, which is useful for static fields when \( \mathbf{J} = 0 \) (for then \( \nabla \times \mathbf{H} = 0 \), which implies that \( \mathbf{H} = -\nabla V^{(m)} \)). One other potential that is occasionally useful is explored in \textbf{P12.12}.

PROBLEMS

\textbf{P12.11.} Verify that Eqs. (12.40) and (12.41) for the potentials in homogeneous, linear matter follow from Eqs. (12.14) and (12.16) provided that the condition in Eq. (12.42) applies.

\textbf{P12.12.} Consider a situation in which there are no free charges or currents but the polarization throughout all space is a specified function of position and time. Let the permeability be that of free space. Show that all of Maxwell’s equations are satisfied if we introduce a single potential \( \Pi \), called the \textit{Hertz potential}\footnote{German physicist Heinrich Rudolf Hertz, b. 22 February 1857 in Hamburg, Germany; d. 1 January 1894 in Bonn, Germany.} or the \textit{polarization potential}, in terms of which

\[ \mathbf{D} = \epsilon_0 \nabla \times (\nabla \times \Pi) \quad ; \quad \mathbf{B} = \mu_0 \epsilon_0 \nabla \times \frac{\partial \Pi}{\partial t} \]
provided $\Pi$ satisfies
\[
\left( \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \Pi = -\frac{P}{\epsilon_0}
\]

Show also that the fields can be derived from the vector and scalar potentials
\[
A = \mu_0 \epsilon_0 \frac{\partial \Pi}{\partial t}, \quad V = -\nabla \Pi
\]

### 12.7 Boundary Conditions at Discontinuities in the Medium

Because Maxwell’s equations are differential equations, the complete statement of a problem in electromagnetism requires the specification of conditions that acceptable solutions must satisfy at the boundaries of the domain of the problem. In one-dimensional problems, these conditions are adequately stated by giving the value of the solution and/or sometimes the values of its derivative at both ends of the region of interest. In $n$ dimensions, values and maybe derivatives must be specified over the $(n-1)$-dimensional surface that bounds the $n$-dimensional volume in which the solution is sought. If, for example, the region of interest is the interior of a sphere of radius $a$, the boundary condition might be that the electric field vanish on the boundary, expressed analytically in spherical coordinates by the equation
\[
E(a, \theta, \phi) = 0 \quad (12.43)
\]

If the volume of interest happens to be infinite, the bounding surface is a surface at infinity. In such cases the boundary condition may be stated not by giving a value but by requiring that the fields approach a particular asymptotic limit as the boundary at infinity is approached; for example, we might require that
\[
E(r, \theta, \phi) \longrightarrow \frac{1}{r^2} \hat{r} \quad (12.44)
\]

as $r \rightarrow \infty$. The particular character of boundary conditions of this type is very specific to the problem being solved and we can say little more without specifying the problem more explicitly.

In principle, the differential form of Maxwell’s equations and the appropriate conditions at the boundaries of some region are sufficient to determine the fields in this region. In fact, such a formulation of an electromagnetic problem is suitable only if the properties of space in the region of interest vary smoothly from point to point. In many problems of interest, however, a boundary in the form of an interface between two media having different properties occurs within the volume of the problem. If we can regard this boundary to be spread out over some region and if we can regard the properties of the medium to change smoothly in some known way from one side of this region to the other, then Maxwell’s equations can be integrated across the boundary with no difficulty. Despite its physical impossibility, however, it is much more convenient to regard an interface between two different media (one of which may be vacuum) to be abrupt. For such an idealistic
boundary conditions at discontinuities in the medium

Figure 12.1: Interface between two media. The pillbox used to obtain boundary conditions on the normal components of \( \mathbf{D} \) and \( \mathbf{B} \) is shown.

interface, the differential form of Maxwell’s equations cannot be easily integrated across the interface and we must find another way to express the connection between the fields on one side of the interface and the fields on the other side of the interface. As we shall show in this section, the integral form of Maxwell’s equations contains the necessary information.

Consider, then, an interface between two different media (Fig. 12.1). In a sufficiently small region about a selected point \( P \) on this interface, the boundary between the two media can be regarded as plane. Introduce a unit vector \( \hat{n} \) normal to the interface at the point \( P \) and, for definiteness, direct the normal vector from medium 1 to medium 2. Finally, imagine a small pillbox with its two plane faces perpendicular to the unit vector \( \hat{n} \) and its remaining side perpendicular to the interface. Let the plane faces of this pillbox have area \( \Delta S \) and let the pillbox enclose a volume \( \Delta v \). Now, apply Gauss’s Law, Eq. (12.14), to this pillbox. The left-hand side of this equation has the more explicit evaluation

\[
\oint \mathbf{D} \cdot dS \approx \mathbf{D}_2(P) \cdot \hat{n} \Delta S + \mathbf{D}_1(P) \cdot (-\hat{n}) \Delta S + \left( \text{contribution from the cylinder} \right) \tag{12.45}
\]

where \( \mathbf{D}_1(P) \) is the displacement vector at \( P \) in medium 1 and \( \mathbf{D}_2(P) \) is the displacement vector at \( P \) in medium 2; the right-hand side has the evaluation

\[
\int \rho dv = \text{total free charge in pillbox} \approx \rho_1(P) \Delta v_1 + \rho_2(P) \Delta v_2 + \sigma(P) \Delta s \tag{12.46}
\]

where \( \rho_1(P) \) and \( \rho_2(P) \) are the free volume charge densities at \( P \) in medium 1 and medium 2, \( \sigma(P) \) is the free surface charge density at \( P \), and \( \Delta v_1 \) and \( \Delta v_2 \) are the portions of \( \Delta v \) lying in medium 1 and medium 2, respectively. All volumes and surfaces have been assumed small enough to justify treating the fields and charge densities as constants throughout the region of integration. Substituting Eqs. (12.45) and (12.46) into Eq. (12.14), we find that

\[
\left( \mathbf{D}_2(P) \cdot \hat{n} - \mathbf{D}_1(P) \cdot \hat{n} \right) \Delta S \approx \rho_1(P) \Delta v_1 + \rho_2(P) \Delta v_2 + \sigma(P) \Delta S \tag{12.47}
\]

Now let the two faces of the pillbox move arbitrarily close to the interface, each remaining on its own side of the interface. In that limit, \( \Delta v_1 \) and \( \Delta v_2 \) approach zero and the contribution...
to the flux from the cylindrical wall of the pillbox also disappears. In this limit, Eq. (12.47) becomes

$$D_2(P) \cdot \hat{n} - D_1(P) \cdot \hat{n} = \sigma(P)$$  \hspace{1cm} (12.48)

Thus, the normal component of the displacement vector is discontinuous across an interface by an amount equal to the free charge density on the interface. If this charge density is zero, then the normal component of the displacement vector is continuous across the boundary.

An essentially identical argument that begins with the application of the integral form of the magnetic flux law, Eq. (12.17), to the pillbox of Fig. 12.1 leads finally to the conclusion that

$$B_2(P) \cdot \hat{n} - B_1(P) \cdot \hat{n} = 0$$  \hspace{1cm} (12.49)

Thus, the normal component of the magnetic induction field is continuous across the interface. Because no free magnetic charge exists, the normal component of the magnetic induction field is never discontinuous.

To obtain two additional equations relating the fields on opposite sides of an interface, we apply the remaining two Maxwell equations (in integral form) to a rectangular path whose plane contains the unit vector $\hat{n}$ normal to the interface at point $P$ (Fig. 12.2). Let the path enclose a (plane) area that is small enough so that two of its sides can be regarded as perpendicular to the interface and two as parallel to the interface, and let $\hat{t}$ be a unit vector tangent to the surface and in the plane of the rectangle (which may be any plane containing the vector $\hat{n}$). Now, apply the Faraday Law, Eq. (12.15), to this path. The left-hand side of this equation has the evaluation

$$\oint E \cdot d\ell \approx E_2(P) \cdot \hat{t} \Delta\ell + E_1(P) \cdot (-\hat{t}) \Delta\ell + \left( \text{contribution from sides 2 and 4} \right)$$  \hspace{1cm} (12.50)

and the right-hand side has the evaluation

$$-\int \frac{\partial B}{\partial t} \cdot dS \approx (\text{some finite factor}) \Delta S$$  \hspace{1cm} (12.51)
12.7. BOUNDARY CONDITIONS AT DISCONTINUITIES IN THE MEDIUM

Substituting Eqs. (12.50) and (12.51) into Faraday’s Law, we find that

\[
\left( E_2(P) \cdot \hat{t} - E_1(P) \cdot \hat{t} \right) \Delta \ell + \left( \begin{array}{c} \text{contribution} \\ \text{from sides 2 and 4} \end{array} \right) = \left( \begin{array}{c} \text{some finite} \\ \text{factor} \end{array} \right) \Delta S \quad (12.52)
\]

Here, the line integral has been evaluated by traversing the path in a direction that makes \( \hat{n} \times \hat{t} \) the direction to assign to the surface bounded by the path. Now let the two sides parallel to the interface move arbitrarily close to the interface, each remaining on its own side of the interface. In this limit, the surface area \( \Delta S \) goes to zero and the contribution to the line integral from sides 2 and 4 of the path also disappears; Eq. (12.52) yields

\[
E_2(P) \cdot \hat{t} - E_1(P) \cdot \hat{t} = 0 \quad (12.53)
\]

where \( \hat{t} \) is perpendicular to \( \hat{n} \) but is otherwise arbitrary. Thus, all tangential components of the electric field (and particularly the two tangential components in mutually orthogonal directions) are continuous across an interface between two different media. The alternative expression

\[
\hat{n} \times \left( E_2(P) - E_1(P) \right) = 0 \quad (12.54)
\]

does this condition combines Eq. (12.53) for all \( \hat{t} \) into a single vector equation and is derived in P12.14.

Finally, application of the circuital law, Eq. (12.16), to the path shown in Fig. 12.2 gives

\[
H_2(P) \cdot \hat{t} \Delta \ell - H_1(P) \cdot \hat{t} \Delta \ell + \left( \begin{array}{c} \text{contribution} \\ \text{from sides 2 and 4} \end{array} \right) = \left( \begin{array}{c} \text{some finite} \\ \text{factor} \end{array} \right) \Delta S + j \cdot (\hat{n} \times \hat{t}) \Delta \ell \quad (12.55)
\]

where the final term represents a contribution to the current across the surface bounded by the path from a possible free surface current on the interface between the two media. Again moving the two sides that are parallel to the interface arbitrarily close to the interface, we can ignore the contributions from sides 2 and 4 and also the contribution multiplied by \( \Delta S \) in Eq. (12.55); Eq. (12.55) then yields that

\[
H_2(P) \cdot \hat{t} - H_1(P) \cdot \hat{t} = j \cdot (\hat{n} \times \hat{t}) \quad (12.56)
\]

which applies for any unit vector \( \hat{t} \) tangent to the interface at the point \( P \). Thus, the two tangential components of the magnetic field intensity may be discontinuous across an interface, but will be discontinuous only if there is a free surface current in the interface. The alternative expression

\[
\hat{n} \times \left( H_2(P) - H_1(P) \right) = j \quad (12.57)
\]

of Eq. (12.56) is derived in P12.14. We remind the reader that, throughout this section \( \hat{n} \)

is a normal vector directed from medium \( \mathbb{1} \) to medium \( \mathbb{2} \).

Although not related directly to the fields, one additional condition, obtained by applying the equation of continuity, Eq. (12.24), to the pillbox in Fig. 12.1, should be included in this section; we find that

\[
J_2(P) \cdot \hat{n} \Delta S + J_1(P) \cdot (-\hat{n}) \Delta S + \left( \begin{array}{c} \text{contribution} \\ \text{from the cylinder} \end{array} \right)
\]
CHAPTER 12. TIME-DEPENDENT FIELDS WHEN MATTER IS PRESENT

\[ \approx -\frac{\partial}{\partial t} \left( \rho_1(P) \Delta v_1 + \rho_2(P) \Delta v_2 + \sigma(P) \Delta S \right) \]  

(12.58)

which in turn results in the condition

\[ \mathbf{J}_2(P) \cdot \mathbf{n} - \mathbf{J}_1(P) \cdot \mathbf{n} = -\frac{\partial \sigma(P)}{\partial t} \]  

(12.59)

when the plane surfaces of the pillbox are allowed to approach arbitrarily close to the interface.

A summary of the boundary conditions that must be imposed on the fields at an abrupt interface between two different media follows. In this summary, we suppress explicit indication of the point \( P \) on the boundary at which the fields are evaluated. Further, the vector \( \mathbf{n} \) is a unit vector directed from medium 1 to medium 2. The relations expressed in this summary have been derived without reference to any special or restrictive conditions and hence are perfectly general (although if the fields vary at very high frequencies, complications will probably arise). In general terms, the necessary boundary conditions are

\[ (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma \]  

(12.60)

\[ \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \]

(12.61)

\[ (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0 \]

(12.62)

\[ \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{j} \]

(12.63)

\[ (\mathbf{J}_2 - \mathbf{J}_1) \cdot \mathbf{n} = -\frac{\partial \sigma}{\partial t} \]

(12.64)

In terms of a specific Cartesian coordinate system whose \( z \) axis coincides with the vector \( \mathbf{n} \) and whose \( x \) and \( y \) axes are tangent to the interface at \( P \), the boundary conditions in Eqs. (12.60)–(12.63) have the more explicit expressions

\[ E_{2x} - E_{1x} = 0 ; \quad E_{2y} - E_{1y} = 0 ; \quad D_{2z} - D_{1z} = \sigma \]

\[ H_{2x} - H_{1x} = j_y ; \quad H_{2y} - H_{1y} = -j_x ; \quad B_{2z} - B_{1z} = 0 \]  

(12.65)

\[ J_{2z} - J_{1z} = -\frac{\partial \sigma}{\partial t} \]

Although the conditions on the normal and tangential components do not involve the same fields, either in the electric or the magnetic domain, \( \mathbf{E} \) and \( \mathbf{D} \) and also \( \mathbf{B} \) and \( \mathbf{H} \) are related on each side of the interface by the properties of the corresponding medium. Taking those relationships into account, we therefore have conditions on all three components of all four field vectors and Eqs. (12.60)–(12.63) fully determine the fields on one side of an interface from those on the other side of the interface.
We shall mention two remaining boundary conditions. When the fields are static, the difference in the scalar potential \( V \) between two points gives the amount of work required to move a unit charge from one point to the other. Since physically the electric field cannot become infinite at an interface, the amount of work required to move a unit charge an infinitesimal distance from one side of the interface to the other must be infinitesimal, and, consequently, with \( P \) a point on the interface,

\[
V_2(P) - V_1(P) = 0
\]  

(12.66)
i.e., the electrostatic potential must be continuous across the interface between two different media. In a way, this condition is not independent of Eq. (12.61), for Eq. (12.61) implies the validity of Eq. (12.66) at all points \( P \) if the potential is continuous at but a single point. (Why?) It almost goes without saying that, when the fields are static and the media are linear, the condition on the normal component of \( \mathbf{D} \) is equivalent to the condition

\[
\left( \epsilon_2 \nabla V_2(P) - \epsilon_1 \nabla V_1(P) \right) \cdot \hat{n} = -\sigma
\]  

(12.67)
on the normal derivative of the potential. (Why?)

To illustrate the use of these boundary conditions, suppose two linear dielectric media with permittivities \( \epsilon_1 \) and \( \epsilon_2 \) meet in a plane interface (Fig. 12.3). Let the electric fields \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) lie in the plane of the paper and make angles \( \theta_1 \) and \( \theta_2 \) with the normal to the interface. We seek a relationship between \( \theta_1 \) and \( \theta_2 \) when there is no free charge on the interface. Continuity of the normal component of \( \mathbf{D} = \epsilon \mathbf{E} \) then requires that

\[
\epsilon_1 E_1 \cos \theta_1 = \epsilon_2 E_2 \cos \theta_2
\]  

(12.68)
and continuity of the tangential component of \( \mathbf{E} \) requires that

\[
E_1 \sin \theta_1 = E_2 \sin \theta_2
\]  

(12.69)
Here \( E_i = |\mathbf{E}_i| \). Thus, dividing Eq. (12.68) by Eq. (12.69), we find that \( \theta_1 \) and \( \theta_2 \) are related by

\[
\epsilon_1 \cot \theta_1 = \epsilon_2 \cot \theta_2
\]  

(12.70)
regardless of the strength of the fields. In effect, an electric field is “refracted” at a dielectric interface.
PROBLEMS

P12.13. The way Eq. (12.48) is written, exchange of what we call medium 1 with what we call medium 2, which interchanges $D_1$ and $D_2$, seems to change the sign on the left side but will certainly not affect the right side. Explain where this argument is incomplete.

P12.14. Derive (a) Eq. (12.54) from Eq. (12.53) and (b) Eq. (12.57) from Eq. (12.56). Hint: See P0.5 and P0.7.

P12.15. Derive Eq. (12.65) from Eqs. (12.60)–(12.64).

P12.16. Solve Eq. (12.70) for $\theta_2$ as a function of $\theta_1$ and, using an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON, plot graphs of $\theta_2$ versus $\theta_1$ over the range $0 \leq \theta_1 \leq 90^\circ$ for various values of $\epsilon_1/\epsilon_2$. Include $\epsilon_1/\epsilon_2 < 1$, $= 1$, and $> 1$ in your study.

P12.17. Verify that all applicable boundary conditions are satisfied by your solution to P10.18.

P12.18. Suppose that the media in Fig. 12.3 are linear magnetic media with permeabilities $\mu_1$ and $\mu_2$. Replace $E_1$ and $E_2$ by $B_1$ and $B_2$ and let there be no free surface currents on the interface. Find a relationship analogous to Eq. (12.70). Optional: Explore this relationship after the pattern suggested in P12.16.

P12.19. Suppose that the toroidal core in Fig. 11.9 has a narrow gap of length $d$ as measured along a center line of the toroid. Determine $H$ and $B$ for points along this center line, both inside and outside the toroid. State clearly any assumptions made. Hint: Ignore fringing in the gap.

12.8 Static Potentials

The electrostatic field in a linear dielectric satisfies $\nabla \times E = 0$ and $\nabla \cdot (\epsilon E) = \rho$, the first of which implies the existence of a scalar potential in terms of which $E = -\nabla V$ and the second of which then requires that this potential satisfy Poisson’s equation,$^2$

$$\nabla^2 V = -\frac{\rho}{\epsilon} \quad (12.71)$$

Further, the electrostatic field in a region containing a uniformly polarized dielectric ($P = \text{constant}$) satisfies $\nabla \times E = 0$ and $\nabla \cdot D = \nabla \cdot (\epsilon_0 E + P) = \nabla \cdot (\epsilon_0 E) = \rho$ (since $\nabla \cdot P = 0$), the first of which implies that $E = -\nabla V$ and the second of which then requires that $V$ satisfy Eq. (12.71) with $\epsilon$ replaced by $\epsilon_0$. Yet again, the magnetostatic field intensity in a region containing linear matter but no free currents satisfies $\nabla \times H = 0$ and $\nabla \cdot (\mu H) = 0$, the first of which implies the existence of a (magnetic) scalar potential $V^{(m)}$ in terms of which $H = -\nabla V^{(m)}$ and the second of which then requires that this potential satisfy Laplace’s equation

$$\nabla^2 V^{(m)} = 0 \quad (12.72)$$

Finally, the magnetostatic field in a region containing uniformly magnetized matter ($M = \text{constant}$) but no free currents satisfies $\nabla \times H = 0$ and $\nabla \cdot B = \nabla \cdot [\mu_0 (H + M)] = \nabla \cdot \mu_0 H = 0$ (since $\nabla \cdot M = 0$), the first of which again implies that $H = -V^{(m)}$ and the second of

$^2$Siméon Denis Poisson, see footnote 14 in Chapter 4, page 110.
12.8. STATIC POTENTIALS

which then requires \( V^{(m)} \) to satisfy Eq. (12.72). (See also P9.4.) Thus, many electrostatic and magnetostatic problems in matter reduce to solving Poisson’s or Laplace’s equation in various regions of space and using the boundary conditions developed in Section 12.7 to match the solutions together at any interfaces. Many of the methods discussed in Chapter 8 can therefore be used even when matter is present.

We shall illustrate this type of problem with a single example. Consider an uncharged sphere made of a linear dielectric and placed in a previously uniform electric field. We seek the resulting field both inside and outside the dielectric. If we take the direction of the uniform field \( \mathbf{E}_0 \) to define the polar axis of a spherical coordinate system so that

\[
\mathbf{E}_0 = E_0 \hat{k}
\]

(12.73)

and further if we place the origin at the center of the sphere, the external field and the sphere are invariant to rotation about the \( z \) axis, nothing can depend on the coordinate \( \phi \), and the general solution of Laplace’s equation has the form of Eq. (8.41). We shall find that all boundary conditions can be satisfied by taking only the terms for which \( n = 0 \) and 1, and we shall not prove explicitly that the other coefficients must be zero. Thus, our solution for the electrostatic potential has two pieces, viz.,

\[
V_{\text{in}}(r, \theta) = \left( a_0 + \frac{b_0}{r} \right) + \left( a_1 r + \frac{b_1}{r^2} \right) \cos \theta
\]

\[
V_{\text{out}}(r, \theta) = \left( A_0 + \frac{B_0}{r} \right) + \left( A_1 r + \frac{B_1}{r^2} \right) \cos \theta
\]

(12.74)

where \( V_{\text{in}} \) refers to the region \( r < a \), with \( a \) the radius of the sphere, and \( V_{\text{out}} \) refers to the region \( r > a \). Several boundary conditions must be satisfied by Eq. (12.74), and each imposes constraints on the undetermined constants. First, \( V_{\text{out}} \) must approach the potential \(-\mathbf{E} \cdot \mathbf{r} = -E_0 r \cos \theta\) of a uniform field as \( r \to \infty \); thus \( A_0 = 0 \) and \( A_1 = -E_0 \). Next, the sphere was required to be uncharged, and hence \( B_0 = 0 \). Thus, \( V_{\text{out}} \) reduces to

\[
V_{\text{out}}(r, \theta) = \left( -E_0 r + \frac{B_1}{r^2} \right) \cos \theta
\]

(12.75)

A third boundary condition requires that \( V_{\text{in}}(r, \theta) \) be finite everywhere, particularly at \( r = 0 \); therefore, \( b_0 = b_1 = 0 \), and \( V_{\text{in}}(r, \theta) \) reduces to

\[
V_{\text{in}}(r, \theta) = a_0 + a_1 r \cos \theta
\]

(12.76)

Finally, we must require that \( V \) and the normal component of \( \mathbf{D} \) be continuous at \( r = a \) for all \( \theta \). Mathematically, the first of these conditions requires that

\[
V_{\text{in}}(a, \theta) = V_{\text{out}}(a, \theta)
\]

\[
\implies a_0 + a_1 a \cos \theta = \left( -E_0 a + \frac{B_1}{a^2} \right) \cos \theta
\]

\[
\implies a_0 = 0, \quad a a_1 = -E_0 a + \frac{B_1}{a^2}
\]

(12.77)

where the final conclusion follows because a condition having this form cannot be satisfied for all \( \theta \) unless the coefficients of \( \cos n\theta \), \( n = 0, 1, 2, \ldots \), are separately equal on both sides
CHAPTER 12. TIME-DEPENDENT FIELDS WHEN MATTER IS PRESENT

of the equation. The condition on the normal component of \( D \) provides a second equation involving \( a_1 \) and \( B_1 \), viz.,

\[
\epsilon \frac{\partial V_{\text{in}}(a, \theta)}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}(a, \theta)}{\partial r}
\]

\[\Rightarrow \epsilon a_1 \cos \theta = \epsilon_0 \left( -E_0 - \frac{2B_1}{a^3} \right) \cos \theta\]

\[\Rightarrow \epsilon a_1 = -\epsilon_0 \left( E_0 + \frac{b_1}{a^3} \right) \tag{12.78}\]

Together, Eqs. (12.77) and (12.78) give values for \( a_1 \) and \( B_1 \), and we find on substituting these values and the value \( a_0 = 0 \) into Eqs. (12.75) and (12.76) that

\[
V_{\text{in}}(r, \theta) = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 r \cos \theta
\]

\[V_{\text{out}}(r, \theta) = -\left( 1 - \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \frac{a^3}{r^3} \right) E_0 r \cos \theta \tag{12.79}\]

In particular, the electric field inside the sphere is uniform and parallel to the original exterior field, as is shown by

\[
E_{\text{in}} = -\nabla V_{\text{in}} = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 \tag{12.80}
\]

(Write \( r \cos \theta = z \) and calculate the gradient directly in Cartesian coordinates.) Other aspects of this field are explored in P12.20.

PROBLEMS

P12.20. (a) Solve Eqs. (12.77) and (12.78) for \( a_1 \) and \( B_1 \) and then obtain Eq. (12.79) from Eqs. (12.75) and (12.76). (b) Find the electric field corresponding to Eq. (12.79) at all points in space and, in particular show that the boundary condition on the tangential components of \( E \) at \( r = a \) is automatically satisfied. (c) Find the equivalent charge distribution with which the dielectric sphere of Section 12.8 might be replaced. (d) Show that the exterior field computed in part (b) consists of the field of a dipole superimposed on the original uniform field, and find the dipole moment describing the dipole contribution. (e) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON or a program such as that described in P4.58 to obtain graphs of field lines and equipotentials for this field.

P12.21. An infinitely long, uncharged cylinder of radius \( a \) made of a linear dielectric having dielectric constant \( K_e \) is placed in a previously uniform electric field \( E = E_0 \hat{i} \) with its axis coincident with the \( z \) axis. Determine the electrostatic potential and the electrostatic field at points both inside and outside the cylinder.

P12.22. An uncharged hollow sphere with inner radius \( a \) and outer radius \( b \) is made of a linear dielectric having dielectric constant \( K_e \). Let this sphere be placed in a uniform external electric field \( E_0 \). (a) Show that the field in the hollow interior is constant and is given by

\[
E_1 = \frac{9K_e}{(2K_e + 1)(K_e + 2) - 2(a/b)^3(K_e - 1)^2} E_0
\]

Note: You do not need to calculate all of the unknown constants in order to determine the required field. You might, in fact, want to use an available symbolic manipulating
12.8. STATIC POTENTIALS

program like MAXIMA, MAPLE, or Mathematica to solve the set of equations given by the pertinent boundary conditions. (b) Show that $E_1$ reduces to $E_0$ when $K_e = 1$. (c) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON to obtain graphs of the coefficient in the above equation as a function of $K_e$ for various values of $a/b$. (Note that physically $a/b < 1$). Suggest a possible use for the arrangement of this problem.

P12.23. A uniform field $\mathbf{E}_m$ is established in a linear dielectric having dielectric constant $K_e$. A spherical cavity is now cut in the dielectric. Show that the field in the cavity is given by $3K_e\mathbf{E}_m/(2K_e + 1)$. Hint: Exchange of $\epsilon$ and $\epsilon_0$ converts the dielectric sphere treated in the text into this problem.

P12.24. A sphere of radius $a$ made of a linear magnetic material having relative permeability $K_m$ is placed with its center at the origin in a previously uniform magnetic induction $\mathbf{B} = B_0\hat{k}$. Determine the magnetic scalar potential and the magnetic induction field at points inside and outside the sphere. Show that the added field is the field of a dipole and determine the magnetic dipole moment of that dipole.

P12.25. Calculate the fields $\mathbf{B}$ and $\mathbf{H}$ both inside and outside a sphere of radius $a$ having uniform magnetization $\mathbf{M} = M\hat{k}$.

P12.26. By solving Laplace’s equation in spherical coordinates, find a series expansion for the magnetic scalar potential at points off the axis of a circular current loop of radius $a$ carrying current $I$. Consider particularly the region $r > a$, where $r$ is the distance from the center of the loop to the observation point. Hints: (1) The $\mathbf{B}$-field on the axis of a loop is given in Eq. (5.15). What should $V^{(m)}$ reduce to on the axis? (2) See P8.18. (3) Let the loop lie in the $xy$-plane with its center at the origin. Optional: Consider the region $r < a$.

SUPPLEMENTARY PROBLEMS

P12.27. Describe an experimental setup in which the $B$-$H$ curve for a ferromagnetic material could be displayed on the screen of a cathode ray oscilloscope.

P12.28. Show that the $i$-th component of the right-hand side of Eq. (12.34) can be written in the form $\sum_j \frac{1}{2} \oint T_{ij} dS_j$, where

$$T_{ij} = E_i D_j + H_i B_j - \frac{1}{2} \delta_{ij}(D \cdot E + H \cdot B)$$

provided the material present is characterized by constant, scalar permittivities and permeabilities. The quantity $T_{ij}$ is called the Maxwell stress tensor. (Compare P6.37.)

P12.29. The depolarization factor $L$ of a dielectric object in a uniform external field $\mathbf{E}_0$ is defined by $\epsilon_0\mathbf{E}_{in} = \epsilon_0\mathbf{E} - LP_{in}$, where $\mathbf{E}_{in}$ and $\mathbf{P}_{in}$ are the electric field and polarization in the object. Find the depolarization factor of a sphere made of a linear dielectric. Note: A similar concept, called the demagnetization factor, is often used in discussing objects made of magnetic material.

P12.30. The method of images can sometimes be extended to solve problems involving dielectrics. Suppose, for example, that two dielectrics of permittivity $\epsilon_1$ and $\epsilon_2$ meet in a plane interface, say the $xz$-plane. Let medium 1 occupy the region $y > 0$ and medium 2 the region $y < 0$. Finally, let a point charge $q$ be placed at $(0, d, 0)$ with $d > 0$. (Suggestion: Sketch a figure.) We seek separate expressions $V_1$ and $V_2$ for the potential in the two media. In seeking $V_1$, we can place image charges in the region $y < 0$ and, further, we can regard that region to be filled with a dielectric of permittivity $\epsilon_1$. Similarly, in seeking $V_2$, we can place image charges in the region $y > 0$ and regard that region to be filled with a dielectric of permittivity $\epsilon_2$. In both cases, the region modified lies outside the
domain in which the resulting potential applies. Thus, in seeking $V_1$ we might place an image charge $q_1$ at $(0, -d_1, 0)$, with $d_1 > 0$, and have

$$V_1(r) = \frac{q}{4\pi \epsilon_1} \frac{1}{|r - d\hat{j}|} + \frac{q_1}{4\pi \epsilon_1} \frac{1}{|r + d_1\hat{j}|}$$

Further, in seeking $V_2$, we might place an image charge $q_2$ at $(0, d_2, 0)$, with $d_2 > 0$, and have

$$V_2(r) = \frac{q_2}{4\pi \epsilon_2} \frac{1}{|r - d_2\hat{j}|}$$

Show that these two potentials will satisfy the necessary boundary conditions at the interface ($y = 0$) for all $x$ and $z$ provided that $d_1$, $d_2$, $q_1$, and $q_2$ are properly chosen, and find these quantities in terms of $d$, $q$, $\epsilon_1$, and $\epsilon_2$. Hint: Write $|r - d\hat{j}|$, for example, as $[x^2 + (y - d)^2 + z^2]^{1/2}$. (e) Use an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON or a program such as that described in P4.58 to obtain graphs of field lines and equipotentials in the $yz$-plane.
Chapter 13

Plane Electromagnetic Waves in Linear Matter

In Chapter 13, we shall transfer our attention from general properties shared by all time-dependent fields to the specific description of fields that vary sinusoidally with time. These monochromatic fields are of particular interest for several reasons: They are analytically simpler than fields having a more general time dependence; they can be superposed to build up more complicated fields; the response of matter to monochromatic fields is often simpler than its response to more general fields; and monochromatic fields (especially monochromatic plane waves) play an important role in applications of the theoretical framework developed in Chapters 1–12 to many physical phenomena, including optical phenomena. We shall begin by reducing Maxwell’s equations and the associated boundary conditions to forms that are convenient for treating monochromatic fields. Next we shall obtain analytic expressions for plane monochromatic waves and illustrate the application of these expressions to problems in macroscopic optics. Then, we shall examine briefly the propagation of monochromatic fields in regions bounded by good conductors. Finally, we shall discuss plane waves produced by the superposition of two or more monochromatic plane waves.

13.1 Maxwell’s Equations for Monochromatic Fields in Linear Matter

A field whose time dependence is sinusoidal or, more specifically, a field whose time dependence is characterized by a single (constant) angular frequency $\omega$ is called a monochromatic field and is conveniently represented as the real part of a complex field. If we introduce the notation

$$\begin{align*}
E(r, t) &= \Re \{\mathcal{E}(r, t)\} \\
D(r, t) &= \Re \{\mathcal{D}(r, t)\} \\
B(r, t) &= \Re \{\mathcal{B}(r, t)\} \\
H(r, t) &= \Re \{\mathcal{H}(r, t)\}
\end{align*}$$

then our assumption that the physical fields $E$, $D$, $B$, and $H$ are monochromatic is expressed analytically by letting the complex fields $\mathcal{E}$, $\mathcal{D}$, $\mathcal{B}$, and $\mathcal{H}$ have the form

$$\begin{align*}
\mathcal{E}(r, t) &= \mathcal{E}_0(r) e^{-i\omega t} \\
\mathcal{D}(r, t) &= \mathcal{D}_0(r) e^{-i\omega t} \\
\mathcal{B}(r, t) &= \mathcal{B}_0(r) e^{-i\omega t} \\
\mathcal{H}(r, t) &= \mathcal{H}_0(r) e^{-i\omega t}
\end{align*}$$

(13.2)
In general, the coefficient multiplying the exponential factor in each equation will be complex. Further, if the fields vary sinusoidally with time, Maxwell’s equations cannot be satisfied unless the free charge and current densities also vary sinusoidally with time \(P13.1\). We therefore represent the physical free current density by the real part of a complex current density

\[
\mathbf{J}(\mathbf{r}, t) = \Re\{\mathbf{J}(\mathbf{r}, t)\}
\]

and we would do the same for the charge density were it not more appropriate to the circumstances of interest to set the volume density of free charge equal to zero,

\[
\rho(\mathbf{r}, t) = 0 \quad (13.4)
\]

(which will be the case if all dielectrics are initially uncharged and the fields vary slowly enough so that all conductors can be assumed to respond instantly and completely to changes in the external field\(^1\)). Conveniently, since Maxwell’s equations are linear, we can require the complex fields to satisfy these equations with the assurance that the physical fields (i.e., the real part of the complex fields) will therefore automatically satisfy them. Thus, substituting Eqs. (13.2)–(13.4) into Maxwell’s equations, Eqs. (12.14)–(12.17), and canceling a common factor of \(e^{-i\omega t}\) in each equation, we find that the coefficients in Eq. (13.2) are determined by the equations

\[
\begin{align*}
\nabla \cdot \mathbf{D}_0(\mathbf{r}) & = 0 \quad (13.5) \\
\nabla \times \mathbf{E}_0(\mathbf{r}) & = i\omega \mathbf{B}_0(\mathbf{r}) \quad (13.6) \\
\nabla \cdot \mathbf{B}_0(\mathbf{r}) & = 0 \quad (13.7) \\
\nabla \times \mathbf{H}_0(\mathbf{r}) & = \mathbf{J}_0(\mathbf{r}) - i\omega \mathbf{D}_0(\mathbf{r}) \quad (13.8)
\end{align*}
\]

Monochromatic fields in matter are simple because, in many media, the constitutive relations between \(\mathbf{E}\) and \(\mathbf{D}\), \(\mathbf{B}\) and \(\mathbf{H}\), and \(\mathbf{J}\) and \(\mathbf{E}\) when the fields vary sinusoidally in time can be written by assuming frequency-dependent permittivities, permeabilities, and conductivities. These media are adequately described by the relationships

\[
\begin{align*}
\mathbf{D}_0(\mathbf{r}) & = \epsilon(\omega) \mathbf{E}_0(\mathbf{r}) \quad (13.9) \\
\mathbf{B}_0(\mathbf{r}) & = \mu(\omega) \mathbf{H}_0(\mathbf{r}) \quad (13.10) \\
\mathbf{J}_0(\mathbf{r}) & = g(\omega) \mathbf{E}_0(\mathbf{r}) \quad (13.11)
\end{align*}
\]

even though the forms \(\mathbf{D}(\mathbf{r}, t) = \epsilon \mathbf{E}(\mathbf{r}, t)\), etc., are not applicable when the fields have a more general time dependence \(P13.46\).

Consider, for example, a simple classical model (Drude theory;\(^2\) see P9.8) that predicts a frequency-dependent conductivity. Suppose that the resistive forces on an electron (mass \(m_e\), charge \(-q_e\)) in a conducting medium can be represented by a viscous damping term

\(^1\)The latter assumption is not as restrictive as it might seem. It was found in P9.5 that a free charge distribution in any conductor decays spontaneously to zero and that, in good conductors, the time scale of that decay is on the order of \(10^{-18}\) s. Thus, as long as \(\omega \ll 2\pi \times 10^{18} \text{ s}^{-1}\), which is easily satisfied even for frequencies in the visible spectrum \((\approx 10^{15} \text{ s}^{-1})\), the assumption of zero volume charge density is realistic. Surface charge densities, of course, may appear, but these will be treated through the boundary conditions.

\(^2\) German physicist Paul Karl Ludwig Drude, b. 12 July 1863 in Braunschweig, Duchy of Brunswick; d. 5 July 1906 in Berlin, Province of Brandenburg.
13.1. MAXWELL’S EQUATIONS FOR MONOCHROMATIC FIELDS

in the equation of motion. When an external electric field \( \mathbf{E} \) is applied to the conducting medium, the individual electrons in the medium then move in accordance with the equation

\[
m_e \frac{d\mathbf{v}}{dt} + b\mathbf{v} = -q_e \mathbf{E}
\]

where \( \mathbf{v} \) is the (complex) electron velocity and \( b \) is the viscous damping constant. Suppose \( \mathbf{E} = E_0 \hat{i} e^{-i\omega t} \) and, ignoring the (short-lived) transient effects associated with turning the field on, assume \( \mathbf{v} = v_0 \hat{i} e^{-i\omega t} \). Then, Eq. (13.12) gives

\[
(-i\omega m_e + b) v_0 = -q_e E_0 \quad \Rightarrow \quad v_0 = -\left( \frac{q_e}{b - i\omega m_e} \right) E_0
\]

where the factor \( e^{-i\omega t} \) common to all terms has been canceled. Thus, the (complex) mobility \( \mu_e \) of the electron is given by

\[
\mu_e = \frac{v_0}{E_0} = -\frac{q_e}{b - i\omega m_e}
\]

and the (complex) conductivity of the material involved is given by

\[
g(\omega) = -q_e n \mu_e = \frac{q_e^2 n}{b - i\omega m_e}
\]

where \( n \) is the density of electrons in the material [Eq. (9.16)]. For this model, the predicted conductivity is not only frequency-dependent but also complex; it is expressed in polar form by the equation

\[
g(\omega) = |g(\omega)| e^{i\theta(\omega)}
\]

where

\[
|g(\omega)| = \frac{q_e^2 n/b}{\sqrt{1 + (\omega m_e/b)^2}} \quad ; \quad \theta(\omega) = \tan^{-1} \frac{\omega m_e}{b}
\]

To obtain a physical interpretation, note that in this example

\[
\mathbf{E} = \Re \{ E_0 \hat{i} e^{-i\omega t} \} = E_0 \hat{i} \cos \omega t
\]

\[
\mathbf{J} = \Re \{ g(\omega) E_0 \hat{i} e^{-i\omega t} \} = |g(\omega)| E_0 \hat{i} \cos[\omega t - \theta(\omega)]
\]

Thus, we see that \( |g(\omega)| \) relates the amplitude of the conduction current density to that of the applied field and \( \theta(\omega) \) expresses a phase difference between the current density and the applied field. For a given applied field, the amplitude of the current density falls off and the phase difference between the current density and the applied field increases in magnitude with increasing \( \omega \). As shown in Fig. 13.1, however, \( |g(\omega)| \approx q_e^2 n/b \) and \( \theta(\omega) \approx 0 \) if \( \omega \ll b/m_e \); at “low” frequencies, the conductivity is real and independent of frequency, and the current density and the applied field are in phase. The numerical value of \( \omega_c = b/m_e \) (which divides “low” frequencies from “high” frequencies) is determined by the properties of the material. For good conductors, \( b \approx 10^{-16} \text{ kg/s} \), and (with \( m_e = 9.1 \times 10^{-31} \text{ kg} \)) \( \omega_c \approx 10^{14} \text{ s}^{-1} \); thus in the microwave region (\( \approx 10^{10} \text{ s}^{-1} \)) and at lower frequencies, we expect the conductivity of good conductors to be real and independent of frequency, but in the visible region (\( \approx 10^{15} \text{ s}^{-1} \)) we expect the conductivity of good conductors to be complex and dependent on frequency. For dielectrics, on the other hand, \( b \) is effectively infinite and \( g(\omega) = 0 \) at all frequencies.
A classical model that yields a frequency-dependent permittivity is explored in P13.2. In part, this model predicts (1) that the static dielectric constant should be given by
\[ K_e(0) = \frac{\epsilon(0)}{\epsilon_0} = 1 + \frac{nq_e^2}{m_e\epsilon_0\omega_0^2} \] (13.19)
where \( n, q_e, \) and \( m_e \) are as in the previous paragraph and \( \omega_0 \) is the natural frequency associated with the individual atoms composing the dielectric; i.e., \( \omega_0 \) is the empirical frequency at which the atoms absorb electromagnetic radiation; (2) that at very high frequencies
\[ K_e(\infty) = 1 \] (13.20)
and (3) that for \( \omega \ll \omega_0, \ K_e(\omega) \approx K_e(0) \) and for \( \omega \gg \omega_0, \ K_e(\omega) \approx 1. \) Thus, according to this model, the dielectric constant and also the permittivity are real and independent of frequency except when \( \omega \approx \omega_0. \) Different substances, of course, are characterized by different values of \( \omega_0. \) Indeed, quantum mechanically, substances exhibit more than one absorption line. Although the classical model then breaks down in detail, we would still expect the dielectric constant to exhibit “funny” behavior only at frequencies near to each of the absorption frequencies.

For the remainder of this chapter, we shall confine our attention to matter for which Eqs. (13.9)–(13.11) provide an adequate description. Further, we shall assume that no free currents other than those induced on conductors by the field \( \mathcal{E}_0(\mathbf{r}) \) are present. Thus, written in terms of \( \mathcal{E}_0 \) and \( \mathcal{H}_0 \) alone, Eqs. (13.5)–(13.8) reduce to
\[
\nabla \cdot \mathcal{E}_0(\mathbf{r}) = 0 \tag{13.21}
\]
\[
\nabla \times \mathcal{E}_0(\mathbf{r}) = i\omega\mu\mathcal{H}_0(\mathbf{r}) \tag{13.22}
\]
\[
\nabla \cdot \mathcal{H}_0(\mathbf{r}) = 0 \tag{13.23}
\]
\[
\nabla \times \mathcal{H}_0(\mathbf{r}) = -i\omega \left( \epsilon + \frac{g}{\omega} \right) \mathcal{E}_0(\mathbf{r}) \tag{13.24}
\]
Explicit indication of the possible dependence of \( \epsilon, \mu, \) and \( g \) on \( \omega \) has been suppressed.) We can also show that the fields satisfy
\[
\left[ \nabla^2 + \mu\epsilon\omega^2 \left( 1 + i\frac{g}{\epsilon\omega} \right) \right] \mathcal{E}_0(\mathbf{r}) = 0 \tag{13.25}
\]
\[
\left[ \nabla^2 + \mu\epsilon\omega^2 \left( 1 + i\frac{g}{\epsilon\omega} \right) \right] \mathcal{H}_0(\mathbf{r}) = 0
\]
(P13.4) and that the time-averaged Poynting vector \( \langle \mathbf{S} \rangle \) and the time-averaged energy density \( \langle u_{\text{EM}} \rangle \) are given for these fields by
\[
\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2} \Re \{ \mathcal{E}_0^* \times \mathcal{H}_0 \} \tag{13.26}
\]
and
\[
\langle u_{\text{EM}} \rangle = \frac{1}{2} \langle \mathbf{E} \cdot \mathbf{D} \rangle + \frac{1}{2} \langle \mathbf{H} \cdot \mathbf{B} \rangle = \frac{1}{4} \Re \{ \epsilon \mathcal{E}_0^* \cdot \mathcal{E}_0 + \mu \mathcal{H}_0^* \cdot \mathcal{H}_0 \} \tag{13.27}
\]
[See Eqs. (12.32) and (12.30) and compare P7.26.] In these equations, explicit evaluation of \( \mathcal{E}_0 \) and \( \mathcal{H}_0 \) at the point \( \mathbf{r} \) has been suppressed.
Figure 13.1: Amplitude and phase of the complex conductivity as functions of $\omega$.

**PROBLEMS**

P13.1. Without making any assumptions about the time dependence of $\mathbf{J}(\mathbf{r}, t)$, substitute the fields given by Eq. (13.2) and show that Eq. (12.16) cannot be satisfied unless $\mathbf{J}(\mathbf{r}, t)$ varies with time as $e^{-i\omega t}$.

P13.2. Suppose that the electrons in a dielectric medium are tied to their nominal equilibrium positions with harmonic forces and also experience damping, so that each electron (charge $-q_e$, mass $m_e$) moves in an electric field $\mathbf{E}$ in accordance with the equation of motion

$$m_e \frac{d^2 \mathbf{r}}{dt^2} + b \frac{d\mathbf{r}}{dt} + k \mathbf{r} = -q_e \mathbf{E}$$
where \( \mathbf{r} \) is a complex vector whose real part gives the displacement of the electron from
its nominal equilibrium position. Let \( \mathbf{E} = \mathbf{E}_0 e^{-i\omega t} \) and, ignoring transients, assume
\( \mathbf{r} = \mathbf{r}_0 e^{-i\omega t} \). Further, remember that the polarization \( \mathbf{P} \) is given by \( n \mathbf{p} \), where \( n \) is the
number of electrons per unit volume and \( \mathbf{p} = -q_e \mathbf{r} \) is the dipole moment of the electron.

(a) Show that the dielectric constant of the material is given by

\[
K_e(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{nq^2_e/m_e}{\omega_0^2 - \omega^2} - i\left(\frac{b\omega}{m_e}\right)
\]

where \( \omega_0 = \sqrt{k/m_e} \) is the natural frequency of the oscillating system. (b) Verify the
two limits given in Eqs. (13.19) and (13.20). (c) Explain what it means physically for the
permittivity \( \epsilon(\omega) \) to be complex and support your explanation with an analytic argument.

(d) Sketch graphs of \( \Re\{K_e(\omega)\} \) and \( \Im\{K_e(\omega)\} \) for various values of \( b \), starting with
\( b = 0 \). (e) Use an available numeric processing program like IDL, MATLAB, OCTAVE,
or PYTHON to obtain careful graphs of \( \Re\{K_e(\omega)\} \) and \( \Im\{K_e(\omega)\} \) versus \( \omega/\omega_0 \) for various
values of the parameters \( nq^2_e/\epsilon_0 k \) and \( b/m_0\omega_0 \). Note in particular that \( \Im\{K_e(\omega)\} \) is related
to the absorption spectrum of the oscillating system and that values of \( b \) can be inferred
from measurements of the width of the absorption curve.

P13.3. Estimate \( \omega_0 \) for hydrogen (a) by setting the ionization potential, 13.6 eV, equal to \( \hbar\omega_0 \),
where \( \hbar \) is Planck’s constant divided by \( 2\pi \), and (b) by determining the force constant
\( k \) using the model discussed in P10.3. (c) Use Eq. (13.19) to predict the static dielectric
constant of hydrogen gas at 100 °C and 1 atmosphere pressure, comparing the results
with the measured value of 1.000264. Is the agreement good or bad? Suggest possible
sources for any discrepancy. **Hints:** (1) Assume that the two atoms in each hydrogen
molecule respond independently to the applied field. (2) The Bohr radius for the ground
state of the hydrogen atom is 0.0528 nm.

P13.4. Verify that Eq. (13.25) follows from Eqs. (13.21)–(13.24). **Hint:** Take the curl of
Eqs. (13.22) and (13.24) and then use Eq. (C.19).

P13.5. Show that the average power dissipated per unit volume by a monochromatic field in a
conducting medium is given by \( \langle P \rangle = \frac{1}{2} \Re\{J_0 \cdot \mathbf{E}_0^*\} \), and then show that \( \langle P \rangle = 0 \) when
\( g(\omega) \) is purely imaginary (i.e., when the electric field and the current are 90° out of phase,
as in the high frequency domain shown in Fig. 13.1).

### 13.2 Boundary Conditions on Monochromatic Fields

As with all differential equations, those in Section 13.1 for the amplitudes of the complex
fields become specific to a particular problem only when supplemented with whatever
boundary conditions the problem imposes. Since in this chapter we are interested ultimately
in the optical problem of reflection and transmission of monochromatic fields at the interface
between two media and in the confinement of monochromatic fields to regions bounded
by conductors, we need in particular to rewrite the general boundary conditions developed
in Section 12.7 in a form applicable to monochromatic fields. Essentially, we must write
the boundary conditions at an interface so that they apply to the complex fields (which is
easy, because the boundary conditions are linear in the fields and hence apply directly to
the complex fields), and we must examine what happens if the medium on one or both sides
of the interface has nonzero or even infinite conductivity. For our present purposes, it is
sufficient to assume a plane interface that we take to define the \( xy \) plane (Fig. 13.2). Then,
assuming that all fields, current densities, and charge densities have the time dependence
Figure 13.2: Plane interface between two different media.

$e^{-i\omega t}$ and recognizing that the frequency must be the same on both sides of the interface,\(^3\) we can reexpress the boundary conditions in Eq. (12.65) in the form\(^4\)

\[
\begin{align*}
\mathcal{E}^{(1)}_{0x} &= \mathcal{E}^{(2)}_{0x} \\
\mathcal{H}^{(2)}_{0y} - \mathcal{H}^{(1)}_{0y} &= \mathcal{j}_{0y} \\
\mathcal{E}^{(1)}_{0y} &= \mathcal{E}^{(2)}_{0y} \\
\mathcal{H}^{(2)}_{0y} - \mathcal{H}^{(1)}_{0y} &= -\mathcal{j}_{0x} \\
e_2\mathcal{E}^{(2)}_{0z} - e_1\mathcal{E}^{(1)}_{0z} &= \sigma_0 \\
\mu_1\mathcal{H}^{(1)}_{0z} &= \mu_2\mathcal{H}^{(2)}_{0z} \\
\mathcal{j}^{(2)}_{0z} - \mathcal{j}^{(1)}_{0z} &= g_2\mathcal{E}^{(2)}_{0z} - g_1\mathcal{E}^{(1)}_{0z} = i\omega\sigma_0
\end{align*}
\]

where we have (1) suppressed explicit indication that all quantities are to be evaluated at a point on the interface,\(^2\) moved the specification of the medium to a superscript to avoid a triple subscript, (3) canceled the factor $e^{-i\omega t}$ from every term, and (4) written $\sigma(r,t) = \Re\{\sigma_0(r)e^{-i\omega t}\}$, $\mathcal{j}(r,t) = \Re\{\mathcal{j}_{0x}(r)e^{-i\omega t}\}$, etc.

\(^3\)The boundary conditions cannot be satisfied for all times if the frequencies are different. Continuity of a tangential component of $\mathbf{E}(r,t)$, for example, requires that $\mathcal{E}^{(1)}_{0x}(r) \exp(-i\omega_1 t) = \mathcal{E}^{(2)}_{0x}(r) \exp(-i\omega_2 t)$ or, equivalently, that $\mathcal{E}^{(1)}_{0x}(r) = \mathcal{E}^{(2)}_{0x}(r) \exp(-i(\omega_2 - \omega_1) t)$. Since the left-hand side of this latter form is independent of $t$, the right-hand side must also be independent of $t$ and will be so only if $\omega_1 = \omega_2$. Q.E.D.

\(^4\)We here use a lower case script $j$, i.e., $\mathcal{j}$, for a complex surface current density.
It is useful to comment specifically on further constraints that more specific stipulation of the two media imposes on Eqs. (13.28)–(13.34).\(^5\) If, for example, both media are nonconductors so that \(g_1 = g_2 = 0\), then there can be neither induced surface charge nor induced surface currents on the interface. Consequently, \(\sigma_0 = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0\) and we conclude that

\[
\begin{align*}
\varepsilon_0^{(1)} &= \varepsilon_0^{(2)} & \sigma_0^{(1)} &= \sigma_0^{(2)} \\
\mathcal{H}_0^{(1)} &= \mathcal{H}_0^{(2)} & \mu_1 \mathcal{H}_0^{(1)} &= \mu_2 \mathcal{H}_0^{(2)} \\

\end{align*}
\]

(13.35)

i.e., the tangential components of \(\varepsilon_0\) and \(\mathcal{H}_0\) and the normal components of \(\varepsilon_0\) and \(\mu \mathcal{H}_0\) must be continuous across an interface between two nonconductors \((g_1 = g_2 = 0)\).

In a second useful special case, \(g_1\) and \(g_2\) may be nonzero but neither is infinite. Since a medium with finite conductivity cannot support surface currents,\(^6\) \(\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0\) for this case. The surface charge density \(\sigma_0\), however, may or may not be zero. If \(\sigma_0 = 0\), then Eqs. (13.32) and (13.34) are contradictory unless \(\sigma_1 g_2 = \sigma_2 g_1\) (Why?) When this particular relationship among \(\varepsilon_1\), \(\varepsilon_2\), \(g_1\), and \(g_2\) applies, the boundary conditions to be imposed are the same as those in Eq. (13.35). More often, however, this special relationship will not be satisfied. Then \(\sigma_0 \neq 0\) and Eqs. (13.32) and (13.34) can be combined to give a single condition that does not involve \(\sigma_0\); except for the special case \(\sigma_1 g_2 = \sigma_2 g_1\), we thus find that, when \(g_1\) and \(g_2\) are finite and at least one differs from zero, the fields must satisfy

\[
\begin{align*}
\varepsilon_0^{(1)} &= \varepsilon_0^{(2)} & \varepsilon_0^{(1)} &= \varepsilon_0^{(2)} \\
\mathcal{H}_0^{(1)} &= \mathcal{H}_0^{(2)} & \mathcal{H}_0^{(1)} &= \mathcal{H}_0^{(2)} \\

(1 + \frac{g_1}{\omega}) \varepsilon_0^{(1)} &= (1 + \frac{g_2}{\omega}) \varepsilon_0^{(2)} & \mu_1 \mathcal{H}_0^{(1)} &= \mu_2 \mathcal{H}_0^{(2)} \\

\end{align*}
\]

(13.36)

In words, the tangential components of \(\varepsilon_0\) and \(\mathcal{H}_0\) and the normal components of \([\varepsilon + i(g/\omega)]\varepsilon_0\) and \(\mu \mathcal{H}_0\) must be continuous across an interface between two imperfect conductors. Once the fields have been found, Eq. (13.32) can be used to calculate the charge density on the interface.

A third special case occurs when one of the media, say medium 2, can be approximated as a perfect conductor \((g_2 = \infty)\) but the other medium is either an imperfect conductor or a nonconductor. In this case both surface currents and surface charges may appear on the interface, and the simplifications lie in the requirement that all fields be zero in the perfect conductor. To prove that \(\varepsilon_0^{(2)} = 0\), for example, suppose first that \(g_2\) is finite. Then Eq. (13.24) gives

\[
\varepsilon_0^{(2)} = \frac{\nabla \times \mathcal{H}_0^{(2)}}{g_2 - i\omega \varepsilon}
\]

(13.37)

Thus, if \(\nabla \times \mathcal{H}_0^{(2)}\) is finite (which must be required on physical grounds), \(\varepsilon_0^{(2)} \to 0\) as \(g_2 \to \infty\). Coupled with the vanishing of \(\varepsilon_0^{(2)}\) in the (perfect) conductor, Eq. (13.22) yields

---


\(^6\)Existence of a surface current requires the volume current density to be infinite, but Ohm’s law \(\mathbf{J} = \sigma \mathbf{E}\) precludes infinite \(\mathbf{J}\) unless \(g = \infty\), since physically \(\mathbf{E}\) must be finite; surface currents are an idealization that can occur only in conjunction with the idealization of a perfect conductor.
We begin by seeking expressions for the solutions themselves. In Chapter 7, we expressed the fields in a sinusoidal plane wave propagating in the $z$ direction as trigonometric functions of the argument $(\kappa z - \omega t)$, where $\kappa$ represents the wave number. More generally, for a plane wave propagating in the direction of the unit vector $\hat{n}$, we expect that the coordinate $\hat{n} \cdot r$, which measures the distance of the point $r$ from a plane through the origin whose normal is $\hat{n}$, will replace the coordinate $z = \hat{k} \cdot r$, which measures the distance of $r$ from the $xy$ plane (whose normal is $\hat{k}$). Thus, we expect the more general plane wave to be described analytically by trigonometric functions of the argument $K \hat{n} \cdot r - \omega t = K \cdot r - \omega t$, where $K$ is analogous to $\kappa$—we use a different symbol because we shall find that $K$ is sometimes complex—and the propagation vector $K$ is defined by

$$K = K \hat{n}$$  \hspace{1cm} (13.39)

Thus, in terms of this vector, whose meaning is yet to be explored, a sinusoidal plane wave propagating in the direction $\hat{n}$ is expressed as the real part of the quantity

$$(\text{constant}) e^{i(K \cdot r - \omega t)}$$  \hspace{1cm} (13.40)

where the constant may be complex and contains information about both the phase and the amplitude of the corresponding wave. The general monochromatic field expressed in

$$E_0(1) = E_0(2) = 0$$

$$\mu_1 H_0(1) = \mu_2 H_0(2) = 0$$

the tangential components of $E_0$ and the normal component of $\mu H_0$ must be zero on both sides of an interface between a general medium and a perfect conductor; the normal component of $\epsilon E_0$ and the tangential components of $H_0$ must be zero in the perfect conductor and will be nonzero in the general medium by amounts determined by the charge and current distributions that the fields themselves induce on the interface. These charge and current densities induced on the interface are, of course, not part of the information given at the beginning of a problem and the above boundary conditions often are used to determine these densities after the fields have been found by other means.
CHAPTER 13. PLANE ELECTROMAGNETIC WAVES

Eqs. (13.2) and (13.3) is therefore reduced to a sinusoidal plane wave by requiring the spatially dependent coefficients to have the form

\[ E_0(\mathbf{r}) = E_0 e^{i \mathbf{K} \cdot \mathbf{r}} \]
\[ D_0(\mathbf{r}) = D_0 e^{i \mathbf{K} \cdot \mathbf{r}} \]
\[ B_0(\mathbf{r}) = B_0 e^{i \mathbf{K} \cdot \mathbf{r}} \]
\[ H_0(\mathbf{r}) = H_0 e^{i \mathbf{K} \cdot \mathbf{r}} \]
\[ J_0(\mathbf{r}) = J_0 e^{i \mathbf{K} \cdot \mathbf{r}} \]

(13.41)

where the amplitudes \( E_0, D_0, B_0, H_0 \), and \( J_0 \) are now constant vectors, although they may have complex components. Further, substitution of Eq. (13.41) into Eqs. (13.9)–(13.11) yields the relationships

\[ D_0 = \epsilon E_0 \]
\[ B_0 = \mu H_0 \]
\[ J_0 = g E_0 \]

(13.42)

among these constant amplitudes and the parameters \( \epsilon \), \( \mu \), and \( g \) that characterize the medium.

Instead of solving Maxwell’s equations directly and systematically, we have in a sense guessed the spatial dependence of the complex fields representing a monochromatic plane wave. We must now determine whether Maxwell’s equations impose any additional constraints on the amplitudes \( E_0, D_0, B_0, \) and \( H_0 \) and the wave vector \( \mathbf{K} \). Note first that

\[
\nabla \left[ e^{i \mathbf{K} \cdot \mathbf{r}} \right] = \nabla \left[ e^{i(K_x x + K_y y + K_z z)} \right] = i (K_x \hat{i} + K_y \hat{j} + K_z \hat{k}) e^{i(K_x x + K_y y + K_z z)} = i \mathbf{K} e^{i \mathbf{K} \cdot \mathbf{r}}
\]

(13.43)

Thus, substitution of Eq. (13.41) into Eqs (13.21)–(13.24), use of the vector identities in Eqs. (C.8) and (C.11), and cancellation of the factor \( e^{i \mathbf{K} \cdot \mathbf{r}} \) yields the relationships

\[
\mathbf{K} \cdot \mathbf{E}_0 = 0
\]
\[
\mathbf{K} \times \mathbf{E}_0 = \omega \mu \mathbf{H}_0
\]
\[
\mathbf{K} \cdot \mathbf{H}_0 = 0
\]
\[
\mathbf{K} \times \mathbf{H}_0 = -\omega \left( \epsilon + \frac{g \omega}{\omega} \right) \mathbf{E}_0
\]

(13.44)

(13.45)

(13.46)

(13.47)

that must be satisfied by the constant vectors \( \mathbf{E}_0 \) and \( \mathbf{H}_0 \) if the assumed fields are to satisfy Maxwell’s equations. The so-called dispersion relation

\[ K^2 = \mu \omega^2 \left( \epsilon + i \frac{g}{\omega} \right) \]

(13.48)

connecting \( K \) and \( \omega \) is implied by Eqs. (13.44)–(13.47). In words, Eqs. (13.44) and (13.46) require that both \( \mathbf{E}_0 \) and \( \mathbf{H}_0 \) be perpendicular to \( \hat{n} \), which is the direction of propagation [see Eq. (13.39)]; the plane monochromatic electromagnetic wave is thus a transverse wave. Further, Eqs. (13.45) and (13.47) require \( \mathbf{H}_0 \) and \( \mathbf{E}_0 \) to be mutually perpendicular. Finally, Eq. (13.48) determines \( K \) if \( \omega \) is given. Paralleling the discussion in Section 7.5, we can think of this plane wave as being defined as follows:

1. Specify the direction of propagation \( \hat{n} \) and the frequency \( \omega \) arbitrarily.
2. Specify \( \mathbf{E}_0 \) arbitrarily, subject only to the constraint that \( \mathbf{E}_0 \) be perpendicular to \( \hat{n} \).
3. Determine \( \mathbf{H}_0 \) from Eq. (13.45) and \( K \) from Eq. (13.48).

\[ \text{Evaluate the cross product of Eq. (13.45) with } \mathbf{K} \text{ and then use Eq. (C.1); see P13.6.} \]
Substituting Eqs. (13.42) and (13.44)–(13.47) into Eq. (13.41) and reinstating the exponential time factor, we find finally that the general plane monochromatic solution to Maxwell’s equations in unbounded, isotropic, homogeneous, linear media is represented analytically by the real part of the complex fields

\[ E(r, t) = D(r, t) = E_0 e^{i(K \cdot r - \omega t)} \]

\[ H(r, t) = B(r, t) = \frac{K}{\mu(\omega)} \times E_0 e^{i(K \cdot r - \omega t)} \]

where \( K \) and \( \omega \) must satisfy Eq. (13.48) and \( E_0 \) must be perpendicular to \( K \). It is shown in P13.8 that the Poynting vector and the energy density for these fields are given by

\[ \langle S \rangle = \frac{|E_0|^2 \Re\{K\}}{2\omega\mu} \hat{n} e^{-2(\hat{n} \cdot r) \Im\{K\}} \]

\[ \langle u_{EM} \rangle = \frac{1}{4}|E_0|^2 \left( \epsilon + \frac{|K|^2}{\mu \omega^2} \right) e^{-2(\hat{n} \cdot r) \Im\{K\}} \]

where \( |E_0|^2 = E_0 \cdot E_0^* \), \( K \) has been written as in Eq. (13.39), and we have assumed that \( \epsilon \) and \( \mu \) are real. Note that whenever \( K \) has a nonzero imaginary part, both \( \langle S \rangle \) and \( \langle u_{EM} \rangle \) (and in fact also the fields—see Section 13.3.2) have a real exponential factor. Since physically the fields cannot increase in amplitude as the wave propagates into the medium, we anticipate that \( \Im\{K\} > 0 \).

### 13.3.2 Interpretation of the Propagation Vector

As we have already recognized, plane waves propagating in a general medium are characterized by a propagation vector \( K \) that is complex. The physical significance of this vector is therefore to be sought in the separate significance of its real and imaginary parts, which we introduce explicitly by writing

\[ K = K \hat{n} = (\kappa + i\alpha) \hat{n} = \kappa + i\alpha \]

where \( \kappa = \kappa \hat{n} \) and \( \alpha = \alpha \hat{n} \), and \( \kappa \) and \( \alpha \) are both real. The exponential factor in Eqs (13.49) and (13.50) then has the more interpretable expression

\[ e^{i(K \cdot r - \omega t)} = e^{i(\kappa \xi - \omega t)} e^{-\alpha \xi} \]

where \( \xi = \hat{n} \cdot r \) is in essence the single spatial coordinate on which the plane wave depends (analogous to \( z \) when the wave propagates in the \( z \) direction). Further, the exponential factor \( e^{-2n \hat{n} \cdot r \Im\{K\}} \) in \( \langle S \rangle \) and \( \langle u_{EM} \rangle \) as given by Eqs. (13.51) and (13.52) reduces to \( e^{-2\alpha \xi} \). Clearly, \( \kappa = \Re\{K\} \) relates to the oscillations of the waves in space and \( \alpha = \Im\{K\} \) describes an attenuation or absorption of the wave. (We assume \( \alpha > 0 \).) More specifically, \( \kappa \) determines the wavelength \( \lambda \) and the velocity of propagation\(^9\) \( v_p \) of the monochromatic wave by

\[ \lambda = \frac{2\pi}{\kappa} ; \quad v_p = \frac{\omega}{\kappa} \]

\(^8\)In some treatments of this subject, the exponential factor in plane waves is written in the form \( \exp[i(\omega t - K \cdot r)] \). In that form, decaying waves occur when \( \Im\{K\} < 0 \).

\(^9\)In Section 13.6, we shall refer to this velocity as the *phase* velocity to distinguish it from the *group* velocity to be defined there. Hence, we use the subscript \( p \) at the outset.
[compare Eqs. (7.21) and (7.20)] and α determines the penetration length or skin depth δ by

$$\delta = \frac{1}{\alpha}$$

(13.56)

where δ is the distance within which the amplitude of the wave decays to 1/e of its value at ξ = 0. With c representing the speed of light in vacuum, we also define the (ordinary, i.e., real) index of refraction of the medium by

$$n = \frac{c}{v_p} = \frac{c\kappa}{\omega}$$

(13.57)

and the complex index of refraction by

$$\eta = \frac{cK}{\omega} = \frac{c(\kappa + i\alpha)}{\omega} = n + \frac{i}{\omega\delta}$$

(13.58)

both of which are in general dependent on frequency. The complex index of refraction is particularly useful in theoretical considerations, because it combines both the (ordinary, i.e., real) index of refraction and the penetration length in a single quantity that reduces to the ordinary index of refraction in the absence of absorption (i.e., when α = 0). However we choose to express these properties analytically, the essential physical features are a wavelength, speed of propagation, and an (ordinary) index of refraction determined by κ = ℜ{K} and an attenuation determined by α = ℑ{K}. Note also that the factor $K \times \mathcal{E}_0 = (\kappa + i\alpha)\hat{n} \times \mathcal{E}_0$ in Eq. (13.50) is complex when $\alpha \neq 0$, and attenuation of the wave is thus accompanied by a phase difference between $\mathcal{E}$ and $\mathcal{H}$.

To connect α and κ more directly with the physical characteristics of the medium, we begin by substituting $K = \kappa + i\alpha$ into the dispersion relation, Eq. (13.48), obtaining

$$\kappa^2 - \alpha^2 + 2i\kappa = \mu\epsilon\omega^2 \left( \epsilon + \frac{g}{\omega} \right)$$

(13.59)

We now explicitly assume that ϵ, μ, and g are real at the particular frequency involved. The real and imaginary parts of Eq. (13.59) are then readily separated to give the two equations

$$\kappa^2 - \alpha^2 = \mu\epsilon \omega^2$$

$$2\alpha\kappa = \mu g \omega$$

(13.60)

which can be solved for κ and α. In terms of the quantity

$$\beta(\omega) = 1 + \sqrt{1 + \left( \frac{g}{\epsilon\omega} \right)^2}$$

(13.61)

the results are

$$\kappa^2 = \frac{1}{2}\mu\epsilon \omega^2 \beta$$

(13.62)

$$\alpha^2 = \frac{\mu\epsilon}{2\beta} \left( \frac{g}{\epsilon} \right)^2$$

(13.63)

(P13.9). Thus, nonzero α and the consequent attenuation of the wave occur physically in media with nonzero conductivity. Further, in terms of the properties of the medium, the

10 An interesting case in which this assumption is violated is treated in P13.58.
wave length $\lambda$, the velocity of propagation $v_p$, the penetration length $\delta$, and the (ordinary) index of refraction $n$ are given by

\begin{align}
\lambda &= \frac{2\pi}{\kappa} = \left(\frac{2\pi}{g\sqrt{\mu}}\right) \frac{g}{\epsilon \omega} \sqrt{\frac{2}{\beta}} \\
v_p &= \frac{\omega}{\kappa} = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{\frac{2}{\beta}} \\
\delta &= \frac{1}{\alpha} = \left(\frac{2}{g\sqrt{\mu}}\right) \sqrt{\frac{\beta}{2}} \\
n &= c \left(\frac{v_p}{c}\right) = \left(\sqrt{\mu \epsilon c}\right) \sqrt{\frac{\beta}{2}}
\end{align}

Equations (13.64)–(13.67) are of particular interest in two special cases. Suppose first that the medium is a good conductor; i.e., suppose that $g$ is large, specifically $g \gg \epsilon \omega$. (Equivalently, we can think of this case as a low-frequency limit, $\omega \ll g/\epsilon$; see P13.16.) In this limit $\beta$ as given by Eq. (13.61) approaches $g/\epsilon \omega$, and Eqs. (13.64)–(13.67) have the limits

\begin{align}
\lambda &\rightarrow 2\pi \sqrt{\frac{2}{\mu g \omega}} \\
v_p &\rightarrow \frac{2\omega}{\mu gc^2} \\
\delta &\rightarrow \sqrt{\frac{2}{\mu g \omega}} \\
n &\rightarrow \sqrt{\frac{\mu gc^2}{2\omega}}
\end{align}

(P13.10). In particular, we expect these limits to apply for good conductors ($\mu \approx \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$, $g \approx 10^7 \text{ mho/m}$) in the microwave region ($\omega \approx 10^{10} \text{ s}^{-1}$; $\lambda_0 = \text{wavelength in vacuum} \approx 20 \text{ cm}$) where $g$ is real (Section 13.1). With these values for the parameters, $\lambda \approx 10^{-4}\lambda_0$, $v_p \approx 10^{-4}c$, $\delta \approx 4 \times 10^{-4} \text{ cm}$, and $n \approx 10^4$. Microwaves in good conductors therefore have much smaller wavelengths than in vacuum, travel slowly (for electromagnetic waves), penetrate only very little, and are characterized by a very large index of refraction. Another aspect of wave propagation in good conductors is explored in P13.12.

The general results in Eqs. (13.64)–(13.67) can also be simplified when the medium is nonconducting ($g = 0$) or approximately so ($g \ll \epsilon \omega$). (Equivalently, we can think of this case as a high-frequency limit, $\omega \gg g/\epsilon$; see P13.16). In this limit, $\beta$ as given by Eq. (13.61) approaches the value 2, and Eqs. (13.64)–(13.67) reduce to

\begin{align}
\lambda &= \frac{2\pi}{\sqrt{\mu \epsilon \omega}} \\
v_p &= \frac{1}{\sqrt{\mu \epsilon}} \\
\delta &= \sqrt{\frac{4\epsilon}{\mu g^2}} \rightarrow \infty
\end{align}
CHAPTER 13. PLANE ELECTROMAGNETIC WAVES

Figure 13.3: Wavelength, velocity of propagation, penetration length, and index of refraction versus frequency for a plane wave in a conducting medium. Graph A shows \( \lambda/(2\pi g \sqrt{\epsilon/\mu}) \), graph B shows \( n/(\sqrt{\mu \epsilon}) \) and \( \delta/(2g \sqrt{\epsilon/\mu}) \), and graph C shows \( v_p \sqrt{\mu \epsilon} \) under the assumption that \( \epsilon, \mu, \) and \( g \) are independent of frequency and are real.

\[
n = \sqrt{\mu \epsilon} \approx \sqrt{K_{m} K_{e}} \approx \sqrt{K_{e}}
\]  

[See Eq. (7.20).] The final form in Eq. (13.75) follows because most optically transparent nonconductors are nonmagnetic and \( K_{m} \) is therefore approximately unity for the cases of interest. Although \( \omega \) appears explicitly only in the expression for \( \lambda \), all of these quantities may nonetheless depend on \( \omega \) because \( \epsilon, \mu, \) and \( g \) may depend on \( \omega \). Tabulated indices of refraction, for example, can therefore be expected to agree with the square roots of tabulated dielectric constants only if the compared values are measured at the same frequency. Numerically, for a typical nonmagnetic transparent dielectric (a glass), for which \( K_{e} \approx 2, K_{m} = 1, \) and \( g \approx 10^{-12} \text{ mho/m} \) in the visible spectrum (\( \omega \approx 4 \times 10^{15} \text{ s}^{-1}, \lambda_{0} = \text{wavelength in vacuum} = 500 \text{ nm} \)), we find that \( \lambda \approx 0.7 \lambda_{0}, v_{p} \approx 0.7c, \delta \approx 7 \times 10^{9} \text{ m(!), and } n \approx 1.4. \)

Graphs of \( \lambda, v_{p}, \delta, \) and \( n \) as given by Eqs. (13.64)–(13.67) are shown in Fig. 13.3. These graphs display the limiting forms in Eqs. (13.68)–(13.75).

13.3.3 Waves in Non-Conducting Media

A further aspect of the limit \( g \approx 0 \) is that \( \alpha \) is small enough to be set equal to zero [Eq. (13.63)]. Thus, the dispersion relation in Eq. (13.48) becomes

\[
\kappa = \sqrt{\mu \epsilon} \omega
\]  

(13.76)
and the vector $\mathbf{K}$, given by Eq. (13.53) as

$$\mathbf{K} = \hat{n} \kappa = \kappa \hat{n} = \kappa$$

(13.77)

becomes real. The results in Eqs. (13.49)–(13.52) therefore reduce to

$$\mathcal{E}(\mathbf{r}, t) = \mathcal{E}_0 e^{i(\mathbf{K} \cdot \mathbf{r} - \omega t)}$$

(13.78)

$$\mathcal{H}(\mathbf{r}, t) = \sqrt{\varepsilon \mu} \hat{n} \times \mathcal{E}_0 e^{i(\mathbf{K} \cdot \mathbf{r} - \omega t)}$$

(13.79)

$$\langle S \rangle = \frac{|\mathcal{E}_0|^2 \kappa}{2\omega \mu} = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} |\mathcal{E}_0|^2 \hat{n}$$

(13.80)

$$\langle u_{EM} \rangle = \frac{1}{2} \varepsilon |\mathcal{E}_0|^2$$

(13.81)

and there is no attenuation.

**PROBLEMS**

**P13.6.** (a) Substitute Eq. (13.41) into Eqs. (13.21)–(13.24) to derive the conditions of Eqs. (13.44)–(13.47) on the constant amplitudes of a plane monochromatic field. (b) Derive the dispersion relation, Eq. (13.48), by the method outlined in the text and also by evaluating the cross product of Eq. (13.47) with $\mathbf{K}$. (c) Show that Eq. (13.48) also follows if the fields in Eq. (13.41) are required to satisfy Eq. (13.25).

**P13.7.** Given the dispersion relation, Eq. (13.48), rewrite the boundary condition on $\mathcal{E}_{0z}$ in Eq. (13.36) in terms of $K^2$.

**P13.8.** Derive Eqs. (13.51) and (13.52) for $\langle S \rangle$ and $\langle u_{EM} \rangle$ from Eqs. (13.26) and (13.27). *Hints:* (1) Allow for the possibility of complex $\mathbf{K}$, so that, for example, $\exp(\mathbf{K} \cdot \mathbf{r})^* = \exp(-\mathbf{K}^* \cdot \mathbf{r})$. (2) See Eq. (D.10).

**P13.9.** Solve Eq. (13.60) for $\kappa$ and $\alpha$, obtaining Eqs. (13.62) and (13.63). *Hints:* (1) Solve $2\alpha \kappa = \mu g \omega$ for $\alpha$ and substitute into $\kappa^2 - \alpha^2 = \mu \varepsilon \omega^2$. The result is an equation quadratic in $\kappa^2$. (2) Physically, $\kappa$ is real and so $\kappa^2 > 0$.

**P13.10.** Verify that Eqs. (13.68)–(13.71) follow from Eqs. (13.64)–(13.67) when $g \gg \varepsilon \omega$.

**P13.11.** The conductivity of sea water is about 4.3 mho/m. Assume $\mu = \mu_0$ and take $\varepsilon \approx 75\varepsilon_0$, which is about the value of $\varepsilon$ for pure water at low frequency. (a) For what frequencies can Eq. (13.70) be applied? (b) Calculate the penetration depth in sea water at the typical radio frequency of $10^5$ Hz ($\omega = 2\pi \times 10^5$ s$^{-1}$) and comment on the suitability of ordinary radio signals as a means of underwater communication, say, between submarines.

**P13.12.** Show that for a good conductor ($g \gg \varepsilon \omega$) $K = \kappa + i\alpha \rightarrow \sqrt{\mu \varepsilon \omega} e^{i\pi/4}$ and hence infer from Eqs. (13.49) and (13.50) that the $\mathcal{H}$- and $\mathcal{E}$-fields are out of phase by 45° in a good conductor.

**P13.13.** Show that when $\varepsilon$, $\mu$, and $g$ are real, the complex index of refraction can be written in the form $\eta = n[1 + (2i/\mu g \omega \delta^2)]$. What does $\eta$ become in the two limits $g \gg \varepsilon \omega$ and $g \ll \varepsilon \omega$?

**P13.14.** Use an appropriate computer tool to produce your own copy of Fig. 13.3.

**P13.15.** Examine $\kappa$ and $\alpha$ as given by Eqs. (13.62) and (13.63) in the limits $g \ll \varepsilon \omega$ and $g \gg \varepsilon \omega$, sketch graphs of $\kappa$ and $\alpha$ versus $\varepsilon \omega/g$, and then use an appropriate computer tool to obtain careful graphs of these relationships.
P13.16. In the context of this section, high and low frequencies are divided by the frequency \( \omega_c = g/\epsilon \). Numerically, what is this frequency for good conductors \( (g \approx 10^7 \text{ mho/m}; \epsilon \approx \epsilon_0) \) and for poor conductors \( (g \approx 10^{-12} \text{ mho/m}; \epsilon \approx 2\epsilon_0) \)? In what region of the spectrum does \( \omega_c \) lie in each case?

P13.17. Show that a plane monochromatic wave having frequency \( \omega \) and wave number \( \kappa \), propagating in a nonconductor in a direction \( \hat{n} \) lying in the \( xz \) plane at an angle \( \theta \) to the \( z \) axis, and linearly polarized in the \( y \) direction is described by the fields

\[
\mathbf{E}(r, t) = E_0 \hat{\mathbf{j}} e^{i[\kappa(z \cos \theta + x \sin \theta) - \omega t]}
\]
\[
\mathbf{H}(r, t) = \sqrt{\frac{\mu}{\epsilon}} E_0 (\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{k}}) e^{i[\kappa(z \cos \theta + x \sin \theta) - \omega t]}
\]

The angle \( \theta \) is positive when measured from the positive \( z \) axis toward the positive \( x \) axis.

13.4 Transmission and Reflection at Plane Interfaces

When a beam of light is incident on the interface between two different media, in general some of the incident energy is reflected back into the medium of incidence and some is transmitted across the interface. An important problem in classical optics therefore is to relate the intensity of the transmitted and reflected beams to the characteristics of the incident beam and to the properties of the two media. Representing beams of light (which we assume to be monochromatic) by electromagnetic waves, we can now bring classical electromagnetic theory to bear on this optical problem.

13.4.1 The Grand Strategy

To be sure, we should represent a physical beam of light by a wave of finite (temporal) duration rather than by a monochromatic wave of infinite duration. Even when we recognize that macroscopic beams of light are usually made up of individual wave trains whose duration is comparable to the lifetime of excited atomic states \( (\approx 10^{-8} \text{ s}) \), however, the individual wave trains contain very many \( (\approx 6 \times 10^7 \text{ at } 500 \text{ nm}) \) complete cycles and each wave train is difficult indeed to distinguish from one that contains an infinite number of cycles. If the “light bulb” used as a source is a laser, in which wave trains from individual atoms are locked together into much longer wave trains, the approximation of infinite wave trains is all the more applicable. We do, however, confine our attention to interfaces whose area is large compared to the square of the wavelength of the light (so we can ignore diffraction) and to wavelengths that are large compared to atomic dimensions (so we can treat matter as a macroscopic continuum). Although we shall consider reflection and transmission only at plane interfaces, in fact (properly interpreted) our results will be valid also for curved interfaces provided the curvature is not significant over distances on the order of several wavelengths. With these restrictions on applicability, we therefore conclude that a suitable reflection problem theoretically involves picturing three plane monochromatic electromagnetic waves: an incident wave, a reflected wave, and a transmitted wave (Fig. 13.4). In a general problem, the amplitude, polarization, direction, and frequency of the incident wave will be given and we seek these characteristics of the reflected and transmitted waves. A suitable general strategy involves the following steps:
13.4. TRANSMISSION AND REFLECTION AT PLANE INTERFACES

Figure 13.4: The three waves present in a reflection/transmission problem. The medium of incidence lies in the region \( z < 0 \) and the medium of transmission lies in the region \( z > 0 \). The plane interface dividing these two media coincides with the \( xy \) plane.

1. Using whichever of the results in Section 13.3 is appropriate, write analytic expressions for the incident, reflected, and transmitted waves. Several unknown amplitudes, directions, etc., will be introduced.

2. Obtain a set of equations for the unknowns by imposing the appropriate boundary conditions at the interface.

3. Solve the resulting equations for the unknowns in terms of the (known) characteristics of the incident wave.

4. Calculate the rates 
   
   \[ I_i = |\langle \mathbf{S}_i \rangle \cdot \hat{k}|, \quad I_r = |\langle \mathbf{S}_r \rangle \cdot \hat{k}|, \quad \text{and} \quad I_t = |\langle \mathbf{S}_t \rangle \cdot \hat{k}|, \]

   at which the incident, reflected, and transmitted waves transport energy in a direction along the normal to the interface. Here, \( \hat{k} \) is the unit vector normal to the interface in Fig. 13.4.

5. Calculate the reflection and transmission coefficients \( R \) and \( T \) by applying the definitions

   \[ R = \left( \frac{\text{fraction of incident energy that is reflected}}{\text{energy that is reflected}} \right) = \frac{I_r}{I_i} = \frac{|\langle \mathbf{S}_r \rangle \cdot \hat{k}|}{|\langle \mathbf{S}_i \rangle \cdot \hat{k}|} \quad (13.82) \]

   \[ T = \left( \frac{\text{fraction of incident energy that is transmitted}}{\text{energy that is transmitted}} \right) = \frac{I_t}{I_i} = \frac{|\langle \mathbf{S}_t \rangle \cdot \hat{k}|}{|\langle \mathbf{S}_i \rangle \cdot \hat{k}|} \quad (13.83) \]
CHAPTER 13. PLANE ELECTROMAGNETIC WAVES

Figure 13.5: A reflection/transmission problem at normal incidence.

\[ \text{Plane interface} \]

\[ \begin{align*}
\text{Reflected wave} & \quad \rightarrow \\
\text{Incident wave} & \quad \rightarrow \\
\text{Transmitted wave} & \quad \leftarrow
\end{align*} \]

\[ x \quad \text{y (out)} \quad z \]

13.4.2 Reflection/Transmission at Normal Incidence

Let us illustrate the approach with a simple example before discussing the more general case of Fig. 13.4. Consider a linearly polarized plane wave propagating in a nonconductor and incident normally on the plane surface of an imperfect conductor. Let the incident direction define the \( z \) axis and let the interface be at \( z = 0 \). The three waves involved are shown in Fig. 13.5. We first write analytic expressions for these three waves. If the incident wave is polarized in the \( x \) direction, we take \( \mathbf{E}_i = E_{i0} \hat{i}, \mathbf{k} = \kappa_i \hat{k}, \) and \( \mathbf{n} = \hat{k} \) in Eqs. (13.78) and (13.79) to find that the incident wave is represented analytically by

\[ \mathbf{E}_i(z,t) = E_{i0} \hat{i} e^{i(\kappa_i z - \omega t)} \]
\[ \mathbf{H}_i(z,t) = \sqrt{\frac{\epsilon_i}{\mu_i}} E_{i0} \hat{j} e^{i(\kappa_i z - \omega t)} \]  

(incident) \hspace{1cm} (13.84)

Now, the frequencies of all three waves must be the same (Section 13.2) and hence by Eq. (13.76) the wave numbers of the incident and reflected waves must be the same. Thus, with \( \mathbf{E}_r = E_{r0} \hat{i}, \mathbf{n} = -\hat{k}, \) and \( \mathbf{k} = -\kappa_i \hat{k}, \) we find from Eqs. (13.78) and (13.79) that the reflected wave is represented analytically by

\[ \mathbf{E}_r(z,t) = E_{r0} \hat{i} e^{-i(\kappa_i z + \omega t)} \]
\[ \mathbf{H}_r(z,t) = -\sqrt{\frac{\epsilon_i}{\mu_i}} E_{r0} \hat{j} e^{-i(\kappa_i z + \omega t)} \]  

(reflected) \hspace{1cm} (13.85)

Finally, the analytic representation

\[ \mathbf{E}_t(z,t) = E_{t0} \hat{i} e^{i(K_t z - \omega t)} \]
\[ \mathbf{H}_t(z,t) = \frac{K_t}{\omega \mu_t} E_{t0} \hat{j} e^{i(K_t z - \omega t)} \]  

(transmitted) \hspace{1cm} (13.86)

for the transmitted wave follows from Eqs. (13.49) and (13.50) on setting \( \mathbf{E}_{t0} = E_{t0} \hat{i} \) and \( \mathbf{K} = K_t \hat{k} \). In Eqs. (13.85) and (13.86), we have made the reasonable (but tacit) “assumption”
13.4. TRANSMISSION AND REFLECTION AT PLANE INTERFACES

that the reflected and transmitted waves are polarized in the same direction as the incident wave.\textsuperscript{11} Throughout Eqs. (13.84)–(13.86), the dispersion relations

\[ \kappa_i = \omega \sqrt{\mu_i \epsilon_i} \; ; \; \quad K_t^2 = \mu_t \omega^2 \left( \epsilon_t + i \frac{g_t}{\omega} \right) \]  

(13.87)

are assumed.

We now find the unknown amplitudes in the reflected and transmitted waves by imposing the proper boundary conditions on the fields at \( z = 0 \). We must, of course, add the incident and reflected waves to find the total field in the medium of incidence. Then, identifying medium 1 with the medium of incidence and medium 2 with the medium of transmission, we find by applying the conditions on \( E_0^x \) and \( H_0^y \) in Eq. (13.36) that

\[ \epsilon_{i0} + \epsilon_{r0} = \epsilon_{t0} \; ; \; \quad \sqrt{\frac{\epsilon_i}{\mu_i}} (\epsilon_{i0} - \epsilon_{r0}) = \frac{K_t}{\omega \mu_t} \epsilon_{t0} \]  

(13.88)

(\textit{Remember:} \( z = 0 \)); all of the remaining conditions in Eq. (13.36) are automatically satisfied by the above fields. The solution of Eq. (13.88) for the unknown amplitudes \( \epsilon_{r0} \) and \( \epsilon_{t0} \) now is

\[ \epsilon_{r0} = \frac{1 - \zeta}{1 + \zeta} \epsilon_{i0} \; ; \; \quad \epsilon_{t0} = \frac{2}{1 + \zeta} \epsilon_{i0} \; ; \; \quad \text{where} \; \zeta = \frac{K_t}{\omega \mu_t} \sqrt{\frac{\mu_t}{\epsilon_t}} \]  

(13.89)

Finally, we find the reflection and transmission coefficients. From Eqs. (13.80) and (13.51) with \( \hat{n} = \pm \hat{k} \), we find first that

\[ I_i = |\langle S_i \rangle \cdot \hat{k}| = \frac{\kappa_i |\epsilon_{i0}|^2}{2 \omega \mu_i} \]

\[ I_r = |\langle S_r \rangle \cdot \hat{k}| = \frac{\kappa_i |\epsilon_{r0}|^2}{2 \omega \mu_i} \]

\[ I_t = |\langle S_t \rangle \cdot \hat{k}| = \frac{|\epsilon_{t0}|^2 \Re\{K_t\}}{2 \omega \mu_t} \]  

(13.90)

(\textit{Remember again:} \( z = \hat{n} \cdot \mathbf{r} = 0 \).) Substituting Eq. (13.90) into the definitions in Eqs. (13.82) and (13.83), we then find the expressions

\[ R = \frac{I_r}{I_i} = \frac{|\epsilon_{r0}|^2}{|\epsilon_{i0}|^2} = \left| \frac{1 - \zeta}{1 + \zeta} \right|^2 \]

\[ T = \frac{I_t}{I_i} = \frac{\mu_t \Re\{K_t\}}{\mu_t \kappa_i} \frac{|\epsilon_{t0}|^2}{|\epsilon_{i0}|^2} = \frac{4 \mu_t \Re\{K_t\}}{\mu_t \kappa_i |1 + \zeta|^2} \]  

(13.91)

for the reflection and transmission coefficients \( R \) and \( T \).

Two limiting cases of these general results are of interest. In the first case, the medium of transmission is a perfect conductor, for which \( g_t \to \infty \) and \( K_t \to \sqrt{\mu_t g_t \omega} \) (P13.12). Thus, \( \zeta \) as given by Eq. (13.89) becomes infinite in proportion to \( \sqrt{g_t} \) and, in accordance with Eqs. (13.89 and (13.91), we find that, as \( g_t \to \infty \),

\[ \epsilon_{r0} \to -\epsilon_{i0} \; ; \; \quad \epsilon_{t0} \to 0 \]  

\[ R \to 1 \; ; \; \quad T \to 0 \]  

(13.92)

(13.93)

\textsuperscript{11}It is shown in P13.18 that, in fact, this requirement follows from the boundary conditions.
Thus, if the medium of transmission is a good conductor, the incident wave is (almost) completely reflected and there is a 180° phase shift between the incident and reflected waves.

The second interesting limiting case occurs when the medium of transmission is a non-conductor, for which \( g_t = 0 \). Then \( K_t = \kappa_t = \omega \sqrt{\mu_t \epsilon_t} \) [Eq. (13.87)], and

\[
\zeta = \sqrt{\frac{\epsilon_t \mu_i}{\epsilon_i \mu_t}} \approx \sqrt{\frac{\epsilon_0}{\epsilon_i \epsilon_0}} = \frac{n_i}{n_t}
\] (13.94)

where \( n_i \) and \( n_t \) are the indices of refraction of the two media. [See Eq. (13.89), recall that \( \mu_i \) and \( \mu_t \approx \mu_0 \) for nonmagnetic, optically transparent dielectrics, and then note Eq. (13.75).] For reflection and transmission at normal incidence on an interface between two dielectrics, we therefore find from Eqs. (13.89) and (13.91) that

\[
E_{r0} = \frac{n_i - n_t}{n_i + n_t} E_{i0} \quad ; \quad E_{t0} = \frac{2n_i}{n_i + n_t} E_{i0}
\] (13.95)

\[
R = \left( \frac{n_i - n_t}{n_i + n_t} \right)^2 \quad ; \quad T = \frac{4n_i n_t}{(n_i + n_t)^2}
\] (13.96)

Thus, at normal incidence on a real interface \( (n_t \neq n_i) \) between two dielectrics, both the reflected and transmitted waves have nonzero amplitude and there is a phase shift of 0° or 180° between the incident and reflected waves depending on whether \( n_i > n_t \) or \( n_i < n_t \), respectively; the transmitted wave is always in phase with the incident wave. Graphs of \( R \) and \( T \) are shown in Fig. 13.6, and it is shown in P13.19 that \( R + T = 1 \).

13.4.3 Reflection/Transmission at Oblique Incidence

We now turn to the more general problem illustrated in Fig. 13.4. Let the interface define the \( xy \) plane; let the incident, reflected, and transmitted waves be characterized, respectively, by propagation vectors \( \mathbf{K}_i, \mathbf{K}_r, \) and \( \mathbf{K}_t \), making angles \( \theta_i, \theta_r, \) and \( \theta_t \) with the normal \( \mathbf{n} \) (here \( \mathbf{k} \)) to the interface; and let the plane of incidence determined by the vectors \( \mathbf{K}_i \) and \( \mathbf{n} \) define the \( yz \) plane. The angles \( \theta_i, \theta_r, \) and \( \theta_t \) are measured positively as shown by the arrows in Fig. 13.7 and are called the angle of incidence, the angle of reflection, and the angle of transmission or the angle of refraction, respectively.

13.4.3.1 General Requirements

Several general requirements on these waves follow from Maxwell’s equations and the boundary conditions:

1. The incident and reflected waves are characterized by the same value of \( K \), i.e., \( K_i = K_r \). We have already argued (Section 13.2) that all frequencies must be the same. In view of the dispersion relation in Eq. (13.48), the quantity \( K \) can therefore change only when the properties of the medium change. But the incident and reflected waves are in the same medium. Q.E.D.
2. The incident, reflected, and transmitted propagation vectors are all parallel to the plane of incidence. Each term in the boundary conditions at \( z = 0 \) will involve a factor of the form \( e^{iK \cdot r} \bigg|_{z=0} = e^{i(K_i x + K_i y)} \), where the vector \( \mathbf{K} \) will refer to one of the three waves. Continuity of the tangential component of \( \mathbf{E} \), for example, will require in part that

\[
\mathbf{E}^{(i)}_x e^{i(K_i x + K_i y)} + \mathbf{E}^{(r)}_x e^{i(K_r x + K_r y)} = \mathbf{E}^{(t)}_x e^{i(K_t x + K_t y)}
\]  

(13.97)

This boundary condition must be satisfied for arbitrary values of \( x \) and \( y \) (Why?), which is equivalent to requiring that the exponential factors cancel from Eq. (13.97). Cancellation will occur for arbitrary \( x \) and \( y \), however, only if

\[
K_{ix} = K_{rx} = K_{tx}
\]  

(13.98)

and

\[
K_{iy} = K_{ry} = K_{ty}
\]  

(13.99)

Now, the incident wave certainly lies in the plane of incidence, so \( K_{ix} = 0 \). Equation (13.98) then requires that \( K_{rx} = K_{tx} = 0 \). Q.E.D.

3. The angle of incidence is equal to the angle of reflection. From the geometry of Fig. 13.7,

\[
K_{iy} = -K_i \sin \theta_i \quad ; \quad K_{ry} = -K_r \sin \theta_r
\]

(13.100)

Since \( K_i = K_r \) [item (1) above], Eqs. (13.99) and (13.100) together yield the proof:

\[
K_{iy} = K_{ry} \quad \implies \quad \sin \theta_i = \sin \theta_r \quad \implies \quad \theta_i = \theta_r
\]  

(13.101)

This property is, of course, the classical law of specular reflection.

4. The angle of incidence and the angle of transmission satisfy Snell’s law,\(^{12}\) \( n_i \sin \theta_i = n_t \sin \theta_t \), where \( n_i \) and \( n_t \) are the complex indices of refraction of the two media. [See

\(^{12}\)Dutch astronomer and mathematician Willebrord Snellius, b. 13 June 1580 in Leiden, Dutch Republic; d. 30 October 1626 in Leiden, Dutch Republic.
Eq. (13.58). Again, combining the geometry of Fig. 13.7 with Eq. (13.99), we find that

\[ K_{iy} = K_{ty} \implies K_i \sin \theta_i = K_t \sin \theta_t \implies \eta_i \sin \theta_i = \eta_t \sin \theta_t \]  

(13.102)

where the final form follows after multiplying the original form by \( c/\omega \) and using Eq. (13.58). Q.E.D. In particular, when both media are nonconductors, \( \eta_i \) and \( \eta_t \) reduce to the ordinary (real) indices of refraction \( n_i \) and \( n_t \), and Eq. (13.102) becomes

\[ n_i \sin \theta_i = n_t \sin \theta_t \]  

(13.103)

Depending on the nature of the two media, Eq. (13.102) or Eq. (13.103) determines \( \theta_t \) when \( \theta_i \) is given. Now, \( \theta_i \) can range from 0 to \( \frac{1}{2} \pi \) and is a real angle. Since \( \eta_i \) and \( \eta_t \) are in general complex, however, Eq. (13.102) cannot be satisfied unless we sometimes interpret \( \theta_t \) as a complex angle. (See Appendix D.) Even when Eq. (13.103) applies and \( n_i \) and \( n_t \) are real, \( \theta_i \) will be complex when \( (n_i/n_t) \sin \theta_i > 1 \), which occurs when \( n_i > n_t \) and \( \sin \theta_i > n_t/n_i \). In the case of two nonconductors, complex \( \theta_t \) corresponds physically to what is called total reflection; there is a “transmitted” wave, but it has a number of curious properties and in particular all of the incident energy appears in the reflected wave (P13.52).

5. The incident wave may be polarized at any direction in a plane perpendicular to \( K_i \), and in particular an incident wave with an arbitrary polarization can be regarded as a superposition of a wave polarized perpendicular to the plane of incidence [Fig. 13.8(a)] and a wave polarized parallel to the plane of incidence [Fig. 13.8(b)]. It is therefore sufficient to treat only these two special cases.
6. If the incident wave is polarized parallel (perpendicular) to the plane of incidence, the reflected and transmitted waves are also polarized parallel (perpendicular) to the plane of incidence, as shown in Fig. 13.8. To prove this property, assume an incident wave containing only one polarization, assume reflected and transmitted waves containing both polarizations, and then show that the boundary conditions at the interface cannot be satisfied unless the “wrong” polarizations in the reflected and transmitted waves have zero amplitude. The details are left to P13.25.

13.4.3.2 The Fresnel Equations

Let the wave now be incident obliquely on the interface. As the first step in the grand strategy of Section 13.4.1, we introduce the (as yet) unknown amplitudes $E_{i0}$, $E_{r0}$, and $E_{t0}$,
in terms of which, for a wave we take to be polarized perpendicular to the plane of incidence, we write that
\[ \mathbf{E}_{i0} = \mathbf{E}_{i0} \hat{i} \quad ; \quad \mathbf{E}_{r0} = \mathbf{E}_{r0} \hat{i} \quad ; \quad \mathbf{E}_{t0} = \mathbf{E}_{t0} \hat{i} \] (13.104)

Further, from the geometry of Fig. 13.7, we conclude that
\[ \begin{align*}
K_i &= K_i \left[ \cos \theta_i \hat{k} - \sin \theta_i \hat{j} \right] \\
K_r &= K_i \left[ -\cos \theta_i \hat{k} - \sin \theta_i \hat{j} \right] \\
K_t &= K_i \left[ \cos \theta_t \hat{k} - \sin \theta_t \hat{j} \right]
\end{align*} \] (13.105)

We now find analytic expressions for the waves by substituting Eqs. (13.104) and (13.105) into Eqs. (13.49) and (13.50); the results are
\[ \begin{align*}
\mathbf{E}_i(r, t) &= \mathbf{E}_{i0} \hat{i} e^{i(K_i\cdot r - \omega t)} \\
\mathbf{H}_i(r, t) &= \frac{K_i}{\omega \mu_i} \mathbf{E}_{i0} \hat{i} e^{i(K_i\cdot r - \omega t)}
\end{align*} \] (13.106)

\[ \begin{align*}
\mathbf{E}_r(r, t) &= \mathbf{E}_{r0} \hat{i} e^{i(K_r\cdot r - \omega t)} \\
\mathbf{H}_r(r, t) &= \frac{K_i}{\omega \mu_i} \mathbf{E}_{r0} \hat{i} e^{i(K_r\cdot r - \omega t)}
\end{align*} \] (13.107)

\[ \begin{align*}
\mathbf{E}_t(r, t) &= \mathbf{E}_{t0} \hat{i} e^{i(K_t\cdot r - \omega t)} \\
\mathbf{H}_t(r, t) &= \frac{K_t}{\omega \mu_t} \mathbf{E}_{t0} \hat{i} e^{i(K_t\cdot r - \omega t)}
\end{align*} \] (13.108)

The boundary conditions in Eq. (13.36) are applicable to the present case, with medium 1 identified with the medium of incidence and medium 2 identified with the medium of transmission. Many of these conditions are automatically satisfied; only the conditions on \( \mathbf{E}_{0x} \), \( \mathbf{H}_{0y} \) and \( \mathbf{H}_{0z} \) are nontrivial, and they yield
\[ \mathbf{E}_{i0} + \mathbf{E}_{r0} = \mathbf{E}_{t0} \] (13.109)
\[ \frac{K_i}{\mu_i} (\mathbf{E}_{i0} - \mathbf{E}_{r0}) \cos \theta_i = \frac{K_t}{\mu_t} \mathbf{E}_{t0} \cos \theta_t \] (13.110)
\[ K_i (\mathbf{E}_{i0} + \mathbf{E}_{r0}) \sin \theta_i = K_t \mathbf{E}_{t0} \sin \theta_t \] (13.111)

where the exponential factors must cancel and will if (as we have already assumed) the conditions discussed in items (2)–(4) of Section 13.4.3 are satisfied. Now, since \( K_i \sin \theta_i = K_t \sin \theta_t \) [Eq. (13.102)], Eq. (13.111) is identical with Eq. (13.109). Thus, Eqs. (13.109) and (13.110) are two equations for the two unknowns \( \mathbf{E}_{r0} \) and \( \mathbf{E}_{i0} \). In terms of the complex index of refraction [Eq. (13.58)], Eq. (13.110) assumes the more convenient form
\[ \mu_t \eta_i (\mathbf{E}_{i0} - \mathbf{E}_{r0}) \cos \theta_i = \mu_t \eta_t \mathbf{E}_{t0} \cos \theta_t \] (13.112)

Finally, simultaneous solution of Eqs. (13.109) and (13.112) yields the results
\[ \begin{align*}
\mathbf{E}_{r0} &= \frac{\mu_i \eta_i \cos \theta_i - \mu_t \eta_t \cos \theta_t}{\mu_i \eta_i \cos \theta_i + \mu_t \eta_t \cos \theta_t} \mathbf{E}_{i0} \quad (13.113)
\end{align*} \]

\[ \begin{align*}
\mathbf{E}_{t0} &= \frac{2 \mu_i \eta_i \cos \theta_i}{\mu_i \eta_i \cos \theta_i + \mu_t \eta_t \cos \theta_t} \mathbf{E}_{i0} \quad (13.114)
\end{align*} \]

\(^{13}\)Remember: \( \theta_r = \theta_i \) and \( K_r = K_i \).
where the superscript $\perp$ has been added as a reminder that these results apply when the incident wave is polarized perpendicular to the plane of incidence. Equations (13.113) and (13.114) are two of Fresnel’s equations, the other two being

$$
\epsilon^{\parallel}_{r0} = \frac{-\mu_i \mu_0 \cos \theta_i + \mu_0 \mu_t \cos \theta_t}{\mu_i \rho_t \cos \theta_i + \mu_t \rho_i \cos \theta_t} \epsilon^{\parallel}_{i0}
$$
(13.115)

$$
\epsilon^{\parallel}_{i0} = \frac{2\mu_i \mu_0 \cos \theta_i}{\mu_i \rho_t \cos \theta_i + \mu_t \rho_i \cos \theta_t} \epsilon^{\parallel}_{i0}
$$
(13.116)

which apply to the case of an incident wave polarized parallel to the plane of incidence provided unit vectors in the direction of the electric field are chosen to point in the directions indicated in Fig. 13.8(b) (P13.26). These equations, which give the amplitudes of the reflected and transmitted waves for the two polarizations, are quite general; they apply not only when $\eta_i$ and $\eta_t$ are complex but also when $\theta_t$ must be interpreted as a complex angle.

### 13.4.3.3 The Consequences of Fresnel’s Equations

We shall examine the predictions of Fresnel’s equations only for the special case in which both media are nonmagnetic ($\mu_i = \mu_t = \mu_0$) and are nonconductors ($\eta_i = n_i$, $\eta_t = n_t$). Further, we shall restrict our considerations to circumstances for which $\theta_t$ is real (i.e., we do not consider total reflection). Under these circumstances, Fresnel’s equations reduce to

$$
\epsilon^{\perp}_{r0} = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \epsilon^{\perp}_{i0}
$$
(13.117)

$$
\epsilon^{\perp}_{i0} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} \epsilon^{\perp}_{i0}
$$
(13.118)

$$
\epsilon^{\parallel}_{r0} = \frac{-n_t \cos \theta_i + n_i \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \epsilon^{\parallel}_{i0}
$$
(13.119)

$$
\epsilon^{\parallel}_{i0} = \frac{2n_i \cos \theta_i}{n_t \cos \theta_i + n_t \cos \theta_t} \epsilon^{\parallel}_{i0}
$$
(13.120)

Further, we find from the definitions in Eqs. (13.82) and (13.83) that the reflection and transmission coefficients for the two polarizations are given by

$$
R_{\perp} = \frac{I_r}{I_i} = \left| \langle \mathbf{S}_r \rangle \cdot \mathbf{k} \right| = \frac{\| \epsilon^{\perp}_{r0} \|^2 \cos \theta_i}{\| \epsilon^{\perp}_{i0} \|^2 \cos \theta_i} = \left( \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \right)^2
$$
(13.121)

$$
T_{\perp} = \frac{I_t}{I_i} = \left| \langle \mathbf{S}_t \rangle \cdot \mathbf{k} \right| = \sqrt{\mu_i \mu_0 \| \epsilon^{\perp}_{i0} \|^2 \cos \theta_i} = \frac{4n_i n_t \cos \theta_i \cos \theta_t}{(n_t \cos \theta_i + n_t \cos \theta_t)^2}
$$
(13.122)

$$
R_{\parallel} = \left( \frac{-n_t \cos \theta_i + n_i \cos \theta_t}{n_t \cos \theta_i + n_t \cos \theta_t} \right)^2
$$
(13.123)

$$
T_{\parallel} = \frac{4n_i n_t \cos \theta_i \cos \theta_t}{(n_t \cos \theta_i + n_t \cos \theta_t)^2}
$$
(13.124)
where the Poynting vectors have been evaluated by substituting Eqs. (13.104) and (13.105) (and the analogous expressions for the parallel polarization) into Eq. (13.80), and the multiplying square root in Eq. (13.122) has been written using the approximations $\mu = \mu_0$ and $\sqrt{\epsilon} = n\sqrt{\epsilon_0}$ for both media. We note now the following properties of these results:

7. For both polarizations, all of these results reduce to Eqs. (13.95) and (13.96) at normal incidence ($\theta_i = \theta_t = 0$), for which special case the plane of incidence is not defined.

8. For both polarizations, the transmitted wave and the incident wave are in phase, because the coefficients multiplying $\xi_{i0}^\perp$ and $\xi_{i0}^\parallel$ in Eqs. (13.118) and (13.120) are real and positive.

9. For the perpendicular polarization, the reflected wave and the incident wave are in phase when $n_t < n_i$ and $180^\circ$ out of phase when $n_t > n_i$. At normal incidence ($\theta_i = \theta_t = 0$), the coefficient in Eq. (13.117) is positive when $n_t < n_i$ and negative when $n_t > n_i$. At oblique incidence, the coefficient will be positive when

$$n_i \cos \theta_i - n_t \cos \theta_t > 0$$

(13.125)

or (in view of Snell’s law and some trigonometric identities) when

$$n_i \cos \theta_i - \frac{n_i \sin \theta_i}{\sin \theta_t} \cos \theta_t = \frac{n_t \sin(\theta_t - \theta_i)}{\sin \theta_t} > 0 \implies \theta_t > \theta_i$$

(13.126)

Again from Snell’s law, however, $\theta_t > \theta_i$ requires that $n_t < n_i$. In reverse, the coefficient will be negative when $\theta_t < \theta_i$ or $n_t > n_i$, and all aspects of property (9) are established.

10. For the parallel polarization, the reflected wave and the incident wave are in phase (a) if $n_t < n_i$ and $\theta_t + \theta_i < \frac{1}{2}\pi$ or (b) if $n_t > n_i$ and $\theta_t + \theta_i > \frac{1}{2}\pi$ and out of phase otherwise. For normal incidence, $\theta_t + \theta_i = 0$ and case (a) applies. At normal incidence, the coefficient in Eq. (13.119) is positive for $n_t < n_i$ and negative for $n_t > n_i$. At oblique incidence, we use Snell’s law (and some trigonometric identities) to rewrite the numerator of that coefficient in the form

$$-n_t \cos \theta_i + n_i \cos \theta_t = \frac{n_i [\sin 2\theta_i - \sin 2\theta_t]}{2 \sin \theta_i} = \frac{n_t \sin(\theta_t - \theta_i) \cos(\theta_t + \theta_i)}{2 \sin \theta_i}$$

(13.127)

which can be positive only if $\theta_t > \theta_i$ ($n_t < n_i$) and $\theta_t + \theta_i < \frac{1}{2}\pi$ or if $\theta_t < \theta_i$ ($n_t > n_i$) and $\theta_t + \theta_i > \frac{1}{2}\pi$. Q.E.D.

11. For the perpendicular polarization, the transmitted wave has zero amplitude only at grazing incidence ($\theta_i = \frac{1}{2}\pi$) and the reflected wave has zero amplitude only in the trivial case of no interface, $n_t = n_i$. From Eq. (13.118), $\xi_{i0}^\perp = 0$ only if $\theta_i = \frac{1}{2}\pi$. For the reflected wave at normal incidence ($\theta_i = \theta_t = 0$), $\xi_{i0}^\perp$ as given by Eq. (13.117) will be zero only if $n_t = n_i$. For the reflected wave at oblique incidence, $\xi_{i0}^\perp$ can be zero only if $\theta_t = \theta_i$ [see Eq. (13.126)], which implies $n_t = n_i$. Q.E.D.

12. For the parallel polarization, the transmitted wave has zero amplitude only at grazing incidence ($\theta_i = \frac{1}{2}\pi$) and the reflected wave has zero amplitude not only for the trivial case of no interface, $n_t = n_i$, but also when $n_t \neq n_i$ and $\theta_t + \theta_i = \frac{1}{2}\pi$, equivalent to
\[ \theta_i = \tan^{-1}(n_t/n_i). \]
The proofs for the transmitted wave and for the reflected wave when \( n_t = n_i \) are similar to those in item (11) above and are left to the reader. The remainder of this property follows from Eq. (13.127), which gives the numerator in Eq. (13.119). When \( n_t \neq n_i \) (\( \theta_t \neq \theta_i \)), Eq. (13.127) yields zero for that numerator if \( \theta_t + \theta_i = \frac{1}{2}\pi \). Q.E.D. Snell’s law in turn then gives

\[ n_i \sin \theta_i = n_t \sin(\frac{1}{2}\pi - \theta_i) = n_t \cos \theta_i \quad \implies \quad \theta_i = \tan^{-1}\left(\frac{n_t}{n_i}\right) \]  

(13.128)

The critical angle \( \theta_B = \tan^{-1}(n_t/n_i) \) is called Brewster’s angle.\(^{15}\) Unpolarized light incident at this angle results in a reflected beam polarized perpendicular to the plane of incidence, and this phenomenon can be used to produce polarized light.

Several graphs showing various aspects of the results in Eqs. (13.117)–(13.124) appear in Fig. 13.9.

### 13.4.3.4 Optics as Part of Electromagnetic Theory

We conclude this discussion with a reminder that all of the results contained herein have been obtained by formal application of the principles of electromagnetic theory. We have made no reference to direct results of optical experiments per se. Thus, as we have obtained them, the results of this section are predictions of optical behavior, not summaries of experimental observations, even though in most cases the optical properties were known long before electromagnetic theory was developed. The agreement of these predictions with observations is therefore substantial indirect evidence supporting the correctness of Maxwell’s equations and represents one of the major achievements of late nineteenth-century physics.

#### PROBLEMS

**P13.18.** Add waves polarized in the \( y \) direction to Eqs. (13.85) and (13.86) and show that the boundary conditions cannot be satisfied unless the amplitudes of these added waves are zero.

**P13.19.** Show that \( R + T = 1 \), with \( R \) and \( T \) given by Eq. (13.96). **Optional:** Show that \( R + T = 1 \), with \( R \) and \( T \) given by Eq. (13.91).

**P13.20.** Find an expression for the transmission coefficient given by Eq. (13.91) in the limit of large (but not infinite) conductivity. Take the medium of incidence to be vacuum and assume \( \mu_t = \mu_0 \). Substitute a typical conductivity for a good conductor (\( \sigma_t \approx 10^7 \) mho/m) and determine numerical values of \( T \) for radio waves, microwaves, and visible light.

**P13.21.** Substitute Eq. (13.92) into Eqs. (13.84)–(13.86) to obtain expressions for the incident and reflected waves when the plane wave in Eq. (13.84) is incident normally on a perfect conductor and then use the boundary conditions in Eq. (13.38) to find the surface charge density and the surface current density induced on the surface of the conductor.

**P13.22.** Use a suitable computer tool to obtain careful graphs of \( \theta_t \) versus \( \theta_i \) as given by Eq. (13.103) for several different values of \( n_i/n_t \). Include values both larger and smaller than unity.

---

\(^{15}\)Scottish physicist David Brewster, b. 11 December 1781 in Canongate, Jedburgh, Roxburghshire; d. 10 February 1868 in Allerly House, Gattonside, Roxburghshire.
Figure 13.9: Reflection and transmission coefficients as a function of angle of incidence. Graphs are shown for both incident polarizations and for several values of \( n_t/n_i \).

**P13.23.** Use a suitable computer tool to reproduce the graphs in Fig. 13.6.

**P13.24.** Assume \( \theta_i \) and \( \theta_t \) are real in Eq. (13.102). Then the real part of the equation is identical with Eq. (13.103). Show that the imaginary part of the equation can be written in the form

\[
\frac{\mu_i g_i}{n_i} \sin \theta_i = \frac{\mu_t g_t}{n_t} \sin \theta_t
\]

and that this form is identical with Eq. (13.103) when both media are good conductors.

**P13.25.** Following the method outlined in the text, prove property (6) for one or the other of the polarizations.

**P13.26.** Write down equations analogous to Eqs. (13.104) and (13.105) but describing waves polarized parallel to the plane of incidence, find the fields in the three waves, and apply the boundary conditions to derive Eqs. (13.115) and (13.116). You may find a symbol manipulating computer program to be useful, but do not use it blindly.

**P13.27.** Derive the law of reflection from a perfect conductor by considering a wave that is linearly polarized perpendicular to the plane of incidence and incident obliquely on the plane surface of a perfect conductor. Find also the surface charge density \( \sigma \) and the surface current density \( j \) induced on the surface of the conductor. You may find a symbol manipulating computer program to be useful, but do not use it blindly. **Optional:** Do the same for an incident wave polarized in the plane of incidence.

**P13.28.** A linearly polarized, monochromatic plane wave having wavelength \( \lambda \) (in vacuum) is incident normally on a thin, transparent dielectric film in vacuum. Let the film have thickness
d and index of refraction n. Determine the transmission and reflection coefficients and sketch graphs of each as functions of λ, assuming n to be independent of frequency. **Hint:** Take the plane of the film to be vertical and introduce five waves: incident and reflected waves in the medium of incidence, left and right traveling waves in the film, and a transmitted wave in the medium of transmission. This problem involves techniques similar to those used in the theory of non-reflecting coatings. You may find a symbol manipulating computer program to be useful, but do not use it blindly.

P13.29. Use a suitable computer tool to obtain graphs of Brewster’s angle and of the critical angle for total reflection as functions of $n_t/n_i$ over the range $0 < n_t/n_i < 10$ and, in particular, show that (when both angles exist) Brewster’s angle is always the smaller one. Note that only real angles in the range $0°$ to $90°$ are physically meaningful.

P13.30. (a) Find the expression to which $R_\perp$ as given by Eq. (13.121) reduces when the angle of incidence is Brewster’s angle. (b) Find Brewster’s angle and $R_\perp$ numerically if $n_i = 1$ and $n_t = 1.5$. (c) Obtain a graph of $R_\perp$ versus $n_t/n_i$ for incidence at Brewster’s angle. (d) Three glass plates with $n = 1.45$, 1.55, and 1.70 are available. Which would you select to build a polarizer to use with incident beams in air? Why?

### 13.5 Wave Guides and Cavity Resonators

We shall consider now some characteristics of monochromatic solutions to Maxwell’s equations in regions that are partially or totally bounded by perfectly conducting surfaces, beginning with a wave guide consisting of an evacuated hollow pipe made of a perfect conductor and having a rectangular cross section (Fig. 13.10). The most general monochromatic electromagnetic field in this wave guide can be very complicated, but it can also be constructed as a superposition of simpler basic fields or modes.

#### 13.5.1 A Simple Transverse Electric Mode

Suppose first that we seek a solution to Maxwell’s equations having an electric field that is (1) independent of $x$, (2) polarized parallel to the $x$ axis, and (3) propagating in the positive $z$ direction along the guide. The (complex) field in such a solution has the analytic representation

$$\mathcal{E}(r, t) = \mathcal{E}_0(y) \hat{i} e^{i(\kappa_z z - \omega t)}$$

and, in accordance with Eq. (13.22), the associated (complex) magnetic intensity is

$$\mathcal{H}(r, t) = \nabla \times \mathcal{E}(r, t) = \frac{1}{i\omega \mu_0} \left( i\kappa_z \mathcal{E}_0(y) \hat{j} - \frac{\partial \mathcal{E}_0(y)}{\partial y} \hat{k} \right) e^{i(\kappa_z z - \omega t)}$$

Now, the field $\mathcal{E}_0(r)$ obtained from Eq. (13.129) by deleting the factor $e^{-i\omega t}$ must satisfy Eq. (13.25) with $\mu_\epsilon = \mu_0 \epsilon_0 = 1/c^2$ and $g = 0$. Thus, if we set

$$\kappa_y^2 = -\kappa_z^2 + \frac{\omega^2}{c^2}$$

then $\mathcal{E}_0(y)$ must satisfy

$$\frac{d^2 \mathcal{E}_0}{dy^2} + \kappa_y^2 \mathcal{E}_0 = 0$$
the most general solution of which is
\[ \mathbf{E}_0(y) = A \sin \kappa_y y + B \cos \kappa_y y \quad (13.133) \]

As in Eq. (13.38), the tangential components of \( \mathbf{E}_0(\mathbf{r}) \) must now be made zero at all (perfectly) conducting surfaces. In particular, the \( x \) component of \( \mathbf{E}_0(\mathbf{r}) \) must be zero in sides 1 and 3 in Fig. 13.10 and will be so only if \( \mathbf{E}_0(0) = \mathbf{E}_0(b) = 0 \) or if \( B = 0 \) and \( \kappa_y b = n\pi \), where \( n = 1, 2, 3, \ldots \). With these restrictions on \( \mathbf{E}_0(y) \), we find finally that the fields in the wave guide are given by

\[
\mathbf{E}_{0n}^{\text{TE}}(\mathbf{r}, t) = A \sin \left( \frac{n\pi y}{b} \right) \hat{i} e^{i(\kappa_z z - \omega t)} 
\]

\[
\mathbf{H}_{0n}^{\text{TE}}(\mathbf{r}, t) = \frac{A}{i\omega \mu_0} \left[ i\kappa_z \sin \left( \frac{n\pi y}{b} \right) \hat{j} - \frac{n\pi}{b} \cos \left( \frac{n\pi y}{b} \right) \hat{k} \right] e^{i(\kappa_z z - \omega t)} 
\]

Here, the subscripts and superscripts have been added to specify the mode to which these fields apply: The superscript TE indicates that the mode is a transverse electric mode, in which the electric field has no component in the direction of propagation along the guide (i.e., the electric field is transverse to this direction of propagation); the subscript \( 0n \) is a conventional two-index subscript that relates to the dependence of the fields on \( x \) and \( y \), the value 0 here for the first index indicating that the fields are those in a mode that does not depend on \( x \). Finally, the equation

\[ \kappa_z^2 = \frac{\omega^2}{c^2} - \left( \frac{n\pi}{b} \right)^2 \quad (13.136) \]

determining \( \kappa_z \) from \( \omega \) for the TE\(_{0n} \) mode follows from Eq. (13.131) when \( \kappa_y \) is set equal to \( n\pi/b \). The reader may now verify that all of the boundary conditions in Eq. (13.38) either are satisfied by the fields in Eqs. (13.134) and (13.135) or yield values for surface currents and surface charges at each conducting boundary of the wave guide. Thus, Eqs. (13.134)–(13.136) certainly express a solution for the fields in the guide, even though they do not express the only solution.

Several properties of this guided wave are of interest. Note first that \( \kappa_z \) as given by Eq. (13.136) is purely imaginary if

\[ \omega < \omega_c = \frac{n\pi c}{b} \quad (13.137) \]
where $\omega_c$ is a (lower) cutoff frequency below which the exponential factor in Eqs. (13.134) and (13.135) expresses a decay rather than a sinusoidal oscillation. Thus, waves for which $\omega < \omega_c$ or, equivalently, waves having a vacuum wavelength $\lambda > \lambda_c = 2b/n$ are attenuated rather than propagated in the guide. Practically, guides whose dimensions are on the order of centimeters are most common. Thus, the typical laboratory guide has a cutoff wavelength on the order of a few centimeters and propagates waves in the microwave region (and at shorter wavelengths) but attenuates waves at longer wavelengths.

A second property of guided waves can be inferred if we write the sine in Eq. (13.134) in its complex exponential form, finding that

$$E_{TE}^{\theta n}(r, t) = \frac{A}{2i} e^{i[(n\pi y/b) + \kappa z - \omega t]} - e^{i[-(n\pi y/b) + \kappa z - \omega t]}$$

(13.138)

We can thus interpret the solution as the superposition of two plane waves, the first having a propagation vector

$$\kappa_1 = \frac{n\pi}{b} \hat{j} + \kappa_z \hat{k}$$

(13.139)

and the second, whose electric field is 180° out of phase with that of the first, having a propagation vector

$$\kappa_2 = -\frac{n\pi}{b} \hat{j} + \kappa_z \hat{k}$$

(13.140)

The relative orientation of these two vectors is shown in Fig. 13.11. Since $\kappa_1$ and $\kappa_2$ have the same $z$ component and have $y$ components differing only in sign, they both make the same angle with the walls, namely

$$\theta_1 = \theta_2 = \tan^{-1} \left( \frac{\kappa_z b}{n\pi} \right)$$

(13.141)
and we can therefore think of either wave as the reflection of the other in one of the walls. Even the phase difference between the two waves is consistent with that view. Propagation of this wave along the guide therefore involves successive reflections of the wave first from wall \( \text{①} \), then from wall \( \text{③} \), then from wall \( \text{①} \) again, and so on. The angle at which the wave is incident on the walls decreases as the mode number \( n \) increases, so higher modes involve more reflections per unit length along the guide than lower modes. At the cutoff frequency for the \( n \)-th mode (\( \kappa_z = 0 \)), Eq. (13.141) gives \( \theta_1 = \theta_2 = 0 \), and the two waves described in this paragraph are incident normally on the walls; \( \kappa_1 \) and \( \kappa_2 \) have no component along the guide and no wave is propagated.

Several different velocities and several different wavelengths are used to describe the guided wave in Eqs. (13.134) and (13.135). As measured in the direction of the vectors \( \kappa_1 \) and \( \kappa_2 \), for example, the wave is characterized by a wavelength \( \lambda_0 \) given by
\[
\lambda_0 = \frac{2\pi}{\sqrt{\kappa_y^2 + \kappa_z^2}} = \frac{2\pi c}{\omega} \tag{13.142}
\]
and propagates with speed \( c \); the wavelength \( \lambda_0 \) is the vacuum wavelength corresponding to the frequency \( \omega \). As measured across the guide, the wave is characterized by a wavelength \( \lambda_\perp \) given by
\[
\lambda_\perp = \frac{2\pi}{\kappa_y} = \frac{2b}{n} \quad \implies \quad b = \frac{1}{2} n \lambda_\perp \tag{13.143}
\]
which supports the conclusion that the allowed wavelengths for a given guide are those that establish standing wave patterns between two of the walls. Finally, as measured along the guide, the wave is characterized by a wavelength \( \lambda_\parallel \) given by
\[
\lambda_\parallel = \frac{2\pi}{\kappa_z} = \frac{2\pi}{\sqrt{\left(\frac{\omega^2}{c^2} - \left(\frac{n\pi}{b}\right)^2\right)}} = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_\perp}\right)^2}} \tag{13.144}
\]
and propagates with a speed \( v_\parallel \) given by
\[
v_\parallel = \frac{\omega}{\kappa_z} = \frac{\omega}{\sqrt{\left(\frac{\omega^2}{c^2} - \left(\frac{n\pi}{b}\right)^2\right)}} = \frac{c}{\sqrt{1 - \left(\frac{n\pi c}{b\omega}\right)^2}} \tag{13.145}
\]
Note that \( \lambda_\parallel > \lambda_0 \) and \( v_\parallel > c \). This apparent contradiction of the limitations imposed by special relativity will be resolved in the next paragraph when we find that information is transmitted along the guide not with speed \( v_\parallel \) but with a different speed that in fact is less than \( c \).

Consider now the energy in the fields in this guide. We find first that the time-averaged Poynting vector is given by
\[
\langle S \rangle = \frac{1}{2} \Re\{\mathbf{E} \times \mathbf{H}^*\} = \frac{\lambda_0}{2\omega\mu_0} |A|^2 \sin^2 \left(\frac{n\pi y}{b}\right) \hat{k} \tag{13.146}
\]
As expected, \( \langle S \rangle \) is directed along the guide. Similarly, we find that the time-averaged energy density is given by
\[
\langle u_{\text{EM}} \rangle = \frac{1}{4} \Re\{\mathbf{E} \cdot \mathbf{D}^* + \mathbf{B} \cdot \mathbf{H}^*\}
= \frac{1}{4} |A|^2 \left[ \left(\varepsilon_0 + \frac{\kappa_z}{\mu_0\omega^2}\right) \sin^2 \left(\frac{n\pi y}{b}\right) + \frac{1}{\mu_0\omega^2} \left(\frac{n\pi}{b}\right)^2 \cos^2 \left(\frac{n\pi y}{b}\right) \right] \tag{13.147}
\]
Neither the energy flux nor the energy density is uniform over the cross section of the guide. We can, however, calculate the average rate at which energy is transported along the guide by integrating \( \langle S \rangle \) over that cross section; we find that

\[
\left( \text{average rate of energy transport} \right) = \int \langle S \rangle \cdot dS = \int_a^b \frac{k_z}{2\omega\mu_0} |A|^2 \sin^2 \left( \frac{n\pi y}{b} \right) (a \, dy) = \frac{abk_z}{4\omega\mu_0} |A|^2 \tag{13.148}
\]

Similarly, we find that the average energy stored in the fields in unit length of the guide is given by

\[
\left( \text{average energy in unit length} \right) = \int_a^b \langle u_{EM} \rangle \, a \, dy = \frac{1}{4}ab\varepsilon_0 |A|^2 \tag{13.149}
\]

If we now think of the average energy given by Eq. (13.149) as propagating down the guide with speed \( v_g \) so as to produce the average energy transport given by Eq. (13.148), it must be that

\[
\left( \text{average rate of energy transport} \right) = v_g \left( \text{average energy in unit length} \right) \tag{13.150}
\]

and thus that

\[
v_g = \frac{k_z}{\omega\mu_0\varepsilon_0} = \frac{c^2}{v_\parallel} \quad \implies \quad v_\parallel v_g = c^2 \tag{13.151}
\]

Since \( v_\parallel > c \), the speed \( v_g \) at which energy is propagated along the guide is smaller than \( c \), in agreement with the limits imposed by special relativity.

Another aspect of energy in guided waves relates to losses arising from the finite (even though large) conductivities of any real material used for the walls; the essential idea of a method for treating this complication is explored in P13.54.

13.5.2 Other Modes

As our notation indicates, we have considered only a very few of the possible waves in a rectangular wave guide. There are additional TE modes depending only on \( x \) and not on \( y \) (the TE\(_{m0}\) modes), there are more involved TE modes depending on both \( x \) and \( y \) (the TE\(_{mn}\) modes), and there is an analogous family of transverse magnetic modes (TM\(_{mn}\)) in which the magnetic intensity \( \mathbf{H} \) is normal to the direction of propagation along the guide. Further, for each mode representing propagation toward \( z = +\infty \) there is a corresponding mode representing propagation toward \( z = -\infty \). The main characteristics of the propagation of guided waves, however, have emerged in our simple example and we therefore leave a treatment of these other modes (and also of wave guides having non-rectangular cross sections) to the problems and to other authors. We leave a discussion of wave guides that turn corners, change dimension, or contain dielectric fillers entirely to other authors.

13.5.3 Cavity Resonators

As a second example of fields in a region bounded by conductors, consider a cavity resonator, which consists of an evacuated volume completely surrounded by conducting walls. The simplest such devices are shaped like rectangular parallelepipeds, right circular cylinders, and spheres. We shall consider here only the first shape (Fig. 13.12). Again we seek
simple solutions in the interior of this region, realizing that more general solutions can be constructed by superposition. Thus, let us assume that

$$\mathbf{E}(r, t) = \mathbf{E}_0(r) e^{-i\omega t}$$

with

$$\mathbf{E}_0(r) = \mathbf{E}_{0x}(r) \hat{i} + \mathbf{E}_{0y}(r) \hat{j} + \mathbf{E}_{0z}(r) \hat{k}$$

(13.153)

Now, $\mathbf{E}_0(r)$ and hence each of its Cartesian components must satisfy Eq. (13.25) with $\mu\epsilon = \mu_0\epsilon_0 = 1/c^2$ and $g = 0$; e.g.,

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) \mathbf{E}_{0x}(x, y, z) = 0$$

(13.154)

Assuming that $\mathbf{E}_{0x}(x, y, z)$ can be factored into the form $X(x) Y(y) Z(z)$ (as in Section 8.4), we find that Eq. (13.154) is equivalent to

$$\frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} + \frac{1}{Z} \frac{d^2Z}{dz^2} + \frac{\omega^2}{c^2} = 0$$

(13.155)

which can be satisfied only if each of the first three terms is a constant by itself. Writing the separation constants as $-\kappa^2_x$, $-\kappa^2_y$, and $-\kappa^2_z$, we thus have that

$$\frac{d^2X}{dx^2} + \kappa^2_x X = 0 \implies X(x) = A \sin \kappa_x x + B \cos \kappa_x x$$

(13.156)

$$\frac{d^2Y}{dy^2} + \kappa^2_y Y = 0 \implies Y(y) = A' \sin \kappa_y y + B' \cos \kappa_y y$$

(13.157)

$$\frac{d^2Z}{dz^2} + \kappa^2_z Z = 0 \implies Z(z) = A'' \sin \kappa_z z + B'' \cos \kappa_z z$$

(13.158)

where the separation constants must satisfy

$$\kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \frac{\omega^2}{c^2}$$

(13.159)
The relevant boundary conditions, Eq. (13.38), require that the tangential components of $\mathbf{E}$ be zero on all conducting surfaces. Since $\mathbf{E}_0(x, y, z)$ is a tangential component on four of the six surfaces, we must then have that

$$
\mathbf{E}_0(x, 0, z) = 0 \implies Y(0) = 0 \implies B' = 0 \quad (13.160)
$$
$$
\mathbf{E}_0(x, b, z) = 0 \implies Y(b) = 0 \implies \kappa_y b = n\pi \quad (13.161)
$$
$$
\mathbf{E}_0(x, y, 0) = 0 \implies Z(0) = 0 \implies B'' = 0 \quad (13.162)
$$
$$
\mathbf{E}_0(x, y, d) = 0 \implies Z(d) = 0 \implies \kappa_z d = p\pi \quad (13.163)
$$

where $n$ and $p$ are non-negative integers. Further, we must require that

$$
\nabla \cdot \mathbf{E}_0 = 0 \implies \frac{\partial \mathbf{E}_0_y}{\partial y} + \frac{\partial \mathbf{E}_0_z}{\partial z} = -\frac{\partial \mathbf{E}_0_x}{\partial x} \quad (13.164)
$$

In particular, this equation must be satisfied when $x = 0^+$ and $x = a^-$, for which coordinates $\mathbf{E}_0_y$ and $\mathbf{E}_0_z$ must both be zero for all $y$ and $z$ because they both are tangential components to the planes $x = 0$ and $x = a$. Consequently, neither $\mathbf{E}_0_y$ nor $\mathbf{E}_0_z$ change with $y$ and $z$ if $x = 0^+$ or $x = a^-$ and Eq. (13.164) reduces to $\frac{\partial \mathbf{E}_0_x}{\partial x} = 0$ at $x = 0$ and $x = a$. This condition in turn requires that

$$
\left. \frac{dX}{dx} \right|_{x=0} = 0 \implies A = 0 \quad (13.165)
$$
$$
\left. \frac{dX}{dx} \right|_{x=a} = 0 \implies \kappa_x a = m\pi \quad (13.166)
$$

where $m$ is a nonnegative integer. Thus, the boundary conditions reduce $X$, $Y$, and $Z$ substantially and their product, which gives $\mathbf{E}_{0x}$, now is

$$
\mathbf{E}_{0x}(x, y, z) = E_x \cos \kappa_x x \sin \kappa_y y \sin \kappa_z z \quad (13.167)
$$

Similar calculations give

$$
\mathbf{E}_{0y}(x, y, z) = E_y \sin \kappa_x x \cos \kappa_y y \sin \kappa_z z \quad (13.168)
$$
$$
\mathbf{E}_{0z}(x, y, z) = E_z \sin \kappa_x x \sin \kappa_y y \cos \kappa_z z \quad (13.169)
$$

for the remaining two components of the $\mathbf{E}$-field. Here $E_x$, $E_y$, and $E_z$ are constants that are arbitrary except for the condition

$$
\kappa_x E_x + \kappa_y E_y + \kappa_z E_z = 0 \quad (13.170)
$$

imposed by Maxwell’s equation $\nabla \cdot \mathbf{E}_0 = 0$. Throughout these expressions, $\kappa_x$, $\kappa_y$, and $\kappa_z$ can assume only the values

$$
\kappa_x = \frac{m\pi}{a} ; \quad \kappa_y = \frac{n\pi}{b} ; \quad \kappa_z = \frac{p\pi}{d} \quad (13.171)
$$

where $m$, $n$, and $p$ are nonnegative integers, at most one of which can be zero. (If two or more are zero, the field is zero.) Evaluation of the $\mathbf{H}$-field is left to P13.38.

Note finally that, when $m = 0$, then $\mathbf{E}_{0y} = \mathbf{E}_{0z} = 0$. Thus, the $\mathbf{E}$-field in the cavity reduces to

$$
\mathbf{E}(\mathbf{r}, t) = E_x \sin \kappa_y y \sin \kappa_z z \hat{i} e^{-i\omega t} = \frac{E_x}{2i} \sin \kappa_y y \hat{i} [e^{i(\kappa_z z - \omega t)} - e^{i(-\kappa_z z - \omega t)}] \quad (13.172)
$$
which can be viewed as a superposition of two waves of the TE$_{0n}$ mode of the rectangular waveguide [Eq. (13.134)], one propagating toward $z = +\infty$ and the other propagating toward $z = -\infty$. The coefficients $(+1, -1)$ in the superposition and the values of $\kappa_z$, however are restricted so that (1) each TE$_{0n}$ wave can be regarded as a reflection of the other in the planes $z = 0$ and $z = d$ and (2) the tangential component of the resulting $E$-field is zero on the planes $z = 0$ and $z = d$. Those more general modes that follow from the above fields when $m \neq 0$ but $E_z = 0$ can also be viewed as a similar superposition of the TE$_{mn}$ modes of the rectangular waveguide and are referred to as the TE$_{mnp}$ modes of the resonator. The remaining modes of the resonator (in which $E_z$ is not zero but $H_z$ is zero) arise from a superposition of TM$_{mn}$ modes of the guide and are called the TM$_{mnp}$ modes of the resonator. The fields in all of these modes can be viewed as forming standing wave patterns in all three coordinate directions.

In contrast to the wave guide, in which all frequencies exceeding the cutoff frequency are allowed, the cavity resonator has a discrete frequency spectrum. Substitution of Eq. (13.171) into Eq. (13.159) gives

$$\omega^2_{mnp} = \pi^2 c^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2} \right) \quad (13.173)$$

for the frequency $\omega_{mnp}$ characterizing the TE$_{mnp}$ mode. Since this frequency also characterizes the TM$_{mnp}$ mode, there are at least two different modes corresponding to the frequency $\omega_{mnp}$ and all frequencies in this cavity are at least doubly degenerate. Additional degeneracies will occur when $a$, $b$, and $d$ are so related the $\omega_{mnp}$ has the same value for two or more different values of $mnp$. (For example, $\omega_{623} = \omega_{433}$ when $a = 2b$.) Finally, since only one of $m$, $n$, and $p$ can be zero, the lowest resonant frequency of the cavity is that one of $\omega_{110}$, $\omega_{101}$, and $\omega_{011}$ for which the zero corresponds to the direction of the smallest dimension of the cavity. Thus, for example, a cavity with dimensions $1 \text{ cm} \times 2 \text{ cm} \times 4 \text{ cm}$ will have a lowest resonant frequency $\omega_f$ given by

$$\omega^2_f = \pi^2 c^2 \left( \frac{1}{2^2} + \frac{1}{4^2} \right) \quad \Rightarrow \quad \omega_f = 5.3 \times 10^{10} \text{ s}^{-1}$$

which corresponds to a vacuum wavelength of 3.5 cm.

As we have seen, wave guides and cavity resonators of reasonable dimensions are characterized by frequencies in the microwave region of the electromagnetic spectrum. This property contributes significantly to the importance of these devices in experimental work involving microwaves, for wave guides and cavity resonators are distinctly superior to conventional circuitry at these frequencies.

**PROBLEMS**

**P13.31.** Derive the final expressions in Eqs. (13.146) and (13.147) for $\langle S \rangle$ and $\langle u_{\text{EM}} \rangle$ in a rectangular guide carrying a TE$_{0n}$ wave.

**P13.32.** Find the (surface) charge and current densities induced on all walls in Fig. 13.10 when the guide carries the TE$_{0n}$ wave.

**P13.33.** (a) Draw three figures like Fig. 13.11 (but without the propagation vectors) and then draw lines along which the physical $E$-field is zero at time $t = 0$ for the modes TE$_{01}$, TE$_{02}$, and TE$_{03}$, marking each region with a $+$ or $-$ sign to indicate the direction of the field in that region. (b) Find the physical $H$-field for these modes and sketch the lines of $H$ in each figure.
P13.34. (a) For the rectangular wave guide of Fig. 13.10, find the complex fields for the transverse electric modes TE$_{m0}$ in which the $E$-field is in the $\hat{j}$ direction and all fields are independent of $y$. (b) Sketch figures showing the fields in the TE$_{10}$, TE$_{20}$, and TE$_{30}$ modes.

P13.35. Find the complex fields for the transverse electric modes TE$_{mn}$ in the rectangular wave guide of Fig. 13.10. Hints: (1) Let $\mathcal{E}(r,t) = [\mathcal{E}_{0x}(x,y) \hat{i} + \mathcal{E}_{0y}(x,y) \hat{j}] \exp[i(\kappa_z z - \omega t)]$ and use separation of variables on the equation analogous to Eq. (13.132). (2) The tangential components of the electric field must be zero at all conducting walls. (3) $\nabla \cdot \mathcal{E} = 0$ for all $x$ and $y$. (4) Find $\mathcal{E}$ first; then find $\mathcal{H}$ from Eq. (13.22).

P13.36. Let a cylindrical wave guide of radius $a$ have its axis along the $z$ axis. Assuming an electric field of the general form

$$\mathcal{E}(r,t) = \mathcal{E}_0(t) e^{i(\kappa_z z - \omega t)}$$

find the fields and then find the longest three cutoff wavelengths if $a = 5$ cm. Hints: (1) Be careful with the Laplacian of a vector in cylindrical coordinates. (2) The equation for $\mathcal{E}_0(t)$ is Bessel’s equation of order one, and the only solution that is finite at the origin is the first-order Bessel function usually denoted by $J_1(x)$, which is a standard, tabulated function. In particular, $J_1(x) = 0$ for $x = 3.8317, 7.0156, 10.1735, \ldots$

P13.37. The systematic development of Eqs. (13.168) and (13.169) by the method used to obtain Eq. (13.167) does not by itself require $\kappa_x$, $\kappa_y$, and $\kappa_z$ to be the same in all three components of the field. Present a supplementary argument proving that $\kappa_x$, $\kappa_y$, and $\kappa_z$ in fact must have the same value in all three components.

P13.38. Find the $\mathcal{H}$-field corresponding to the $\mathcal{E}$-field in Eqs. (13.167)–(13.169).

P13.39. Suppose $a = 1$ cm, $b = 2$ cm, and $d = 3$ cm. Use a computer to calculate $\omega_{mnp}$ for $0 \leq m, n, p \leq 4$, and show the allowed frequencies as horizontal lines in a diagram with a vertical frequency axis, labeling each line with the number(s) of the mode(s) to which it corresponds. This diagram is analogous to a quantum mechanical energy level diagram.

13.6 Dispersion

Plane monochromatic waves in matter can be superposed in all of the ways that were discussed in Sections 7.3 and 7.4 for plane monochromatic waves in vacuum. Waves in matter therefore can be circularly and elliptically polarized, and they can interfere with one another. They can also be superposed by summing or integrating over a spectrum of wave numbers to produce plane waves that are no longer sinusoidal. In this section, we shall consider briefly a phenomenon that occurs in matter but not in vacuum, restricting our consideration to unattenuated waves and therefore to nonconducting matter. The phenomenon is called dispersion, and it arises because in matter the velocity of propagation of a purely monochromatic wave depends on frequency. Thus, the different frequencies composing a polychromatic wave travel through matter at different speeds and, as a result, the overall wave form changes shape as the wave propagates through the matter. Further, the speed of propagation of the envelope of a polychromatic pulse will be different from—perhaps even quite different from—the speed of propagation of any one of its component frequencies. We must therefore distinguish two velocities: (1) the phase velocity $v_p$, which is the speed at which a purely monochromatic wave propagates and is given by

$$v_p = \frac{\omega}{\kappa}$$  \hspace{1cm} (13.174)
and (2) the group velocity \( v_g \), which is the speed at which the envelope of a polychromatic pulse propagates.

An expression for the group velocity is quickly obtained by considering a simple superposition of two linearly polarized waves having the same amplitude but slightly different frequencies and propagating toward \( z = +\infty \). If the wave number \( \kappa \) depends on frequency, then two different wave numbers

\[
\kappa = \kappa(\omega) \tag{13.175}
\]

and

\[
\kappa' = \kappa(\omega') = \kappa(\omega + \Delta \omega) = \kappa + \Delta \kappa \tag{13.176}
\]

must be introduced. The electric field in this superposed wave is then given by

\[
\mathbf{E}(z, t) = E_0 \left[ e^{i(\kappa z - \omega t)} + e^{i(\kappa' z - \omega' t)} \right]
\]

where

\[
\kappa = \frac{1}{2}(\kappa + \kappa') \tag{13.178}
\]

and

\[
\omega = \frac{1}{2}(\omega + \omega') \tag{13.179}
\]

Physically, Eq. (13.177) represents a sinusoidal wave of frequency \( \overline{\omega} \) propagating with a phase velocity

\[
v_p = \frac{\omega + \omega'}{\kappa + \kappa'} \rightarrow \frac{\omega}{\kappa} \tag{13.180}
\]

on which has been superimposed an envelope (the cosine factor) that modulates the wave (Fig. 13.13). The envelope, however, propagates with the group velocity

\[
v_g = \frac{\Delta \omega}{\Delta \kappa} \rightarrow 0 \frac{d\omega}{dk} \tag{13.181}
\]

which is equal to the phase velocity when \( \omega \) is proportional to \( \kappa \) (P13.41) but more generally may be either larger or smaller than the phase velocity. Since a sinusoidal wave can be considered to transmit information only to the extent that it is modulated, i.e., contains many frequencies, we conclude that any information resides in the modulation and hence must be propagated through matter with the group velocity.

We shall conclude this section by examining briefly the propagation of a linearly polarized pulse in a medium in which \( v_p \) depends on frequency, i.e., in a dispersive medium. In general, we can represent the electric field in a pulse propagating toward \( z = +\infty \) and linearly polarized in the \( x \) direction by the superposition

\[
\mathbf{E}(z, t) = \hat{i} \int_0^\infty \varepsilon_0(\kappa) e^{i(\kappa z - \omega t)} d\kappa \tag{13.182}
\]

\[
= \hat{i} \int_0^\infty \varepsilon_0(\kappa) e^{i(\kappa z - v_pt)} d\kappa \tag{13.183}
\]

\[\text{[16]} \text{We deal here only with frequency modulation (FM).}\]
13.6. Dispersion

Figure 13.13: Superposition of two sinusoidal waves of slightly different frequency. Part (a) shows the real part of the rapidly oscillating factor in Eq. (13.177) as a function of $z$ at $t = 0$; part (b) shows the modulating envelope at $t = 0$; and part (c) shows the superposition [i.e., the product of (a) and (b)]. These graphs are drawn for $\Delta \kappa / \kappa = 0.1$. The rapid oscillations in part (a) propagate with speed $v_p$ and the envelope in part (b) propagates with speed $v_g$.

![Figure 13.13](image)

where we assume that each frequency present propagates at the appropriate phase velocity (which now, however, we must think of as a function of $\kappa$). For our present purposes, we shall not need the accompanying magnetic intensity. As in Chapter 7, it is here convenient to express the field as an integral over the range $-\infty < \kappa < +\infty$, which we accomplish by extracting the physical field, finding

$$
\mathbf{E}(z, t) = \frac{1}{2} [\mathcal{E}(z, t) + \mathcal{E}^*(z, t)]
$$

$$
= \frac{1}{2} \hat{i} \int_0^\infty \mathcal{E}_0(\kappa) e^{i\kappa[z-v_p(\kappa)t]} d\kappa + \frac{1}{2} \hat{i} \int_0^\infty \mathcal{E}_0^*(\kappa) e^{-i\kappa[z-v_p(\kappa)t]} d\kappa
$$

$$
= \frac{1}{2} \hat{i} \int_0^\infty \mathcal{E}_0(\kappa) e^{i\kappa[z-v_p(\kappa)t]} d\kappa + \frac{1}{2} \hat{i} \int_0^\infty \mathcal{E}_0^*(-\kappa) e^{i\kappa[z-v_p(-\kappa)t]} d\kappa
$$

$$
= \hat{i} \int_{-\infty}^{\infty} A(\kappa) e^{i\kappa(z-V_p(\kappa)t)} \frac{d\kappa}{2\pi}
$$

(13.184)
where we have defined the spectral function $A(\kappa)$ as in Eq. (7.63),

$$A(\kappa) = \begin{cases} \pi \mathcal{E}_0(\kappa), & \kappa > 0 \\ \pi \mathcal{E}_0^*(-\kappa), & \kappa < 0 \end{cases}$$  \hspace{1cm} (13.185)$$

and we have defined a phase velocity $V_p(\kappa)$ that applies over the entire range of $\kappa$ by

$$V_p(\kappa) = \begin{cases} v_p(\kappa), & \kappa > 0 \\ v_p(-\kappa), & \kappa < 0 \end{cases}$$  \hspace{1cm} (13.186)$$

This “capital” phase velocity is thus the even extension of $v_p(\kappa)$ into the region $\kappa < 0$. Given Eq. (13.184), we conclude in particular that the field at $t = 0$ is given by

$$E(z, 0) = \hat{i} \int_{-\infty}^{\infty} A(\kappa) e^{i\kappa z} \frac{d\kappa}{2\pi}$$  \hspace{1cm} (13.187)$$

which by Fourier inversion (Appendix D) yields the expression

$$A(\kappa) = \int_{-\infty}^{\infty} E_x(z, 0) e^{-i\kappa z} dz$$  \hspace{1cm} (13.188)$$

where $E(z, 0) = E_x(z, 0) \hat{i}$, for $A(\kappa)$. Thus, if we know the field $E_x(z, 0)$ at time zero and we know the phase velocity $V_p(\kappa)$ as a function of $\kappa$, we can find the field at other times by calculating $A(\kappa)$ from Eq. (13.188) and then calculating $E_x(z, t)$ from Eq. (13.184). In general, numerical methods will be needed for at least part of this calculation.\(^\text{17}\)

To illustrate even more clearly the nature of wave propagation in a dispersive medium, suppose that at $t = 0$ the pulse in the previous paragraph is the triangular pulse for which

$$E_x(z, 0) = \begin{cases} 10 (1 - |z|), & |z| < 1 \\ 0, & |z| > 1 \end{cases}$$  \hspace{1cm} (13.189)$$

as shown in Fig. 13.14(a) and is propagating toward $z = +\infty$. By using Eq. (13.188), we find that the spectral function is

$$A(\kappa) = 10 \int_{-1}^{1} (1 - |z|) e^{-i\kappa z} dz = 20 \frac{1 - \cos \kappa}{\kappa^2}$$  \hspace{1cm} (13.190)$$

which happens for this pulse to be a real, even function of $\kappa$; it is graphed in Fig. 13.14(b). Continuing, we then find from Eq. (13.184) that

$$E_x(z, t) = 20 \int_{-\infty}^{\infty} \frac{1 - \cos \kappa}{\kappa^2} e^{i\kappa[z - v_p(\kappa)t]} \frac{d\kappa}{2\pi}$$

$$= \frac{20}{\pi} \int_{0}^{\infty} \frac{1 - \cos \kappa}{\kappa^2} \cos(\kappa[z - v_p(\kappa)t]) d\kappa$$  \hspace{1cm} (13.191)$$

where the second form follows by writing the exponent as $\cos(\cdots) + i \sin(\cdots)$ and then noting that the real part of the resulting integrand is even in $\kappa$ and the imaginary part is odd in $\kappa$. Although analytically Eq. (13.191) is essentially intractable for all but the

\(^{17}\)See, for example, J. R. Merrill, Am. J. Phys. 39, 539 (1971).
13.6. DISPERSION

Figure 13.14: Propagation of a pulse in various dispersive media. (a) The initial pulse. (b) Its spectral function. (c)–(f) Propagation for phase velocities as labeled.

simplest functions \( v_p(\kappa) \), numerical integration yields \( E_x(z,t) \) quite easily for any function \( v_p(\kappa) \), provided the infinite interval is truncated to some finite range, say \( 0 < \kappa < \kappa_{\text{max}} \), outside of which \( A(\kappa) \) is approximately zero. From the many possible formulas for numerical integration, we select Simpson’s rule,\(^{18}\) which gives\(^{19}\)

\[
E_x(z,t) \approx \frac{1}{3} \Delta \kappa \sum_{i=0}^{N} f_i g(\kappa_i, z, t) \tag{13.192}
\]

where \( N \) (which must be even) is the number of subintervals into which \( 0 < \kappa < \kappa_{\text{max}} \) is divided, \( \Delta \kappa = \kappa_{\text{max}}/N \), \( \kappa_i = i \Delta \kappa \), \( f_i = 1, 4, 2, 4, 2, \ldots, 2, 4, 1 \) as \( i \) increases from \( i = 0 \) to \( i = N \), and

\[
g(\kappa, z, t) = \frac{20}{\pi} \frac{1 - \cos \kappa}{\kappa^2} \cos(\kappa[z - v_p(\kappa)t]) \tag{13.193}
\]

The writing of a computer program to evaluate this sum as a function of \( z \) for selected \( t \) and \( v_p \) is left to P13.44. Figure 13.14(c)–(f) shows the output of such a program when \( v_p = 1 \) (no dispersion), when \( v_p = 1 + 0.05\kappa \) (phase velocity increases steadily over the

\(^{18}\)British mathematician Thomas Simpson, b. 20 August 1710; d. 14 May 1761. Note that Johannes Kepler found this rule 100 years before Simpson.

Figure 13.14: (Continued)

(c) \( E_x(z, t) \) for \( v_p = 1 \)

\( t = 0 \) \hspace{1cm} t = 4 \hspace{1cm} t = 8 \)

(d) \( E_x(z, t) \) for \( v_p = 1 + 0.05\kappa \)

\( t = 0 \) \hspace{1cm} t = 4 \hspace{1cm} t = 8 \)

Figure 13.14: (Continued)

(e) \( E_x(z, t) \) for \( v_p = 1 - 0.05\kappa \)

\( t = 0 \) \hspace{1cm} t = 4 \hspace{1cm} t = 8 \)

(f) \( E_x(z, t) \) for \( v_p = 1 + 0.02(\kappa - 2.5)^2 \)

\( t = 0 \) \hspace{1cm} t = 4 \hspace{1cm} t = 8 \)
range $0 < \kappa < \kappa_{\text{max}}$), when $v_p = 1 - 0.05\kappa$ (phase velocity decreases steadily over the range $0 < \kappa < \kappa_{\text{max}}$), and when $v_p = 1 + 0.02(\kappa - 2.5)^2$ (phase velocity has a minimum at $\kappa = 2.5$). As a pulse propagates into a dispersive medium, the pulse becomes less sharp and oscillatory leading and/or trailing “edges” become more pronounced.

**PROBLEMS**

**P13.40.** Show that the reflected and transmitted waves resulting when a circularly polarized wave is incident obliquely on a dielectric interface are in general elliptically polarized.

**P13.41.** Let the dispersion relation for a particular medium be $\omega = a\kappa$, where $a$ is a constant. Show that $v_g = v_p = a$. In particular, this dispersion relation describes electromagnetic waves in vacuum if $a = c$.

**P13.42.** The group velocity of electromagnetic waves can be expressed in many different ways. Show that $v_g = v_p = a$, where $a$ is a constant. In particular, this dispersion relation describes electromagnetic waves in vacuum if $a = c$.

**P13.43.** In the visible region of the spectrum, the index of refraction $n$ of some media can be adequately represented as a function of wavelength by the two-constant Cauchy equation $n = A + B/\lambda^2$. (a) Find the phase and group velocities as functions of $\lambda$. (b) For barium flint glass, $n = 1.58848$ at $\lambda = 656.3$ nm and $n = 1.60870$ at $\lambda = 398.8$ nm. Find $A$ and $B$ and then use a suitable computer tool to plot graphs of $n$, $v_g$ and $v_p$ versus $\lambda$ in the visible spectrum.

**P13.44.** Write a program to evaluate the sum in Eq. (13.192) as a function of $z$ for specified $t$, determine suitable values of $N$ and $\kappa_{\text{max}}$, and run your program on an available computer to reproduce some of the results in Fig. 13.14. Hint: The function $(1 - \cos \kappa)/\kappa^2$ is indeterminate at $\kappa = 0$. Show that

$$
\frac{20}{\pi} \frac{1 - \cos \kappa}{\kappa^2} = \frac{10}{\pi} \left[ 1 - \frac{\kappa^2}{12} \left( 1 - \frac{\kappa^2}{30} \right) \right]
$$

to within about $\pm 2 \times 10^{-10}$ when $|\kappa| < 0.1$ and use this series for that range of $\kappa$. Optional: Try other values of $N$ and $\kappa_{\text{max}}$ and other functions $v_p(\kappa)$. (2) Try to increase the running speed of your program by thinking of ways to minimize the number of calls to cosine and sine routines. Note, for example, that $\cos(\kappa + \Delta \kappa)z = \cos(\kappa z) \cos(\Delta \kappa z) - \sin(\kappa z) \sin(\Delta \kappa z)$. How can this and similar identities be used to determine $\cos(m \Delta \kappa z)$ for all $m$ using only two calls to the sine and cosine routines? Note also that cosines and sines that are used repeatedly can be calculated once and stored.

**P13.45.** Without evaluating any integrals, sketch a labeled graph of the likely spectral function for the pulse shown in Fig. 13.15. How would increasing the number of cycles to 10 and to 100 affect the spectral function?

**SUPPLEMENTARY PROBLEMS**

**P13.46.** At a particular point in space, the electric field is given as a function of time by $\mathcal{E}(t) = \mathcal{E}(t) \hat{i}$, where $\hat{i}$ can be expressed in terms of a Fourier transform

$$
\hat{\mathcal{E}}(\omega) = \int_{-\infty}^{\infty} \mathcal{E}(t) e^{i\omega t} dt
$$
Suppose the medium is such that $\mathbf{D}$ and $\mathbf{E}$ are related by a frequency-dependent permittivity $\tilde{\varepsilon}(\omega)$ so that $\tilde{\mathbf{D}}(\omega) = \tilde{\varepsilon}(\omega)\tilde{\mathbf{E}}(\omega)$. Show that

$$\mathbf{D}(t) = \int_{-\infty}^{\infty} \epsilon(t') \mathbf{E}(t-t') dt'$$

where

$$\epsilon(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\varepsilon}(\omega) e^{-i\omega t} d\omega$$

etc. The relationship between $\mathbf{D}(t)$ and $\mathbf{E}(t)$ in this case is far less simple than the relationship between $\tilde{\mathbf{D}}(\omega)$ and $\tilde{\mathbf{E}}(\omega)$. The first of the above integrals is called a convolution integral.

**P13.47.** The $Q$ of a medium is defined as the absolute value of the ratio of the displacement current density to the conduction current density. Consider a linear medium carrying a monochromatic plane wave. (a) Show that $Q = \epsilon \omega / g$. (b) Express $\kappa$ and $\alpha$ as given in Eqs. (13.62) and (13.63) in terms of $Q$ by eliminating $\omega$ and then express $K = \kappa + i\alpha$ in the form $|K|e^{i\phi}$, finding $|K|$ and $\phi$ in terms of $Q$. (c) Sketch graphs of $\kappa$, $\alpha$, $|K|$, and $\phi$ as functions of $Q$, and then confirm your sketch by using a suitable computer tool to obtain a careful graph. (d) Determine the ratio of the electric to the magnetic energy densities in the wave of Eqs. (13.49) and (13.50), expressing the result in terms of $Q$, sketch a graph of this ratio as a function of $Q$, and then again use a suitable computer tool to obtain a careful graph.

**P13.48.** Letting $Q = \epsilon_i \omega / g_t$ (compare P13.47) and $\gamma = \sqrt{\mu_t \epsilon_t / \mu_i \epsilon_i}$, and taking $K_t = \kappa_t + i\alpha_t$ with $\kappa_t$ and $\alpha_t$ given by Eqs. (13.62) and (13.63), find $\zeta$ [Eq. (13.89)] and then obtain graphs of $R$ and $T$ [Eq. (13.91)] as functions of $Q$ for typical values of $\gamma$, including $\gamma = 1$.

**P13.49.** Show that for a good nonmagnetic conductor Eq. (13.91) for the reflection coefficient at normal incidence reduces to $R = 1 - \sqrt{8\epsilon_i \omega / g_t}$. Hint: Evaluate $T$ first; then find $R$ from $R = 1 - T$. 
P13.50. Using a suitable computer tool, obtain a few graphs of $R_\perp$, $T_\perp$, $R_\parallel$, and $T_\parallel$ as given by Eqs. (13.121)–(13.124) for various values of $n_t/n_i$. Graph also $R_\perp + T_\perp$ and $R_\parallel + T_\parallel$ to verify that $R_\perp + T_\perp = 1$ and $R_\parallel + T_\parallel = 1$. Compare your results with Fig. 13.9.

P13.51. If the media of incidence and transmission are both nonmagnetic, show that Eqs. (13.113) and (13.114) can be written in the form

$$
\varepsilon_\perp r_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \varepsilon_\perp \varepsilon_\perp \varepsilon_\perp = \frac{2 \cos \theta_t \sin \theta_i}{\sin(\theta_t + \theta_i)} \varepsilon_\perp \varepsilon_\perp \varepsilon_\perp
$$

and find similar expressions for $\varepsilon_\parallel r_\parallel$ and $\varepsilon_\parallel t_\parallel$.

P13.52. In the case of total reflection at an interface between two nonmagnetic nonconductors, Snell’s law requires $\theta_t$ to be complex and

$$
\cos \theta_t = \sqrt{1 - \sin^2 \theta_t} = i \sqrt{\left(\frac{n_i}{n_t}\right)^2 \sin^2 \theta_i - 1}
$$

Show from Eqs. (13.113) and (13.115) that the reflection coefficients for both polarizations are unity.

P13.53. A plane wave in air is incident on a nonmagnetic nonconducting slab as shown in Fig. 13.16, producing a reflected wave and also a transmitted wave. The transmitted wave is subsequently partially reflected and partially transmitted at the second interface. Let the incident wave be polarized perpendicular to the plane of incidence and have amplitude $a$. Finally, introduce reflection and transmission coefficients $r$, $t$, $r'$, and $t'$ at the two interfaces so that the amplitudes of the several waves are as shown in the figure. Using Eqs. (13.113) and (13.114), find $r$, $t$, $r'$, and $t'$ in terms of $\theta_i$, $\theta_t$, and $n$ and then show that $r = -r'$ and that $tt' = rr' = 1$. Do these final two results also apply for parallel polarization? Defend your answer.

P13.54. The following procedure is sometimes used to calculate energy losses when plane waves are incident on nearly perfect conductors: (1) Solve the problem assuming the conductor
CHAPTER 13. PLANE ELECTROMAGNETIC WAVES

to be perfect; (2) determine the surface current density \( j \) in the perfect conductor and then, assuming that the same total current appears for the real conductor and that that current is distributed in a layer of thickness \( \delta \) at the surface, take the current density \( \mathbf{J} \) in that layer to satisfy \( j = \delta \mathbf{J} \); (3) estimate the electric field in that layer from Ohm’s law, \( \mathbf{J} = q \mathbf{E} \); and (4) estimate the magnetic intensity in that layer by requiring continuity of the tangential component of \( \mathbf{H} \) across the interface. (a) Apply this procedure to estimate \( \mathbf{J} \), \( \mathbf{E} \), and \( \mathbf{H} \) in the surface layer of a nearly perfect conductor when the incident wave strikes the surface normally. (b) Find the average power dissipated in an area \( A \) of the conducting surface. (c) Consider an area \( A \) of the conducting surface bounded by a rectangle having sides \( L \) and \( W \), with the sides of length \( L \) directed perpendicular to the direction of \( \mathbf{J} \). Find the root mean square (rms) value of the total current flowing between the two sides of length \( L \) and then, expressing the average power dissipated in the area bounded by this rectangle in terms of the rms current, identify a resistance to associate with the surface. What is the surface resistance of a square sheet between two parallel edges? Can you understand why this resistance does not depend on the size of the sheet? (Note: The quantity \( 1/g \delta \) is sometimes called the surface resistivity.)

**P13.55.** Determine the fields in the TM\(_{mnp}\) modes of the cavity resonator in Fig. 13.12. **Hint:** See P13.38 and require \( \mathcal{H}_{0z} \) to be zero.

**P13.56.** Suppose a portion of the cylindrical wave guide in P13.36 is closed off by conducting planes at \( z = 0 \) and \( z = b \) to make a cylindrical cavity resonator. What are the frequencies of the resonant modes corresponding to the traveling modes found in P13.36? Choose reasonable dimensions for the cavity, calculate several of the lowest resonant frequencies, and plot a “frequency level” diagram as described in P13.39.

**P13.57.** Let a general monochromatic wave in an evacuated, rectangular wave guide be given by

\[
\begin{align*}
\mathcal{E}(r, t) &= \mathcal{E}_0(x, y) e^{i(k_z z - \omega t)}; \\
\mathcal{H}(r, t) &= \mathcal{H}_0(x, y) e^{i(k_z z - \omega t)}
\end{align*}
\]

where \( \mathcal{E}_0 \) and \( \mathcal{H}_0 \) may have components in all three coordinate directions. Show from Maxwell’s equations [Eqs. (13.5)–(13.8)] that

\[
\mathcal{E}_{0z} = \frac{i}{\omega_0} \left( 1 - \frac{\kappa^2 c^2}{\omega^2} \right)^{-1} \left( \frac{\partial \mathcal{H}_{0z}}{\partial y} + \frac{\kappa z}{\omega \mu_0} \frac{\partial \mathcal{E}_{0z}}{\partial x} \right)
\]

find similar expressions determining \( \mathcal{E}_{0y}, \mathcal{H}_{0y}, \) and \( \mathcal{H}_{0y} \) from \( \mathcal{E}_{0z} \) and \( \mathcal{H}_{0z} \), and hence conclude that all components of a guided wave are known if the \( z \) components of the two fields are known. In particular, TE modes have \( \mathcal{H}_{0z} = 0 \) and TM modes have \( \mathcal{H}_{0z} = 0 \).

**P13.58.** In a tenuous plasma, collisions between particles are infrequent enough that resistive forces can be neglected and \( \varepsilon \approx \varepsilon_0, \mu \approx \mu_0 \). In such a case \( b = 0 \) in Eq. (13.12) and the conductivity given by Eq. (13.15) becomes \( \sigma = i q_e^2 N/\omega m_c \) and is purely imaginary. (We switch from \( n \) to \( N \) for particle density in order to avoid confusion with the index of refraction \( n \).) (a) What is the dispersion relation [Eq. (13.48)] for this plasma? Write your result in terms of the so-called plasma frequency \( \omega_p \) defined by \( \omega_p^2 = q_e^2 N/m_c \varepsilon_0 \). (b) Sketch a graph of \( \kappa \) versus \( \omega \). (c) Describe qualitatively the nature of the waves given by Eqs. (13.49) and (13.50) for this plasma. Consider both \( \omega > \omega_p \) and \( \omega < \omega_p \). (d) In what region of the electromagnetic spectrum does the frequency \( \omega_p \) lie when \( N \approx 10^{11} \) particles/m\(^3\), a value characteristic of the ionosphere? What limitations does the ionosphere impose on the frequencies used for communication with moon-bound astronauts? (e) Recognizing that \( \kappa = n\omega/c, \) where \( n \) is the index of refraction, find \( n \) for this plasma.

**P13.59.** Starting with Eqs. (13.104) and (13.105), use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to work through the procedure outlined in the text to derive the Fresnel equations as given by Eqs. (13.113) and (13.114).
Chapter 14

Radiation from Prescribed Sources in Vacuum

Maxwell’s equations not only predict that electromagnetic waves propagate through space but also provide the theoretical framework for relating those waves to their ultimate source in some charge and current distribution. This relationship is the subject of this chapter. The discussion is confined to radiation produced by free charges and currents in space free of matter, and Maxwell’s equations therefore assume the form

\[ \nabla \cdot E = \frac{\rho}{\epsilon_0} \quad (14.1) \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \quad (14.2) \]
\[ \nabla \cdot B = 0 \quad (14.3) \]
\[ \nabla \times B = \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t} \quad (14.4) \]

where \(1/c^2 = \mu_0 \epsilon_0\). In treating radiation problems, however, it is usually convenient to seek first the scalar and vector potentials \(V\) and \(A\) and then derive the fields by applying the relationships

\[ E = -\nabla V - \frac{\partial A}{\partial t} \quad (14.5) \]
\[ B = \nabla \times A \quad (14.6) \]

obtained in Section 6.6. We elect to work in a Lorentz gauge. The potentials therefore satisfy the inhomogeneous wave equations,

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A = -\mu_0 J \quad (14.7) \]
\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) V = -\frac{1}{\epsilon_0} \rho \quad (14.8) \]
and, in addition, are related by the Lorentz condition,

\[ \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \] (14.9)

[Compare Eqs. (6.73)–(6.75).] In brief, the objectives of this chapter are (1) to solve Eqs. (14.7) and (14.8) for the potentials and (2) to examine these solutions and the corresponding fields for a few representative source distributions.

### 14.1 The General Solution of the Inhomogeneous Wave Equation: Retardation

In this section, we shall seek general solutions to Eqs. (14.7) and (14.8). The results, of course, will be general solutions to Maxwell’s equation in vacuum and, once we have obtained these solutions, all problems of electromagnetism in vacuum are solved, at least in principle. We have already found this general solution when the potentials are static, namely

\[ A(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{v}' \] (14.10)

\[ V(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{v}' \] (14.11)

[Compare Eqs. (5.48 and (4.57).] The solution for the time-dependent potentials, however, is more complicated. Because of the time derivatives in the basic equations, we cannot obtain the solutions in the time-dependent case simply by allowing the sources in Eqs. (14.10) and (14.11) to become time-dependent. The effect of the time derivatives is more subtle, but it is not at all surprising once it has been obtained. Any one of several methods of solution might be pursued. Mathematically elegant methods exist but cannot be used here without an extended digression to develop the necessary mathematical techniques (but see P14.26). We adopt instead an approach that draws on no more mathematics than has already been introduced.

Consider first a small, spherically symmetric charge distribution, which we shall ultimately allow to approach a point. If we can find the potentials established by this distribution, we can obtain the solution for a more realistic distribution by superposition. Let the charge be located at the origin (we can translate coordinates any time we wish) and, disregarding temporarily the requirements of charge conservation, suppose that the total charge in the distribution varies with time in accordance with

\[ q = q(t) \] (14.12)

If this distribution is confined to a region of radius \( a \) centered at the origin, then \( \rho(\mathbf{r}, t) \) is zero outside this region and the scalar potential satisfies

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) V = 0 \] (14.13)
when $r > a$. Since the supposed spherical symmetry of the charge distribution implies that $V$ depends only on $r$ and $t$, we can write this equation more simply in spherical coordinates, obtaining

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0$$

(14.14)

The substitution $V(r,t) = \frac{h(r,t)}{r}$ reduces Eq. (14.14) to

$$\frac{\partial^2 h}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 h}{\partial t^2}$$

(14.15)

whose solution

$$h(r,t) = f(r - ct) + g(r + ct)$$

(14.16)

where $f$ and $g$ are arbitrary functions, is immediate (P7.1). Thus, the general solution to Eq. (14.14) is

$$V(r,t) = \frac{1}{r} f(r - ct) + \frac{1}{r} g(r + ct)$$

(14.17)

(Compare P7.27.)

Two physical conditions must yet be imposed on this solution. First we must set $g = 0$, for that term represents a wave propagating inward and such a wave is incompatible with our expectation on physical grounds that the electromagnetic wave produced by a changing charge cannot be present at any point in space before the charge has changed.\(^1\) Thus, the most general physically acceptable solution to Eq. (14.14) is

$$V(r,t) = \frac{1}{r} f(r - ct)$$

(14.18)

The second condition to be imposed on $V(r,t)$ determines the function $f$. Essentially, we must require that the solution, Eq. (14.18), reflect the presence of the charge $q(t)$ within the (small) sphere $r < a$. If $a$ is small enough, however, the field established by the changing charge in the region $r < a$ propagates to the surface $r = a$ in negligible time. Thus, if $a$ is small enough, the potential at and just outside of $r = a$ follows changes in the charge with negligible time lag. At each instant, the potential at and just outside of $r = a$ is therefore equal to the static potential appropriate to the value of the charge at that instant. More specifically, near $r = a$, the time-dependent potential should be given by

$$V(r,t) = \frac{1}{4\pi\epsilon_0} \frac{q(t)}{r} \quad (r \approx a \rightarrow 0)$$

(14.19)

As $r \to 0$, however, Eq. (14.18) becomes

$$V(r,t) = \frac{f(-ct)}{r} \quad (r \approx a \rightarrow 0)$$

(14.20)

Equations (14.19) and (14.20) are the same only if

$$f(-ct) = \frac{1}{4\pi\epsilon_0} q(t) \quad \Rightarrow \quad f(\xi) = \frac{1}{4\pi\epsilon_0} q \left( -\frac{\xi}{c} \right)$$

(14.21)

\(^1\)This general conviction of the physicist that event $A$ at $(r_1, t_1)$ is a possible cause of event $B$ at point $(r_2, t_2)$ only if $t_1 < t_2$ is usually called the principle of causality, and it plays a particularly significant role in some aspects of quantum field theory.
and the function $f$ is determined for any argument $\xi$. The potential expressed in Eq. (14.18) now assumes the more explicit form\(^2\)

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} q \left( t - \frac{r}{c} \right)$$ \hspace{1cm} (14.22)

More generally, if the localized (point) charge is at $\mathbf{r}'$, we would interpret $r$ in Eq. (14.22) as $|\mathbf{r} - \mathbf{r}'|$ and conclude that

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0|\mathbf{r} - \mathbf{r}'|} q \left( t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'| \right)$$ \hspace{1cm} (14.23)

A potential of this form is referred to as a *retarded potential* because the potential at time $t$ is determined by the state of the charge at the so-called *retarded time* $t'$ given by

$$t' = t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'|$$ \hspace{1cm} (14.24)

which, not surprisingly, is earlier than time $t$ by an interval precisely sufficient for an electromagnetic wave to have propagated from the source point to the observation point by the time of observation. In effect a change in a charge at $\mathbf{r}'$ is not detected at some other point $\mathbf{r}$ until sufficient time has elapsed after the change for an electromagnetic signal to propagate from $\mathbf{r}'$ to $\mathbf{r}$.

Since Eq. (14.8) is linear, we find its general solution by adding up elemental contributions having the form of Eq. (14.23). Because different points in space have associated with them different retarded times, however, it is here more suitable to imagine space (rather than the charge distribution itself) to be divided into infinitesimal elements, with the element of volume $d\mathbf{v}'$ being centered at $\mathbf{r}'$ and having retarded time $t'$ associated with it. If the charge density at $(\mathbf{r}, t)$ is $\rho(\mathbf{r}, t)$, then the charge $dq$ in $d\mathbf{v}'$ at the retarded time $t'$ is given by $\rho(\mathbf{r}', t') d\mathbf{v}'$ and the contribution made by the charge in this volume element to the potential at $(\mathbf{r}, t)$ is given by

$$dV(\mathbf{r}, t) = \frac{\rho(\mathbf{r}', t')}{4\pi\varepsilon_0|\mathbf{r} - \mathbf{r}'|} d\mathbf{v}'$$ \hspace{1cm} (14.25)

Finally, the total potential at $(\mathbf{r}, t)$ is obtained by integrating Eq. (14.25) over all volume elements; we find that

$$V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{v}'$$ \hspace{1cm} (14.26)

By similar arguments, we find that the general solution to Eq. (14.7) is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{v}'$$ \hspace{1cm} (14.27)

Extension of the solutions in Eqs. (14.10) and (14.11) to time-dependent potentials thus not only involves allowing the sources to become time-dependent but also requires evaluating these sources at the *retarded* time. Unfortunately for the more explicit evaluation of Eqs. (14.26) and (14.27), evaluation of $\rho$ and $\mathbf{J}$ at the retarded time complicates the expressions considerably for at least two reasons. First, since we seek potentials at a fixed

\(^2\)Note that $t - r/c$ is the argument of the function $q(\ldots)$, not a factor multiplying a constant $q$. 
(even though general) observation time \( t \), the integrals must be evaluated at fixed \( t \). For fixed \( t \), however, \( t' \) as given by Eq. (14.24) varies from point to point in the distribution. Hence, \( \rho(\mathbf{r}', t') \) and \( \mathbf{J}(\mathbf{r}', t') \) depend not only explicitly on \( \mathbf{r}' \) but also implicitly on \( \mathbf{r}' \) through a dependence of \( t' \) on \( \mathbf{r}' \). Even for simple charge distributions, the integrands in Eqs. (14.26) and (14.27) may be complicated functions of \( \mathbf{r}' \).

The appearance of the retarded time also complicates finding the region of space over which the nominally infinite integrals in Eqs. (14.26) and (14.27) actually extend. Generally charge distributions are specified by giving \( \rho(\mathbf{r}', t') \) and \( \mathbf{J}(\mathbf{r}', t') \) as functions of \( \mathbf{r}' \) for fixed \( t' \), which in part involves specifying the regions, say \( R' \) and \( S' \), within which \( \rho(\mathbf{r}', t') \) and \( \mathbf{J}(\mathbf{r}', t') \) differ from zero at fixed \( t' \). The integrals of interest, however, extend over the regions, say \( R \) and \( S \), in which \( \rho(\mathbf{r}', t') \) and \( \mathbf{J}(\mathbf{r}', t') \) differ from zero at fixed \( t \). Since the relationship between \( t \) and \( t' \) is generally complicated, the relationship between the (probably simple) regions \( R' \) and \( S' \) and the regions \( R \) and \( S \) is also complicated.

In summary, we have in this section found general solutions for the potentials established by arbitrary charge and current distributions in vacuum. In principle we therefore have found solutions to all electromagnetic problems in vacuum. In practice, however, the phenomenon of retardation makes the integrands and regions of integration so complicated that explicit evaluation of these general solutions for particular sources is typically very difficult. Some specific situations that can be treated fairly simply are discussed in the remainder of this chapter.

**PROBLEM**

**P14.1.** Show that the potentials given by Eqs. (14.26) and (14.27) satisfy the Lorentz condition, Eq. (14.9).

### 14.2 Radiation from Monochromatic Sources: The Oscillating Electric Dipole

If in particular the source distribution varies sinusoidally (monochromatically) with time, it is convenient to express the charge and current densities as the real parts of the complex densities

\[
\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) e^{-i\omega t}; \quad \mathbf{J}(\mathbf{r}, t) = \mathbf{J}_0(\mathbf{r}) e^{-i\omega t} \tag{14.28}
\]

the potentials as the real parts of the complex potentials

\[
V(\mathbf{r}, t) = V_0(\mathbf{r}) e^{-i\omega t}; \quad \mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}) e^{-i\omega t} \tag{14.29}
\]

and the fields as the real parts of the complex fields

\[
\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) e^{-i\omega t}; \quad \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) e^{-i\omega t} \tag{14.30}
\]

Equations (14.24), (14.26), and (14.27) then yield that the potentials are given by

\[
V_0(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_0(\mathbf{r}') e^{i\omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \, dv' \tag{14.31}
\]
and
\[ \mathcal{A}_0(r) = \frac{\mu_0}{4\pi} \int \mathcal{J}_0(r') \frac{e^{i\kappa|r-r'|}}{|r-r'|} dv' \] (14.32)
where \( \kappa = \omega/c \). Further, from Eqs. (14.5) and (14.6) we find that the fields are given by
\[ \begin{align*}
\mathcal{E}_0(r) &= -\nabla V_0(r) + i\omega \mathcal{A}_0(r) \\
\mathcal{B}_0(r) &= \nabla \times \mathcal{A}_0(r)
\end{align*} \] (14.33) (14.34)
although outside the source (where \( \mathcal{J} = 0 \)) the electric field can often be more readily obtained from the expression
\[ \mathcal{E}_0(r) = \frac{ic}{\kappa} \nabla \times \mathcal{B}_0(r) \] (14.35)
derived from Eq. (14.4).

Consider now the special case of the oscillating electric dipole shown in Fig. 14.1. Let the charge at \( z = \frac{1}{2}a \) vary sinusoidally in accordance with the expression \( \mathcal{Q}_1 = Q_0 e^{-i\omega t} \), let the charge at \( z = -\frac{1}{2}a \) vary in accordance with \( \mathcal{Q}_2 = -Q_0 e^{-i\omega t} \), and let the wire connecting the two charges run along the \( z \) axis and carry the current \( \mathcal{I} = I_0 e^{-i\omega t} \). The equation of continuity applied to a spherical surface surrounding the charge at \( z = \frac{1}{2}a \) requires that \( \mathcal{I}_0 = -i\omega Q_0 \). Replacing \( \mathcal{J}_0(r') dv' \) by \( \mathcal{I}_0(r') dl' \) in Eq. (14.32), we find that
\[ \mathcal{A}_0(r) = \frac{\mu_0}{4\pi} \int_{-a/2}^{a/2} \frac{\mathcal{I}_0 e^{i\kappa|r-z'|}}{|r-z'|} dz' \hat{k} \] (14.36)
We now confine our attention to distant observation points, i.e., to the region \( r \gg a \), in which case \( z'/r \ll 1 \) for all values of \( z' \) within the range of integration. Thus, ignoring
14.2. RADIATION FROM THE OSCILLATING ELECTRIC DIPOLE

terms of order \((a/r)^2\), we find the expansion

\[
\mathcal{A}_0(r) = \frac{\mu_0 q_0}{4\pi r} \hat{k} \int_{-a/2}^{a/2} \frac{e^{ikr[1-(z'/r)\cos \theta + \cdots]}}{1 - (z'/r)\cos \theta + \cdots} \, dz' 
\]

(14.37)

\[
\mathcal{A}_0(r) = \frac{\mu_0 q_0}{4\pi r} e^{ikr} \hat{k} \int_{-a/2}^{a/2} \left(1 + \frac{z'}{r} \cos \theta + \cdots\right) e^{-ikz'\cos \theta + \cdots} \, dz' 
\]

(14.38)

where \(\cos \theta = \hat{k} \cdot \hat{r}\). Further, we confine our attention to radiated wavelengths \(\lambda\) that are long compared to the dimensions of the dipole, i.e. to wavelengths for which \(\kappa a = 2\pi a/\lambda \ll 1\), in which case \(\kappa z' \ll 1\) within the range of integration. Thus, the Taylor expansion

\[
e^{-i\kappa z'\cos \theta} = 1 - i\kappa z'\cos \theta + \cdots
\]

(14.39)

can be employed. Under the conditions \(r \gg a\) and \(\kappa a \ll 1\), the vector potential is therefore given approximately by

\[
\mathcal{A}_0(r) = \frac{\mu_0 q_0}{4\pi r} e^{ikr} \hat{k} \int_{-a/2}^{a/2} \left[1 + \frac{z'}{r} \cos \theta + O(a^2/r^2)\right] \left[1 + i\kappa z'\cos \theta + O(\kappa^2 a^2)/r\right] \, dz'
\]

(14.40)

where in the second expression \(q_0\) has been replaced by its evaluation in terms of \(Q_0\), namely, by \(-i\omega Q_0\) and \(p_0 = Q_0 a\hat{k}\) is the maximum dipole moment of the oscillating distribution. In both expressions the notation \(O(\cdots)\) indicates the lowest order of the terms that have been neglected. Finally, using Eqs. (14.34) and (14.35), we find that the fields are given by

\[
\mathcal{B}_0 = \frac{\mu_0}{4\pi} \omega K e^{ikr} \hat{k} \left(1 - \frac{1}{i\kappa r}\right) \hat{r} \times p_0
\]

(14.41)

\[
\mathcal{E}_0 = \frac{\mu_0}{4\pi} \omega K e^{ikr} \hat{k} \frac{p_0}{r} \times \hat{r} + \frac{1}{4\pi \epsilon_0 r^3} \left[3(\hat{r} \cdot p_0)\hat{r} - p_0\right]
\]

(14.42)

Note that \(\mathcal{B}_0\) (and hence \(\mathbf{B}\)) is perpendicular to \(\hat{r}\) at all points but that \(\mathcal{E}_0\) in general has a component parallel to \(\hat{r}\).

To simplify the discussion of these fields, we have assumed (1) that the dimensions of the source are small compared to the observation distance, \(a \ll r\), and (2) that the dimensions of the source are small compared to the wavelength of the emitted radiation, \(a \ll \lambda = 2\pi/\kappa\). These conditions do not impose any constraint on the relationship between the wavelength and the distance of observation. It is useful to distinguish three zones as follows:

\(\kappa r \ll 1\): the near (static) zone, where \(r \ll \lambda\)
\(\kappa r \approx 1\): the intermediate (induction) zone, where \(r \approx \lambda\)
\(\kappa r \gg 1\): the far (radiation) zone, where \(r \gg \lambda\)

In the near zone, the fields have the limiting expressions

\[
\mathcal{B}_0 = \frac{i\omega \mu_0}{4\pi} \frac{\hat{r} \times p_0}{r^2}
\]

(14.43)

\[
\mathcal{E}_0 = \frac{1}{4\pi \epsilon_0 r^3} \left[3(\hat{r} \cdot p_0)\hat{r} - p_0\right]
\]

(14.44)
Since $\kappa r \ll 1$ and $r$ is therefore small in the near zone, the additional power of $r$ in the denominator of Eq. (14.44) means that the electric field dominates the magnetic induction in the near zone. Further, apart from the sinusoidal time dependence whose implicit appearance in Eqs. (14.43) and (14.44) has been suppressed [Compare Eq. (14.30)], the field given by Eq. (14.44) is the field established by a static electric dipole. Thus at any instant in the near zone the fields are essentially those of a static charge distribution in which the charge density is the charge density in the distribution at that instant. In the particular limit $\kappa \rightarrow 0$ (in which case $\omega \rightarrow 0$ also) the fields are truly static and the near zone extends to $r = \infty$.

In contrast to Eqs. (14.43) and (14.44), the fields in the far zone have the limiting expressions

$$B_0 = \frac{\mu_0}{4\pi} \omega \kappa \frac{e^{i\kappa r}}{r} \hat{r} \times p_0$$

$$E_0 = \frac{\mu_0}{4\pi} \omega \kappa c \frac{e^{i\kappa r}}{r} \hat{r} \times (\hat{r} \times p_0) \times \hat{r} = c B_0 \times \hat{r}$$

Both are perpendicular to $\hat{r}$, and each is perpendicular to the other; both vary as $1/r$; and the electric field at a particular observation point is polarized in the direction given by $(\hat{r} \times p_0) \times \hat{r}$, which is the direction of $-\hat{\theta}$ if $p_0$ defines the polar axis. Further, since the time average of the Poynting vector in the far zone is given by

$$\langle S \rangle = \frac{1}{2\mu_0} \Re \{ E_0 \times B_0^* \} = \frac{\mu_0 \omega^4 |\hat{r} \times p_0|^2}{32\pi^2 c r^2} \hat{r}$$

(see P7.26), the fields in the far zone describe a transport of energy radially away from the dipole; the time average of the total radiated power, which is given by the integral of Eq. (14.47) over a large sphere centered at the dipole, then has the explicit evaluation

$$\langle \text{radiated power} \rangle = \lim_{r \to \infty} \frac{1}{r^2} \int_0^{2\pi} \int_0^\pi (\langle S \rangle \cdot \hat{r}) r^2 \sin \theta \, d\theta \, d\phi$$

$$= \frac{4}{3} \left( \frac{\mu_0 \omega^4 |p_0|^2}{16\pi c} \right)$$

(14.48)

(14.49)

where a coordinate system in which $p_0$ defines the polar axis ($p_0 = p_0 \hat{k}$) has been chosen for evaluating the integral. Note particularly the dependence of the radiated power on the fourth power of the frequency; broadcasting at low frequencies is difficult by comparison with broadcasting at higher frequencies. Further, we conclude from Eq. (14.48) that the time average of the power radiated per unit solid angle by this dipole is given as a function of $\theta$ and $\phi$ by

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \lim_{r \to \infty} r^2 \langle S \rangle \cdot \hat{r}$$

$$= \frac{\mu_0 \omega^4}{32\pi^2 c} |\hat{r} \times p_0|^2 = \frac{\mu_0 \omega^4 |p_0|^2}{32\pi^2 c} \sin^2 \theta$$

(14.50)

(14.51)

where $d\Omega = \sin \theta \, d\theta \, d\phi$ is an element of solid angle.\(^3\) Because of their connection with emitted radiation, the fields given by Eqs. (14.45) and (14.46) are referred to as the radiation fields of an oscillating dipole.

\(^3\)Although Eq. (14.50) can be applied to any monochromatic field, it is not correct for fields exhibiting a more general time dependence. For these more general fields, the phenomenon of retardation must be more carefully handled; see Section 14.5.
In the intermediate zone, the fields undergo transition from the quasistatic fields of the near zone to the radiation fields of the far zone, but the details of that transition are very involved and will not be treated here.

### PROBLEMS

**P14.2.** Verify that Eqs. (14.31) and (14.32) follow from Eqs. (14.26) and (14.27).

**P14.3.** Letting \( p_0 = p_0 \hat{k} \) with \( p_0 \) real, work out the spherical components of the physical fields \( \mathbf{E}(\mathbf{r}, t) \) and \( \mathbf{B}(\mathbf{r}, t) \) represented by the complex fields in Eqs. (14.45) and (14.46). Describe the polarization of the radiation fields. A sketch, as, for example, Fig. 5-10 in J. M. Stone, *Radiation and Optics* (McGraw-Hill Book Company, New York, 1963), might be helpful.

**P14.4.** Starting with Eq. (14.51) and using an available numeric processing program like IDL, MATLAB, OCTAVE, or PYTHON, draw a careful polar graph of the angular distribution of the radiation emitted by an oscillating dipole. Are the lobes in the graph circles?

**P14.5.** Several aspects of the derivation of the fields of an oscillating dipole were not worked out in detail in the text. To complete this derivation, (a) derive Eqs. (14.33)–(14.35) from the starting points indicated in the text, (b) derive the fields given in Eqs. (14.41) and (14.42) from the potential in Eq. (14.40) by using Eqs. (14.34) and (14.35), (c) derive the final term in Eq. (14.47), and (d) carry out the integration to obtain Eq. (14.49).

**P14.6.** From the point of view of an oscillating dipole, radiation to distant space looks like power dissipation. Express the total power given by Eq. (14.49) in terms of the current in the dipole and, recalling that the power dissipated in a resistive circuit can be expressed as the product of a resistance and the square of the rms value of the current, infer that an oscillating dipole can be thought of as radiating into a resistance \( R_{\text{eff}} \) given by

\[
R_{\text{eff}} = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{a}{\lambda}\right)^2 = 790 \left(\frac{a}{\lambda}\right)^2 \text{ ohms}
\]

**P14.7.** Let the source of an electromagnetic field be a sinusoidally varying current of amplitude \( I_0 \) in a circular loop of radius \( a \), and let the loop lie in the \( xy \) plane with its center at the origin. Following the pattern illustrated in Section 14.2, find the magnetic vector potential \( \mathbf{A}_0 \) established by this source in the region where \( r \gg a \) and \( \kappa a \ll 1 \), find the corresponding fields \( \mathbf{B}_0 \) and \( \mathbf{E}_0 \), and compare the results with Eqs. (14.41) and (14.42). Finally, determine the fields in the far zone and compare the angular distribution of energy and the polarization of the radiation fields for this oscillating magnetic dipole with those of an oscillating electric dipole.

### 14.3 The Liénard-Wiechert Potentials

One of the few situations for which the scalar and vector potentials given by Eqs. (14.26) and (14.27) can be evaluated in closed form occurs when the source distribution is confined to a small region of space and is observed from far away. (Ultimately, we shall reduce the distribution to a point charge and all observation points will be far away.) For such an arrangement, the integral in Eq. (14.26), for example, extends over a small region of space centered about the nominal retarded location \( \mathbf{R}(t') \) of the charge distribution and, since
r' \approx \mathbf{R}(t')$ throughout the region of integration, we can approximate $|\mathbf{r} - \mathbf{r}'|$ by $|\mathbf{r} - \mathbf{R}(t')|$ to find that

$$V(\mathbf{r}, t) \approx \frac{1}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{R}(t')|} \int \rho(\mathbf{r}', t') \, dV'$$ \hfill (14.52)

The approximation, of course, becomes better as the region occupied by the charge distribution becomes smaller. It is now tempting to replace the integral in Eq. (14.52) with the total charge in the distribution. This evaluation would be correct, however, only if the integral extended over the region of space in which $\rho(\mathbf{r}', t')$ is nonzero at some fixed retarded time $t'$. Unfortunately (as we remarked in Section 14.1), the integral extends over the volume in which $\rho(\mathbf{r}', t')$ differs from zero at some fixed observation time $t$. This integral is therefore more complicated than one might at first sight suppose, and (as we shall see) the complications do not become negligible as the charge distribution shrinks to a point.

To obtain a correct evaluation of the integral under debate, suppose first that the charge distribution (at fixed $t'$) has the shape of a rectangular parallelepiped and that (at the retarded time) it is moving directly toward a remote observation point along the path shown in Fig. 14.2. Further, let this charge distribution be observed from the observation point at observation time $t$. In Fig. 14.2, the points labeled (1) and (2) at distances $\mathcal{R} + \Delta\mathcal{R}$ and $\mathcal{R}$ from point (3) are the positions at which an observer at point (3) sees the far and near surfaces of the charge distributions to be located at the fixed observation time $t$. These two points therefore bound the region of space within which $\rho(\mathbf{r}', t')$ differs from zero at
the fixed observation time \( t \) and hence also bound the volume over which the integral in Eq. (14.52) extends. These two points do not, however, bound the region within which the charge distribution is confined at any fixed instant of retarded time, for the observer at a fixed observation time \( t \) sees the far surface of the charge distribution where it was at the retarded time

\[
t'_1 = t - \frac{R + \Delta R}{c} \tag{14.53}
\]

and the near surface of the charge distribution where it was at the later (retarded) time

\[
t'_2 = t - \frac{R}{c} = t'_1 + \frac{\Delta R}{c} \tag{14.54}
\]

In the time interval \( \Delta t' = t'_2 - t'_1 = \Delta R/c \), the charge distribution moves (by our above hypothesis) toward the observer. Thus, the region within which \( \rho(r', t') \) is nonzero at the fixed observation time \( t \) is larger than the region within which \( \rho(r', t') \) is nonzero at any fixed retarded time \( t' \), the latter region at the retarded time \( t'_1 \) being bounded by points (1) and (4) in Fig. 14.2. Quantitatively, if points (1) and (4) are separated by a distance \( a \) and \( \Delta t' = \Delta R/c \) is small enough to justify approximating the velocity \( v(t') \) over the interval \( t'_1 < t' < t'_2 \) by \( v(t'_1) \), then the charge distribution moves (approximately) a distance \( v(t'_1) \Delta R/c \) in the time \( \Delta t' \) and

\[
\Delta R \approx a + v(t'_1) \frac{\Delta R}{c} \quad \Rightarrow \quad \Delta R \approx \frac{a}{1 - v(t'_1)/c} \tag{14.55}
\]

Thus, if the observer at time \( t \) sees a charge density \( \rho_0 \) (assumed uniform) in the region between points (1) and (2), we conclude that

\[
\int_{\text{fixed } t} \rho(r', t') \, dv' = \rho_0 \Delta R \Delta S \approx \rho_0 a \frac{\Delta S}{1 - v(t'_1)/c} \tag{14.56}
\]

where \( \Delta S \) is the cross-sectional area of the charge distribution in a plane at right angles to the direction of view. Finally, however, the charge density \( \rho_0 \) seen by the observer is exactly the same as the charge density existing within the charge itself. Thus, \( \rho_0 a \Delta S \) is the actual total charge \( q \) in the distribution (as reckoned by an observation made at a fixed retarded time), and we find that

\[
\int_{\text{fixed } t} \rho(r', t') \, dv' \approx \frac{q}{1 - v(t'_1)/c} \tag{14.57}
\]

Substituting Eq. (14.57) into Eq. (14.52), we find then that the scalar potential established by a small charge distribution is given approximately by

\[
V(r, t) \approx \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{R}(t')|[1 - v(t'_1)/c]} \tag{14.58}
\]

---

4The region bounded by points (1) and (4) is the region over which the integral in Eq. (14.52) would extend if it were evaluated at the fixed retarded time \( t'_1 \).

5In looking at a fixed observation time \( t \), the observer is not seeing the same total charge merely distributed over a larger volume. She is seeing the combination of several portions of this charge at different retarded times. Since this charge is moving, the portions that the observer sees at different retarded times will contain some of the same fundamental charges. Thus, when she looks at a fixed observation time, the observer will in general see some of the same charges in more than one place (if the charges are moving toward her). Hence, the apparent total charge at a fixed observation time will be larger than the actual total charge at a fixed retarded time. Anyone who has tried to count monkeys at the zoo has encountered a similar phenomenon.
where \( t' \) is a retarded time somewhere in the interval \( t'_1 \leq t' \leq t'_2 \). This approximation, of course, becomes progressively more accurate as the dimensions of the charge distribution shrink to zero; it becomes exact for an idealized point charge, for which \( t'_1 \to t'_2 \) and

\[
V(r, t) = \frac{1}{4\pi \varepsilon_0} \left(\frac{q}{|r - \mathbf{R}(t')|[1 - v(t')/c]}\right)
\]

(14.59)

where \( \mathbf{R}(t') \) is the position and \( v(t') \) the speed of the charge \( q \) at the retarded time \( t' \) determined (as a function of \( t \)) by the equation

\[
t' = t - \frac{1}{c}|r - \mathbf{R}(t')|
\]

(14.60)

[Compare Eq. (14.24).]

The generalization of Eq. (14.59) to point charges not moving directly toward the observation point is immediate. We replace the speed \( v(t') \) by the component of the velocity in the direction of the observation point; i.e., we make the replacement

\[
v(t') \longrightarrow v(t') \cdot \frac{\mathbf{r} - \mathbf{R}(t')}{|\mathbf{r} - \mathbf{R}(t')|} = v(t') \cdot \hat{\mathbf{n}}(t')
\]

(14.61)

where

\[
\hat{\mathbf{n}}(t') = \frac{\mathbf{r} - \mathbf{R}(t')}{|\mathbf{r} - \mathbf{R}(t')|}
\]

(14.62)

is a unit vector directed toward the observation point from the (retarded) position of the charge. Thus, defining

\[
\mathcal{R} = |\mathbf{r} - \mathbf{R}(t')|
\]

(14.63)

\[
\xi = 1 - \frac{v(t')}{c} \cdot \hat{\mathbf{n}}(t')
\]

(14.64)

we find finally from Eq. (14.59) that the scalar potential established at \((r, t)\) by a moving point charge is given by

\[
V(r, t) = \frac{1}{4\pi \varepsilon_0} \frac{q}{|r - \mathbf{R}(t')|} \left(1 - \frac{v(t')}{c} \cdot \hat{\mathbf{n}}(t')\right) = \frac{q}{4\pi \varepsilon_0 \mathcal{R} \xi}
\]

(14.65)

Explicit indication that \( \mathcal{R} \) and \( \xi \) are to be evaluated at the retarded time is suppressed.

By arguments similar to those presented above, we find that a moving point charge establishes a vector potential at \((r, t)\) given by

\[
\mathbf{A}(r, t) = \frac{\mu_0 q v(t')}{4\pi \mathcal{R} \xi}
\]

(14.66)

Together, the potentials given by Eqs. (14.65) and (14.66) are called the Liénard-Wiechert potentials and they apply specifically to a moving point charge."
14.4. THE FIELDS OF A MOVING POINT CHARGE

Once the potentials established by a particular source distribution have been determined, finding the fields “merely” involves substituting the potentials into Eqs. (14.5) and (14.6). In applying this method to determine the fields established by a moving point charge, however, we must exercise considerable mathematical care, first, because Eqs. (14.5) and (14.6) require spatial derivatives at constant observation time \( t \) (not at constant retarded time \( t' \)) and, second, because the derivative \( \partial/\partial t \) (not \( \partial/\partial t' \)) is needed. In evaluating the necessary derivatives of the potential, we must therefore not overlook the dependence of the retarded time (in terms of which the potentials have been expressed) on both \( \mathbf{r} \) and \( t \). [See Eq. (14.60).] In this section, we shall summarize the mathematical manipulations that yield the fields when the potentials in Eqs. (14.65) and (14.66) are substituted into Eqs. (14.5) and (14.6); several of the detailed mathematical evaluations, however, are relegated to P14.12 and P14.13.

Let us begin by evaluating the negative gradient of \( V(\mathbf{r}, t) \). From Eq. (14.65) and then Eqs. (14.62)–(14.64), we find that
\[ -\nabla V = -\frac{q}{4\pi \epsilon_0} \nabla \left( \frac{1}{R\xi} \right) = \frac{q}{4\pi \epsilon_0 R^2 \xi^2} \nabla (R\xi) \]

\[ = \frac{q}{4\pi \epsilon_0 (R\xi)^2} \nabla \left( |r - R(t')| - \frac{v(t')}{c} \cdot |r - R(t')| \right) \quad (14.67) \]

which breaks into two terms. Writing the first of these terms in the form

\[ \nabla |r - R| = \frac{1}{2|r - R|} \nabla |r - R|^2 = \frac{1}{2|r - R|} \nabla [(r - R) \cdot (r - R)] \quad (14.68) \]

[see Eq. (C.6)] and then using the vector identity in Eq. (C.15) and the two identities \( \nabla \times r = 0 \) and \((\hat{n} \cdot \nabla)r = \hat{n} (P14.12)\), we find that

\[ \nabla |r - R| = (\hat{n} \cdot \nabla)(r - R) + \hat{n} \times [\nabla \times (r - R)] \]

\[ = \hat{n} - (\hat{n} \cdot \nabla)R - \hat{n} \times (\nabla \times R) \quad (14.69) \]

\(\text{where} \ \hat{n} \ \text{is given by Eq. (14.62).} \ \text{Further, remembering that} \ R \ \text{is a function of} \ t' \ \text{and is therefore implicitly a function of} \ r, \ \text{we find that} \)

\[ (\hat{n} \cdot \nabla)R = \sum_{i,j=1}^{3} n_{i} \frac{\partial}{\partial x_{i}} R_{j} \hat{e}_{j} = \sum_{i,j=1}^{3} n_{i} \frac{dR_{j}}{dt'} \frac{\partial}{\partial x_{i}} \hat{e}_{j} = (\hat{n} \cdot \nabla) \dot{v} \quad (14.70) \]

\(\text{where} \ v \ \text{is the velocity of the particle evaluated at the retarded time.} \ \text{Similarly, we find that} \)

\[ \nabla \times R = \nabla t' \times v \quad (14.71) \]

\(\text{(P14.12).} \ \text{Finally, substituting Eqs. (14.70) and (14.71) into Eq. (14.69) and using the vector identity in Eq. (C.1), we find that the first part of the gradient in Eq. (14.67) has the alternative expression} \)

\[ \nabla |r - R| = \hat{n} - (\hat{n} \cdot \nabla) \dot{v} \quad (14.72) \]

Turning now to the second part of the gradient in Eq. (14.67) and using vector identities already referred to above, we find that

\[ \nabla \left[ \frac{v}{c} \cdot (r - R) \right] = \left( \frac{v}{c} \cdot \nabla \right) (r - R) + [(r - R) \cdot \nabla] \frac{v}{c} \]

\[ + \frac{v}{c} \times [\nabla \times (r - R)] + (r - R) \times \left[ \nabla \times \frac{v}{c} \right] \]

\[ = \frac{v}{c} - \left( \frac{v}{c} \cdot \nabla \right) R + [(r - R) \nabla] \frac{v}{c} \]

\[ - \frac{v}{c} \times [\nabla \times R] + (r - R) \times \left[ \nabla \times \frac{v}{c} \right] \quad (14.73) \]

Then, using results similar to Eqs. (14.70) and (14.71), we find that

\[ \nabla \left[ \frac{v}{c} \cdot (r - R) \right] = \frac{v}{c} - \left( \frac{v}{c} \cdot \nabla \dot{v} \right) \frac{v}{c} + (\hat{R} \hat{n}) \cdot \nabla \dot{v} \frac{\dot{v}}{c} \]

\[ - \frac{v}{c} \times (\nabla \dot{v} \times \dot{v}) + \hat{R} \hat{n} \times \left( \nabla \dot{v} \times \frac{\dot{v}}{c} \right) \]

\[ = \frac{v}{c} - \frac{v}{c} \nabla \dot{v} + (\hat{R} \hat{n}) \cdot \nabla \dot{v} \frac{\dot{v}}{c} + \hat{R} \hat{n} \times \left( \nabla \dot{v} \times \frac{\dot{v}}{c} \right) \quad (14.74) \]
where $\mathcal{R}$ is given by Eq. (14.63) and

$$\mathbf{v} = \frac{d\mathbf{v}(t')}{dt'}$$

(14.75)

is the acceleration of the particle at the retarded time. At last, substituting Eqs. (14.72) and (14.73) into Eq. (14.67), we find that

$$-\nabla V = \frac{q}{4\pi\varepsilon_0 R^2 \xi^2} \left[ \hat{n} - \frac{\mathbf{v}}{c} - \left( \hat{n} \cdot \mathbf{v} - \frac{v^2}{c} \right) \nabla t' \right]$$

$$-\frac{q}{4\pi\varepsilon_0 R \xi^2} \left[ (\hat{n} \cdot \nabla t') \frac{\mathbf{v}}{c} + \hat{n} \times \left( \nabla t' \times \frac{\nabla}{c} \right) \right]$$

(14.76)

By a similar calculation, we find from Eq. (14.66) that the time derivative of the vector potential is given by

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{A}}{\partial t'} \frac{\partial t'}{\partial t}$$

$$= \frac{q}{4\pi\varepsilon_0 R^2 \xi^3 c^2} \left[ (\hat{n} \cdot \mathbf{v}) \mathbf{v} - \frac{v^2}{c} \mathbf{v} \right] \frac{\partial t'}{\partial t}$$

$$+ \frac{q}{4\pi\varepsilon_0 R \xi^2 c^2} \left[ \mathbf{v} \left( 1 - \frac{\mathbf{v} \cdot \hat{n}}{c} \right) + \mathbf{v} \left( \frac{\mathbf{v} \cdot \hat{n}}{c} \right) \right] \frac{\partial t'}{\partial t}$$

(14.77)

(See P14.12.) Further, we find from Eq. (14.60) not only that

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c} \frac{\partial}{\partial t'} |\mathbf{r} - \mathbf{R}(t')| \frac{\partial t'}{\partial t}$$

$$= 1 + \frac{\hat{n} \cdot \mathbf{v} \partial t'}{c \partial t} \implies \frac{\partial t'}{\partial t} = \frac{1}{1 - \hat{n} \cdot \mathbf{v}/c} = \frac{1}{\xi}$$

(14.78)

but also that

$$\nabla t' = -\frac{1}{c} \nabla |\mathbf{r} - \mathbf{R}(t')| = -\frac{1}{c} \left( \hat{n} - (\hat{n} \cdot \mathbf{v}) \nabla t' \right)$$

$$\implies \nabla t' = \frac{-\hat{n}/c}{1 - \hat{n} \cdot \mathbf{v}/c} = \frac{\hat{n}}{\xi c}$$

(14.79)

Thus, substituting Eqs. (14.78) and (14.79) into Eqs. (14.76) and (14.77), substituting those results in turn into Eq. (14.5), and finally using a number of by now familiar vector identities, we find that the electric field established by a point charge is given by

$$\mathbf{E}(\mathbf{r}, t) = -\nabla V(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = \frac{qN}{4\pi\varepsilon_0 R^2 \xi^3} + \frac{qM}{4\pi\varepsilon_0 c^2 R \xi^3}$$

(14.80)

where, for brevity, we have introduced the quantities

$$\mathbf{N} = \left( \hat{n} - \frac{\mathbf{v}}{c} \right) \left( 1 - \frac{v^2}{c^2} \right)$$

(14.81)
and
\[ M = \hat{n} \times \left[ \left( \hat{n} - \frac{\mathbf{v}}{c} \right) \times \dot{\mathbf{v}} \right] \] (14.82)

(See P14.12.) An equally lengthy but similar argument leads ultimately to the expression
\[ \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \hat{n} \times \mathbf{E}(\mathbf{r}, t) \] (14.83)

for the magnetic induction field accompanying the electric field in Eq. (14.80). (See P14.13.) We remind the reader that all quantities with unspecified time arguments in Eqs. (14.80)–(14.83) are evaluated at the retarded time.


**PROBLEMS**

**P14.12.** To complete the derivation of the electric field established by a moving point charge, (a) show that \( \left( \hat{n} \cdot \nabla \right) \mathbf{r} = \hat{n} \), (b) derive Eq. (14.71), (c) derive Eq. (14.77), and (d) combine the expressions indicated in the text to derive Eq. (14.80).

**P14.13.** Derive Eq. (14.83). Shorten the labor as much as possible by using results already obtained.

**P14.14.** Show that the fields established by a particle moving non-relativistically are given by
\[
\begin{align*}
\mathbf{E}(\mathbf{r}, t) &= \frac{q}{4\pi\epsilon_0} \left[ \frac{\hat{n}}{R^2} \left( 1 - \frac{v^2}{c^2} \right) \hat{n} - \frac{\mathbf{v} \times \hat{n}}{c} \right] + \frac{\dot{\mathbf{n}} \times \left( \hat{n} \times \mathbf{v} \right)}{c^2 R^2} \\
\mathbf{B}(\mathbf{r}, t) &= \frac{q}{4\pi\epsilon_0} \left[ \frac{\mathbf{v} \times \hat{n}}{c^2 R^2} + \frac{\dot{\mathbf{n}} \times \hat{n}}{c^3 R^2} \right]
\end{align*}
\]


**P14.15.** The position of a charge \( q \) as a function of time \( t \) is given by \( \mathbf{R} = vt \hat{k} \), where \( v \) is a positive constant and \( \hat{k} \) is a unit vector in the \( z \) direction. Let the observation point have coordinates \((r, \phi, z, t)\) in cylindrical coordinates. (a) Show that the fields established by this charge and current distribution are given by
\[
\begin{align*}
\mathbf{E}(r, \phi, z, t) &= \frac{q}{4\pi\epsilon_0} \left( 1 - \frac{v^2}{c^2} \right) \hat{r} + \frac{z \hat{v} + (z - vt) \hat{k}}{c} \\
\mathbf{B}(r, \phi, z, t) &= \frac{qv}{c^2 R^2} \left( 1 - \frac{v^2}{c^2} \right) + \frac{z \hat{v}}{c^2 R^2}
\end{align*}
\]

(b) Calculate \( |\mathbf{E}(r, \phi, z, t = 0)| \), express the result in terms of the *spherical* polar coordinates of the observation point and, using a suitable computer tool, draw careful polar graphs of this quantity as a function of the polar angle for fixed radial coordinate and
for several different values of $v/c$. Describe how this graph is altered as $v$ moves from non-relativistic values to relativistic values. Optional: Derive the above fields by applying Eqs. (14.5) and (14.6) to the results of P14.11.

\section*{14.5 Radiation from Accelerated Point Charges}

The energy transported by any electromagnetic field is determined by the Poynting vector $\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0$. For the fields of a point charge [Eqs. (14.80) and (14.83)], the Poynting vector has the more explicit evaluation

\begin{equation}
\mathbf{S} = \left( \frac{q}{4\pi\varepsilon_0 c^2} \right)^2 \frac{1}{\mu_0 c} \left[ \frac{\mathbf{N}}{\mathcal{R}^2} + \frac{\mathbf{M}}{c^2 \mathcal{R}} \right] \times \left[ \hat{n} \times \mathbf{N} \frac{1}{\mathcal{R}^2} + \hat{n} \times \mathbf{M} \frac{1}{c^2 \mathcal{R}} \right]
\end{equation}

\begin{equation}
= \frac{q^2 [M \times (\mathbf{M} \times \mathbf{n})]}{16\pi^2\varepsilon_0 c^2 \mathcal{R}^2 \xi^6} + O\left( \frac{1}{\mathcal{R}^3} \right)
\end{equation}

\begin{equation}
= \frac{q^2 [M^2 \mathbf{n} - (\mathbf{M} \cdot \mathbf{n}) \mathbf{M}]}{16\pi^2\varepsilon_0 c^2 \mathcal{R}^2 \xi^6} + O\left( \frac{1}{\mathcal{R}^3} \right)
\end{equation}

\begin{equation}
= \frac{q^2 M^2}{16\pi^2\varepsilon_0 c^2 \mathcal{R}^2 \xi^6} \hat{n} + O\left( \frac{1}{\mathcal{R}^3} \right)
\end{equation}

(14.84)

where (1) the final form follows because $\mathbf{M}$ and $\mathbf{n}$ are perpendicular [see Eq. (14.82)] and (2) terms in $1/\mathcal{R}^3$ and $1/\mathcal{R}^4$ have been indicated only symbolically because they will prove to play no role in determining the radiation emitted by the charge. As we expected, the dominant term in Eq. (14.84) is directed away from the (retarded) position of the charge.

To connect the Poynting vector more explicitly with the distribution of radiated energy, note first that an observer who measures the total energy $\Delta W$ transported across a small surface of area $\Delta S$ during the (observation) time interval $t_1 \leq t \leq t_2$ will obtain a result given by

\begin{equation}
\Delta W = \int_{t_1}^{t_2} (\mathbf{S} \cdot \Delta \mathbf{S}) \, dt
\end{equation}

(14.85)

It is, however, more appropriate for our present application to rewrite this integral as an integral over the retarded time interval $t'_1 \leq t' \leq t'_2$, where

\begin{equation}
t'_1 = t_1 - \frac{1}{c} |\mathbf{r} - \mathbf{R}(t'_1)| \quad ; \quad t'_2 = t_2 - \frac{1}{c} |\mathbf{r} - \mathbf{R}(t'_2)|
\end{equation}

(14.86)

and $\mathbf{r}$ is the position of the area element $\Delta \mathbf{S}$. Since

\begin{equation}
\frac{dt}{dt'} \frac{dt'}{dt} = \xi \, dt'
\end{equation}

(14.87)

[see Eq. (14.78)], we find that

\begin{equation}
\Delta W = \int_{t'_1}^{t'_2} \xi (\mathbf{S} \cdot \Delta \mathbf{S}) \, dt'
\end{equation}

(14.88)
from which we conclude that the rate at which energy is radiated toward the surface $\Delta S$ [or equivalently the power $\Delta P(t')$ radiated toward $\Delta S$ at the retarded time $t'$] is given by

$$\Delta P(t') = \xi \mathbf{S} \cdot \Delta S$$  \hspace{1cm} (14.89)

Now, let $\Delta S$ specifically represent a surface of area $|\Delta S|$ oriented with its plane perpendicular to the vector from the retarded position of the charge to the surface; i.e., let $\Delta S = \hat{n}|\Delta S|$. Then, $|\Delta S| = R^2 \Delta \Omega$, where $\Delta \Omega$ is the solid angle subtended by the surface as viewed from the retarded position of the charge (Section 2.3), and we find from Eq. (14.89) that

$$\Delta P(t') = \xi \mathbf{S} \cdot \hat{n} R^2 \Delta \Omega$$

$$\Rightarrow \frac{dP(t')}{d\Omega} = \xi R^2 \mathbf{S} \cdot \hat{n}$$  \hspace{1cm} (14.90)

where $dP(t')/d\Omega$ gives the power emitted per unit solid angle at the retarded time $t'$. More usefully, if $R \to \infty$, Eq. (14.90) then gives that portion of the power emitted per unit solid angle at the retarded time $t'$ that is destined to escape from the charge altogether; i.e.,

$$\frac{dP(t')}{d\Omega} = \lim_{R \to \infty} \xi R^2 \mathbf{S} \cdot \hat{n}$$  \hspace{1cm} (14.91)

gives the angular distribution of the power actually radiated away from the charge. Specifically, from the Poynting vector in Eq. (14.84), we find that

$$\frac{dP(t')}{d\omega} = \frac{q^2 M^2}{16\pi^3 \epsilon_0 c^3 \xi^5} \frac{q^2 \hat{n} \times \left[ \left( \hat{n} - \frac{v}{c} \right) \times v \right]^2}{16\pi^2 \epsilon_0 c^3 \xi^5}$$  \hspace{1cm} (14.92)

By way of reminder, all time-dependent quantities in this equation are evaluated at the retarded time. Further, the vector whose square magnitude appears has the same direction as the radiative part of the electric field given in Eq. (14.80) and hence conveys the polarization of the radiation at the observation point. Finally, note that there is no radiated power at all ($dP/d\Omega = 0$) unless the charged particle is accelerated.

Equation (14.92) assumes its simplest form in the non-relativistic limit $(v/c \ll 1)$, in which case $\xi \to 1$ [Eq. (14.64)]. Further, if we assume that the particle is near the origin, then $\hat{n} = \hat{r}$ and Eq. (14.92) reduces to

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \hat{r} \times \hat{v}^2}{16\pi^2 \epsilon_0 c^3} = \frac{q^2 v^2 \sin^2 \theta}{16\pi^2 \epsilon_0 c^3}$$  \hspace{1cm} (14.93)

where (in the final form) $\theta$ is the angle between the direction of the acceleration and the direction to the observation point. The radiated power is therefore distributed in space as shown in Fig. 14.3. Further, the total power emitted into all directions is given by the integral of Eq. (14.93) over all solid angles, viz., by

$$P(t') = \int_0^{2\pi} \int_0^\pi \frac{dP(t')}{d\Omega} \sin \theta \, d\theta \, d\phi = \frac{q^2 v^2}{6\pi \epsilon_0 c^3}$$  \hspace{1cm} (14.94)

which is known as Larmor’s formula;\(^8\) its generalization to include relativistic motion of the charge is

$$P(t') = \frac{q^2}{6\pi \epsilon_0 c^3} \frac{1}{[1 - (v/c)^2]^{3/2}} \left[ v^2 - \frac{|v \times \hat{v}|^2}{c^2} \right]$$  \hspace{1cm} (14.95)

\(^8\)Irish physicist and mathematician Joseph Larmor, b. 11 July 1857 in Magheragall, County Antrim, Ireland; d. 19 May 1942 in Holywood, County Down, Northern Ireland.
14.5. RADIATION FROM ACCELERATED POINT CHARGES

Figure 14.3: Angular distribution of the radiation from an accelerated point charge when the motion is non-relativistic. The distance from the origin to the curve in the direction at an angle $\theta$ to the direction of the acceleration is proportional to $dP(t')/d\Omega$ at that angle.

and is known as Liénard’s formula.$^9$

Equation (14.85) also provides the starting point for determining the frequency distribution of the emitted radiation. Again, we take $\Delta S = \hat{n} R^2 \Delta \Omega$ [see the sentences following Eq. (14.89)] but this time we retain the observer’s time and extend the integral over all time. We find that the total energy emitted per unit solid angle is given by

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \lim_{R \to \infty} (S \cdot \hat{n}) R^2 dt$$

(14.96)

Substituting from Eq. (14.84), however, we find more explicitly that

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |Q(t)|^2 dt$$

(14.97)

where

$$Q(t) = \left( \frac{q^2}{16\pi^2 \epsilon_0 c^3} \right)^{1/2} \frac{M}{\xi^3}$$

(14.98)

and $M$ and $\xi$ continue to be evaluated at the retarded time. To obtain the frequency distribution of the radiated energy, we first introduce the Fourier transform

$$\tilde{Q}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} Q(t) dt$$

(14.99)

(See Appendix D, but note that, in keeping with the established convention for this topic, we use the exponential $e^{i\omega t}$ rather than $e^{-i\omega t}$ in defining the transform.) Then, we use Parseval’s

theorem\(^\text{10}\) [Eq. (D.33), which holds no matter which sign is taken in the exponential in Eq. (14.99)] to write Eq. (14.97) as the integral
\[
\frac{dW}{d\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{Q}(\omega)|^2 d\omega \tag{14.100}
\]
where \(|\tilde{Q}(\omega)|^2 = \tilde{Q}^*(\omega) \cdot \tilde{Q}(\omega)\). Next, we write Eq. (14.100) as an integral over positive frequencies only, i.e., as
\[
\frac{dW}{d\Omega} = \frac{1}{2\pi} \int_{0}^{\infty} \left[ |\tilde{Q}(\omega)|^2 + |\tilde{Q}(-\omega)|^2 \right] d\omega \tag{14.101}
\]
which simplifies to
\[
\frac{dW}{d\Omega} = \frac{1}{\pi} \int_{0}^{\infty} |\tilde{Q}(\omega)|^2 d\omega \tag{14.102}
\]
because \(Q(t)\) is real and Eq. (14.99) then yields that
\[
\tilde{Q}^*(\omega) = \tilde{Q}(-\omega) \implies |\tilde{Q}(-\omega)|^2 = |\tilde{Q}(\omega)|^2 \tag{14.103}
\]
Finally, we note that the integrand in Eq. (14.102) is just the distribution of \(dW/d\Omega\) in frequency. Thus, the frequency distribution of the total energy emitted per unit solid angle is given by
\[
\frac{dW}{d\Omega d\omega} = \frac{1}{\pi} |\tilde{Q}(\omega)|^2 \tag{14.104}
\]
Evaluation of \(\tilde{Q}(\omega)\) can be simplified in two ways. First, we substitute from Eq. (14.98) and then rewrite Eq. (14.99) as an integral over retarded times, obtaining
\[
\tilde{Q}(\omega) = \left( \frac{q^2}{16\pi^2\epsilon_0 c^3} \right)^{1/2} \int_{-\infty}^{\infty} \frac{M(t')}{\xi^2} e^{i\omega(t' + |r - R(t')|/c)} dt' \tag{14.105}
\]
[Compare Eq. (14.87).] Second, we assume that the observation point is remote from the charge and that motion of the charge is confined to a region close to the origin of coordinates. Under these conditions, \(\hat{n}\) is approximately constant and \(r \approx r\hat{n}\) so that
\[
|r - R(t')| \approx r \left| \hat{n} - \frac{R(t')}{r} \right| \approx r - \hat{n} \cdot R(t') \tag{14.106}
\]
and, apart from a phase factor \(e^{i\omega r/c}\) [which we can ignore because only the square magnitude of \(\tilde{Q}(\omega)\) is given a physical interpretation], we find from Eqs. (14.105), (14.82), and (14.64) that
\[
\tilde{Q}(\omega) = \left( \frac{q^2}{16\pi^2\epsilon_0 c^3} \right)^{1/2} \int_{-\infty}^{\infty} \hat{n} \times \left[ \left( \hat{n} - \frac{v}{c} \right) \times \dot{v} \right] e^{i\omega(t' - \hat{n} \cdot R(t')/c)} dt' \tag{14.107}
\]
in which \(R\), \(v\), \(\dot{v}\), and \(\hat{n}\) are all evaluated at the retarded time \(t'\). Thus, in principle, the problem of finding the distribution in frequency and in solid angle of the energy emitted by a charged particle following a prescribed trajectory is solved: We evaluate \(\tilde{Q}(\omega)\) from Eq. (14.107) and substitute the result into Eq. (14.104). In practice the task is rarely trivial; two examples may be found in P14.21 and P14.22.

\(^{10}\) French mathematician Marc-Antoine Parseval, b. 27 April 1755; d. 16 August 1836.
PROBLEMS

P14.16. Do the necessary integration and obtain Eq. (14.94).

P14.17. (a) Show that an electron moving non-relativistically in a constant magnetic induction field of magnitude \( B \) no larger than about 10,000 G radiates a very small fraction of its total energy per cycle of its orbit. (b) Obtain a differential equation for the energy \( E \) of the particle as a function of time and integrate the equation. **Hint:** The total power radiated is equal to the rate of change of kinetic energy. Why?

P14.18. A particle experiences a constant acceleration in the direction of its velocity. (a) Determine the angular distribution of the emitted power as given by Eq. (14.92). Do not assume the motion to be non-relativistic. (b) Show the results in a polar graph and indicate how this graph changes as the particle velocity approaches the speed of light. (c) Determine the angle at which the maximum power appears and draw a graph of that angle as a function of \( v/c \).

P14.19. Consider an electron moving (perhaps relativistically) along a straight line, as, for example, in a linear accelerator, and let the line define the \( x \) axis. (a) Taking the momentum of the electron to be given by \( p = mv/\sqrt{1-(v/c)^2} \), show from Eq. (14.95) that the total power radiated is given by

\[
P = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} \left( \frac{dp}{dt} \right)^2
\]

(b) Show for this particle that the rate \( P \) at which energy is radiated bears the ratio

\[
\frac{\text{power radiated}}{\text{power input}} = \frac{P}{dE/dt} = \frac{q^2}{6\pi\varepsilon_0 c^3 m^2 v} \frac{dE}{dx}
\]
to the rate \( dE/dt \) at which external forces do work on the particle. **Hint:** Show first that \( dp/dt = dE/dx \). (c) In a typical linear accelerator, \( dE/dx \leq 10 \text{ MeV/m} \). Estimate \( P/(dE/dt) \) numerically for such an accelerator, taking \( v \approx c \), and comment on the importance of radiative losses in linear accelerators.

P14.20. Once the speed \( v \) of a particle moving in a circle of radius \( r \) approaches \( c \), the acceleration of the particle is approximately perpendicular to the velocity and has magnitude \( v^2/r \). (a) Show from Eq. (14.95) that the energy radiated per revolution \( \delta W \) is given by

\[
\delta W = \frac{q^2}{3\varepsilon_0 r} \left( \frac{v}{c} \right)^3 \left( \frac{E}{mc^2} \right)^4
\]

where \( E = mc^2/\sqrt{1-(v/c)^2} \) and \( m \) is the (rest) mass of the particle. (b) To what maximum energy (in MeV) can an electron be accelerated in a synchrotron having a radius of 10 m and capable of supplying about 5 MeV/revolution to the particle? Comment on the importance of radiative losses in circular particle accelerators. **Hint:** Assume that the maximum energy is high enough that \( v \approx c \).

P14.21. Consider a charged particle constrained to move along the \( z \) axis so that the acceleration and the velocity necessarily are either parallel or antiparallel. Suppose further that the velocity of the particle is as follows

\[
v(t') = \begin{cases} v_0 \hat{k}, & t' < 0 \\ \frac{1}{2}v_0 \left[ 1 + \cos \left( \frac{\pi t'}{T} \right) \right] \hat{k}, & 0 < t' < T \\ 0 \hat{k}, & t' > T \end{cases}
\]

which represents a smooth deceleration from some initial (constant) velocity to a condition of rest at time \( T \). Calculate the distribution of the emitted energy in frequency and in
solid angle; i.e., calculate \( dW/d\Omega \ d\omega \). Then calculate the integral of this quantity over all solid angles, sketch a graph of the result as a function of \( \omega \), and determine how this graph is affected by changes in the time \( T \) required for the particle to be brought to rest. Assume that the motion is non-relativistic. Hint: For non-relativistic motion, \( v/c \ll 1 \), and, in particular \( \mathbf{R}(t') \) is on the order of \( \langle v \rangle T \), where \( \langle v \rangle \) is a characteristic velocity of the particle. Thus, \( \hat{n} \cdot \mathbf{R}(t')/c \approx \langle v \rangle t'/c \) and can be ignored relative to \( t' \) in the exponent in Eq. (14.107). This neglect of \( \hat{n} \cdot \mathbf{R} \) is called the dipole approximation.

P14.22. A charged particle moves in a circular path described by the trajectory
\[
\mathbf{R}(t') = R_0 (\cos \omega_0 t' \hat{i} + \sin \omega_0 t' \hat{j})
\]
during the time interval \( -T < t < T \) and is at rest for \( |t| > T \). Assume that the motion is non-relativistic. (a) Show that the term \( \hat{n} \cdot \mathbf{R}(t')/c \) in the exponent in Eq. (14.107) can be neglected. Hint: See the hint in P14.21. (b) Find \( dW/d\Omega \ d\omega \) and the integral of this quantity over all solid angles. Sketch the latter distribution as a function of \( \omega \) for several different values of \( T \) and describe the emitted spectrum in words. Radiation by this particle is called (non-relativistic) cyclotron radiation.

P14.23. Assume that the electron in the hydrogen atom moves non-relativistically in a circular orbit of radius \( r \) and that the amount of energy radiated per cycle is a small fraction of the total energy. (a) Show that the total energy \( E \) of the electron and the total power radiated \( P \) are given by
\[
E = -\frac{q^2}{8\pi \varepsilon_0 r} ; \quad P = \frac{q^5}{96\pi^3 \varepsilon_0^3 c m^2} \frac{1}{r^4}
\]
where \( q \) and \( m \) are the charge and mass of the electron. (b) Argue that \( dE/dt = -P \), obtain a differential equation for \( r \), and solve that equation for \( r \) as a function of \( t \). (c) How long does it take for the initial radius—say \( r_0 \)—to be reduced to zero? Insert appropriate numerical values for all quantities and estimate the (classical) lifetime of the hydrogen atom.

14.6 The Radiation Reaction

In the previous sections, we have confined our attention to calculating the radiation emitted by a charged particle following a prescribed trajectory. In those cases (e.g., P14.17 and P14.23) where the trajectory was determined by forces on the particle from external electric and magnetic fields, we first calculated the trajectory by assuming that there was no radiation and then we calculated the radiation by assuming that the trajectory in the presence of radiation differs only slightly from the trajectory in the absence of radiation. Frequently, this two-step solution of a complicated radiation problem yields a valid approximation, because the energy radiated in some characteristic time is often a small fraction of the total energy of the particle and changes in the trajectory arising because of radiative losses can therefore be treated quasi-statically. Strictly, however, this two-step procedure is not correct, for the radiated field carries off energy and momentum and hence must result in a back force or a radiative force (usually called the radiation reaction) on the radiating particle. This radiation reaction, of course, affects the trajectory of the particle and should be taken into account when that trajectory is calculated. The complete problem, however, is extremely difficult and—to the author’s knowledge—no fully satisfactory solution has yet
been developed. In this book, we shall be content with a very simple, approximate treatment that is restricted to non-relativistic motion. Essentially, we add a radiation reaction \( \mathbf{F}_{\text{rad}} \) to the external forces \( \mathbf{F} \) on the particle and write

\[
m\ddot{\mathbf{v}} = \mathbf{F} + \mathbf{F}_{\text{rad}}
\]

for the equation of motion. We then determine \( \mathbf{F}_{\text{rad}} \) so that it accounts properly for the total energy radiated; i.e., we choose \( \mathbf{F}_{\text{rad}} \) so that

\[
\int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v} \, dt = - \int_{t_1}^{t_2} \frac{q^2}{6\pi\varepsilon_0 c^3} \dot{\mathbf{v}} \cdot \dot{\mathbf{v}} \, dt
\]

(14.109)

where the left-hand side is the work done on the particle by \( \mathbf{F}_{\text{rad}} \) and the right-hand side is the negative of the energy radiated away [Compare Eq. (14.94)] in the time interval \( t_1 \leq t \leq t_2 \). Integration of the right-hand side of Eq. (14.109) by parts gives

\[
\int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v} \, dt = - \int_{t_1}^{t_2} \frac{q^2}{6\pi\varepsilon_0 c^3} \left( \mathbf{v} \right) \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{q^2}{6\pi\varepsilon_0 c^3} \dot{\mathbf{v}} \cdot \mathbf{v} \, dt
\]

(14.110)

We now assume that the interval \( t_1 \leq t \leq t_2 \) covers the entire time during which the particle is accelerated so that \( \dot{\mathbf{v}}(t_1) = \dot{\mathbf{v}}(t_2) = 0 \). Then, the first term on the right in Eq. (14.110) is zero and we find that the total energy radiated throughout the motion is given by either side of the equation

\[
\int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v} \, dt = \int_{t_1}^{t_2} \frac{q^2}{6\pi\varepsilon_0 c^3} \dot{\mathbf{v}} \cdot \mathbf{v} \, dt
\]

(14.111)

This equation in turn suggests that we take

\[
\mathbf{F}_{\text{rad}} = \frac{q^2}{6\pi\varepsilon_0 c^3} \dot{\mathbf{v}}
\]

(14.112)

and thus the equation of motion for a radiating charged particle [Eq. (14.108)] becomes

\[
m \left( \ddot{\mathbf{v}} - \frac{q^2}{6\pi\varepsilon_0 mc^3} \ddot{\mathbf{v}} \right) = \mathbf{F}
\]

(14.113)

which is called the Abraham-Lorentz equation of motion.\(^{11}\) By our derivation, of course, we cannot expect Eq. (14.112) to be correct except in the average sense expressed by Eq. (14.111), but a more detailed treatment must be left to other authors.\(^ {12}\) A few properties of the solutions to Eq. (14.113) are explored in P14.24 and P14.25.

**PROBLEMS**

**P14.24.** (a) Dimensionally, the quantity \( q^2/6\pi\varepsilon_0 mc^3 = \tau \) in Eq. (14.113) must be a time. What is the numerical value of \( \tau \) for an electron? (b) Assuming that the force \( \mathbf{F} \) in Eq. (14.113) is zero, show that the runaway solution \( \ddot{\mathbf{v}} \) = (constant) \( e^{t/\tau} \) exists and discuss whether this solution is physically admissible.

---

\(^{11}\)German physicist Max Abraham, b. 26 March 1975 in Danzig, Germany (now Gdansk, Poland); d. 16 November 1922 in Munich, Germany.

P14.25. Assume that the force $F$ in Eq. (14.113) is known as a function of time and let
\[
\tau = \frac{q^2}{6\pi\varepsilon_0 mc^3}.
\]
Show that
\[
m\ddot{v} = \frac{e^{t/\tau}}{\tau} \int_0^\infty e^{-t'/\tau} F(t') \, dt'
\]
where the upper limit has been chosen so that as $\tau \to 0$ (no radiation) the equation approaches $m\ddot{v} = F$. Discuss what it means for the acceleration at a given time $t$ to be determined by the forces acting at times into the indefinite future.

SUPPLEMENTARY PROBLEMS

P14.26. The four-dimensional Fourier transform of a function $\phi(r, t)$ is defined by
\[
\tilde{\phi}(\kappa, \omega) = \int \phi(r, t) e^{-i(\kappa \cdot r - \omega t)} \, dr \, dt
\]
where the integral extends over all space and time. (a) Apply the Fourier inversion theorem to all four variables of integration independently to show that
\[
\phi(r, t) = \frac{1}{(2\pi)^4} \int \tilde{\phi}(\kappa, \omega) e^{i(\kappa \cdot r - \omega t)} \, d\kappa_x \, d\kappa_y \, d\kappa_z \, d\omega
\]
(b) Multiply the wave equation for the scalar potential, Eq. (14.8), by $e^{-i(\kappa \cdot r - \omega t)}$ and integrate the result over all space and time to obtain an equation from which an explicit solution for the Fourier transform of the potential can be found. (c) Finally, express the solution for the potential itself as an integral in $(\kappa, \omega)$-space. Note: Strictly, the result obtained here is only a start toward a solution, because the integral obtained in part (c) is an improper integral. The careful treatment of this integral is most easily accomplished using the theory of contour integration in the complex plane and involves also an examination of the requirements of causality (that cause precede effect). Correctly manipulated, the result in part (c) leads directly to Eq. (14.26).

P14.27. A charged particle interacts with an incident linearly polarized plane electromagnetic wave whose electric field is
\[
E = E_0 (\cos \psi \hat{i} + \sin \psi \hat{j}) e^{i(\kappa z - \omega t')}
\]
and hence experiences an acceleration given by
\[
\ddot{\psi} = \frac{q}{m} E_0 (\cos \psi \hat{i} + \sin \psi \hat{j}) e^{i(\kappa z - \omega t')}
\]
where $q$ and $m$ are the charge and mass of the particle. Assume that the motion is non-relativistic, note that the particle moves in the plane $z = 0$ (why?), and let $\theta$ and $\phi$ be the polar and azimuthal angles of a distant observation point in a coordinate system with the particle near the origin. (a) Calculate $(dP(t')/d\Omega)$ for this particle, where the average is evaluated over a single cycle of the motion. (b) The differential cross section for scattering of incident light by a free particle is defined by
\[
\frac{d\sigma}{d\Omega} = \text{power radiated/unit solid angle =} \frac{\langle dP/d\Omega \rangle}{\text{incident energy/unit area/unit time}} = \frac{|\langle \mathbf{S} \rangle|}{|d\mathbf{S}|}
\]
where $\mathbf{S}$ is the Poynting vector of the incident wave. Evaluate $d\sigma/d\Omega$ for the present case. (c) The cross section for scattering of unpolarized incident light is obtained by averaging the result of part (b) over $\psi$. Determine this cross section and sketch a graph showing this cross section as a function of $\theta$. The resulting expression is referred to as the Thomson formula. (d) The total cross section $\sigma$ for scattering is obtained by integrating $d\sigma/d\Omega$ over all solid angles. Obtain the total cross section corresponding to the result of part (c).
result is referred to as the Thomson cross section.\textsuperscript{13} Evaluate the Thomson cross section numerically if the particle is an electron. \textit{Note}: The discussion of this problem is valid only at low frequencies. At higher frequencies, when the photons in the electromagnetic wave have energies comparable to or exceeding the rest energy of the particle, quantum and relativistic effects become important and classical scattering goes over into Compton scattering, the cross section for which is given by the so-called Klein-Nishina formula. [See, for example, J. M. Jauch and F. M. Rohrlich, \textit{The Theory of Photons and Electrons} (Addison-Wesley Publishing Company, Inc., Reading, MA, 1955), Chapter 11.]

\textsuperscript{13}English physicist Joseph John (J.J.) Thomson, b. 18 December 1856 in Cheetham Hill, Manchester, Lancashire, England; d. 30 August 1940 in Cambridge, England.
Relativistic Form of Maxwell’s Equations

When Maxwell’s equations were first presented in the late nineteenth century, physicists were convinced that all physical phenomena would ultimately be explained by applying Newton’s laws and known (or to be discovered) forces to predict the behavior of suitable particles. Within this mechanical view, any wave phenomenon required a medium to support and transmit the wave. The late nineteenth-century physicist therefore introduced a medium to transmit electromagnetic waves. This ether was imagined to pervade all of space, the electromagnetic field was interpreted as a stress or distortion of the ether from its normal equilibrium “state” in the absence of a field, and electromagnetic waves were thought of as propagating “oscillations” in the ether. Collectively, physicists invested considerable effort in attempts to convert this qualitative view of the electromagnetic field into a satisfactory, quantitative model. But there were many difficulties with this model, and, by the early twentieth century, the ether (and with it the hope of a completely mechanical physical world) had generally been abandoned. The electromagnetic field had been accepted as a physical manifestation that could exist in otherwise empty space—a view that we have, in fact, adopted without serious question throughout this book.

To set the stage for this chapter, we shall describe briefly just one of the fundamental difficulties that plagued the concept of a mechanical ether. Regardless of the state of motion of its source, a classical, mechanical wave propagating in a medium moves through that medium with a fixed velocity, say $v_w$, relative to the medium, and the velocity of the wave relative to an observer at rest in the medium is also $v_w$. If, however, this (classical) observer is moving relative to the medium with velocity $v_o$, the speed of the wave relative to this observer is $v_w - v_o$. This result is, of course, in complete agreement (P15.1) with the predictions of the (classical) Galilean transformation

$$r' = r - vt$$
$$t' = t$$

relating the coordinates of an event in space-time as measured in two different frames of reference, where the origins and axes of the two frames are assumed to coincide at the common time $t = t' = 0$ but (as shown in Fig. 15.1) the primed frame is in uniform

1Italian physicist, mathematician, astronomer Galileo Galilei, b. 15 February 1564 in Pisa, Duchy of Florence, Italy; d. 8 January 1642 in Arcetri, Grand Duchy of Tuscany, Italy.
motion relative to the unprimed frame with velocity $\mathbf{v}$. If the electromagnetic wave is in fact a classical wave in a mechanical ether, it should propagate relative to the ether with a particular speed, the speed of light relative to the medium. Relative to an observer who is moving relative to the ether, however, electromagnetic waves should have a speed different from their speed relative to the ether. Indeed, measurements of the speed of light relative to a particular observer should permit determination of the speed of that observer relative to the ether! Equivalently, it should be possible to locate a frame of reference fixed with respect to the ether (and hence to locate a viable candidate for the absolute frame of reference that Newton assumed was fixed with respect to distant “fixed” stars). Furthermore, if the speed of electromagnetic waves is different for observers in relative motion, Maxwell’s equations as we have developed them cannot be correct for all observers, because the correctness of these equations for all observers would imply that the speed of electromagnetic waves should be the same for all observers, a clear contradiction of the above-outlined properties of classical waves. Thus, the classical physicist inferred that there must be a preferred frame of reference (presumably a frame fixed with respect to the ether) in which Maxwell’s equations are valid, and a search for Newton’s absolute frame of reference is then also a search for that preferred frame of reference in which Maxwell’s equations are valid.

One of the early crucial experiments bearing on the search for this preferred frame of reference was the Michelson-Morley experiment,$^{2,3,4}$ which used the interference of light in attempts to detect any difference between the speeds of light (relative to the earth) in two directions at right angles to one another. Measurements were taken for various “absolute” orientations of the apparatus and at all times of the year in order to make sure that all possible motions of the earth relative to the ether were examined. In repetitions of the

---

2American Journal of Science 34, 333 (1887); Phil. Mag. 24, 449 (1887).
3American physicist Albert Abraham Michelson, b. 19 December 1852 in Strzelno, Prussian Partition of Poland; d. 9 May 1931 in Pasadena, California.
4American scientist Edward Williams Morley, b. 29 January 1838 in Newark, New Jersey; d. 24 February 1923 in West Hartford, Connecticut.
experiment by others, both stellar and terrestrial sources of light have been used. Overall, the experiment was very carefully designed and was more than adequate to detect differences in the speed of light comparable to the orbital speed of the earth (about 1 part in 10,000), and the results of numerous measurements were unequivocal: The speed of light relative to the earth is the same in all directions and at all times of the year. This startling observation means either (1) that the earth is at all times at rest relative to the ether and the experiment provides no information about the form of Maxwell’s equations in a frame of reference moving relative to the ether or (2) that Maxwell’s equations are, in fact, valid in all frames of reference in uniform relative motion and something is wrong with the Galilean transformation (and hence also with Newtonian mechanics). The first of these conclusions can be accepted only if the earth is imagined to drag the ether along with it; accepting the second conclusion requires drastic conceptual changes in a long-standing and very successful world view. Neither alternative was particularly palatable when the results of the Michelson-Morley experiment were first presented.

There is, however, at least one further pertinent experimental observation: The apparent direction from the earth to any star is slightly different at different times of the year, even after correction for the different positions of the earth with each observation. This phenomenon, called aberration (P15.7 and P15.39), is readily and quantitatively explained if the earth moves through the ether but cannot be understood at all easily if the earth drags the ether along. Apparently, we are forced to conclusion (2) by direct experimental observations.

The full story is, of course, not quite that simple. It is more accurate to say that, although numerous ingenious attempts were made to preserve the ether, Newtonian mechanics, and the Galilean transformation, the complexity and artificiality of these attempts gradually convinced the physical community that conclusion (2) was the simpler and the more attractive alternative. After a few decades of hope that a preferred frame of reference in absolute rest might at last be located, attempts to locate this frame were abandoned and physicists returned to an earlier conception that no experiment performed in a given frame of reference could determine the absolute motion of that frame of reference. Absolute motion once again became undetectable and the Galilean principle of relativity, which asserts that the laws of physics are the same for all observers in uniform relative motion, was returned to the roster of basic beliefs about the nature of the physical world. This principle, however, did not emerge unchanged from the several decades of turmoil over the ether. Prior to these decades, the principle applied to the laws of mechanics; following this period, the principle was taken to apply to all laws of physics, including in particular the laws of mechanics and the laws of electromagnetism, and in this extended form the principle is more properly called the Einsteinian principle of relativity.\footnote{In fairness to the earlier form of the principle, it should be noted, however, that in the mechanical view of its time, the laws of mechanics were the only laws of physics; everything was ultimately mechanical. Thus, the Galilean principle of relativity by implication really did include the laws of electromagnetism. The real difference resulting from the years of turmoil was the discovery that electromagnetism could not be interpreted (easily) as a mechanical phenomenon.}

In summary, we have decided in this introduction to adopt the point of view that all laws of physics must conform to the Einsteinian principle of relativity. Further, from the observation that the speed of light is the same for all observers regardless of their relative motion, we are led to assume that Maxwell’s equations as we have developed them are already consistent with this broad principle. If that is the case, however, the Galilean trans-
formation (which contradicts the universal validity of Maxwell’s equations) cannot be the correct transformation relating coordinates in two frames of reference that are in uniform relative motion. Still further, if the Galilean transformation is replaced by a new transformation that is consistent with the universal validity of Maxwell’s equations, this new transformation is almost certain to be inconsistent with the assumption that Newton’s laws have universal validity. Thus, replacing the Galilean transformation with another transformation to “save” Maxwell’s equations will almost certainly “lose” Newton’s laws in their classical form and, in addition to seeking a replacement for the Galilean transformation, we should also seek at least a modified form of Newton’s laws to preserve their invariance under whatever transformation replaces the Galilean transformation. We assume here that the reader is already acquainted with the concepts of Einstein’s theory of special relativity and knows that the correct transformation relating coordinates in two frames of reference in uniform motion is the Lorentz transformation. In this chapter, we shall summarize the Lorentz transformation, describe some of the mathematical machinery that is useful in working with the Lorentz transformation, and then develop a reexpression of Maxwell’s equations that makes their invariance to the Lorentz transformation (and hence their conformity with the principle of relativity) manifestly obvious. We shall also find how several electromagnetic quantities, such as charge density, current density, vector and scalar potentials, and the electromagnetic field itself, transform when one’s point of view shifts from one frame of reference to a second in uniform motion relative to the first. We shall leave to other authors a description of how Newton’s laws must be modified to make them invariant to the Lorentz transformation instead of to the Galilean transformation.

PROBLEMS

P15.1. A particle follows a trajectory given by \( \mathbf{R}(t) \) in the unprimed frame of reference and \( \mathbf{R}'(t') \) in the primed frame of reference, where coordinates in the two frames of reference are related by the Galilean transformation, Eq. (15.1). Show that the velocities \( \mathbf{V} \) and \( \mathbf{V}' \) and accelerations \( \mathbf{A} \) and \( \mathbf{A}' \) as determined in the two frames are related by

\[
\mathbf{V}' = \mathbf{V} - \mathbf{v} \quad ; \quad \mathbf{A}' = \mathbf{A}
\]

and hence argue that force defined by Newton’s second law is a Galilean invariant. What assumptions did you make about the behavior of mass under a Galilean transformation?

P15.2. Explain why it should be Maxwell’s equations and not Newton’s laws that turn out to be consistent automatically with the Einsteinian principle of relativity.

P15.3. Suppose a scalar function \( u(r, t) \) satisfies the wave equation

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u = 0
\]

in the unprimed frame of reference in Eq. (15.1). What equation does this function satisfy in the primed frame of reference? \textit{Hint:} Assume that the \( x, y, \) and \( z \) axes are parallel to the \( x', y', \) and \( z' \) axes, respectively, and let the \( x \) axis be chosen so that \( \mathbf{v} = v \hat{i} \).

15.1 A Review of Special Relativity

Although Lorentz is credited with discovering a coordinate transformation and an associated transformation of the electromagnetic field that together leave Maxwell’s equations
unchanged in form, the theory of special relativity advanced by Einstein\textsuperscript{6,7} in 1905 placed this transformation on a firm theoretical and philosophical footing by deriving it directly from the fundamental postulates described in the above introductory paragraphs. The usual statement of these postulates involves the notion of inertial frames of reference, which in totality consist of any single frame of reference in which the laws of physics have their simplest form and all other frames of reference in translation at constant velocity relative to the first frame of reference. The postulates of special relativity themselves are:

1. that the laws of physics are the same in all inertial frames of reference, and

2. that the speed of light is a universal constant, the same in all inertial frames of reference regardless of the motion of these frames relative to the source of the light.

Together these postulates dictate uniquely the transformation that must relate the spatial and temporal coordinates of an event observed from two different inertial frames of reference. If (for simplicity) the two frames of reference are related as in Fig. 15.2, with the $x$, $y$, and $z$ axes parallel to the $x'$, $y'$, and $z'$ axes, respectively, and with their origins coinciding when $t = t' = 0$, then the resulting Lorentz transformation between the coordinates $(x, y, z, t)$ and the coordinates $(x', y', z', t')$ is

\begin{align}
  x' &= \gamma(x - \beta ct) \\
  y' &= y \\
  z' &= z \\
  ct' &= \gamma(ct - \beta x)
\end{align}

\textsuperscript{6}Annalen der Physik 17 (1905). This original paper is extremely readable and is available in English translation in a collection of reprints entitled The Principle of Relativity (Dover Publications, Inc., New York).

\textsuperscript{7}German physicist Albert Einstein, b. 14 March 1879 in Ulm, Kingdom of W"urtemberg, German Empire; d. 18 April 1955 in Princeton, New Jersey.
where
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad ; \quad \beta = \frac{v}{c} \]  
(15.3)
and \( v \) is the velocity of the primed frame relative to the unprimed frame, this velocity being in the (common) \( x \) and \( x' \) directions.\(^8\)

For our purposes, it is useful to stress that the Lorentz transformation is a linear transformation between the four primed and four unprimed coordinates and to express that transformation in a matrix form. (The algebra of matrices is discussed in Appendix A.) Two choices for the four coordinates are common; the first,
\[ x_0 = ct \quad x_1 = x \quad x_2 = y \quad x_3 = z \]  
(15.4)
involves entirely real numbers and the second
\[ x_1 = x \quad x_2 = y \quad x_3 = z \quad x_4 = ic t \]  
(15.5)
introduces a complex coordinate. We shall adopt the second alternative, in terms of which the Lorentz transformation, Eq. (15.2), has the expression
\[ x'_1 = \gamma(x_1 + i\beta x_4) \quad x'_3 = x_3 \]  
\[ x'_2 = x_2 \quad x'_4 = \gamma(x_4 - i\beta x_1) \]  
(15.6)
or, equivalently,
\[
\begin{pmatrix}
    x'_1 \\
    x'_2 \\
    x'_3 \\
    x'_4
\end{pmatrix} =
\begin{pmatrix}
    \gamma & 0 & 0 & i\gamma\beta \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -i\gamma\beta & 0 & 0 & \gamma
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{pmatrix}
\]  
(15.7)
Thus, if we introduce the \( 4 \times 4 \) transformation matrix
\[
\mathcal{L} (\beta) =
\begin{pmatrix}
    \gamma & 0 & 0 & i\gamma\beta \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -i\gamma\beta & 0 & 0 & \gamma
\end{pmatrix}
\]  
(15.8)
whose elements we denote by \( \mathcal{L}_{\mu\nu} (\beta) \) or by \( \mathcal{L}_{\mu\nu} \), we can express the Lorentz transformation in the form
\[ x'_\mu = \mathcal{L}_{\mu\nu} (\beta) x_\nu \ ; \quad \mathbf{x}' = \mathcal{L} (\beta) \mathbf{x} \]  
(15.9)
where we have adopted the summation convention in which a sum from 1 to 4 is understood on every repeated Greek index. Further, we shall use boldface sans serif characters (e.g., \( \mathbf{x} \)) for four-component vectors. Finally, we understand without explicit mention that any index appearing only once in a given term assumes all four possible values in succession; the first member of Eq. (15.9), for example, expresses four separate equations.

Physically, the inverse transformation (from the primed to the unprimed coordinates) must be obtainable by replacing \( v \) by \( -v \) (or \( \beta \) by \( -\beta \)) in the matrix \( \mathcal{L} \). Thus, we must have that
\[ \mathbf{x} = [\mathcal{L}(\beta)]^{-1} \mathbf{x}' = \mathcal{L}(\beta) \mathbf{x}' \]  
\[ \implies [\mathcal{L}(\beta)]^{-1} = \mathcal{L}(\beta) \]  
(15.10)
\(^8\)A full deduction of this transformation from the basic postulates may be found in many sources, including, for example, the original paper of Einstein (see footnote 6; R. Resnick, Introduction to Special Relativity (John Wiley and Sons, Inc., New York, 1968), Section 2.2; and J. D. Jackson, Classical Electrodynamics, Second Edition, (John Wiley and Sons, Inc., New York, 1975), Section 11.2.
15.1. A REVIEW OF SPECIAL RELATIVITY

Direct observation of Eq. (15.8), however, reveals that \( \mathcal{L}(-\beta) = [\mathcal{L}(\beta)]^T \), the superscript \( T \) denoting the transposed matrix. We conclude that

\[
[\mathcal{L}(\beta)]^{-1} = [\mathcal{L}(\beta)]^T
\]

and the matrix \( \mathcal{L} \) is therefore a (complex) orthogonal matrix (Appendix A). Thus, the Lorentz transformation in Eq. (15.7) can be interpreted as a rigid rotation of the four coordinate axes in a four-dimensional space. (Compare PA.14 in Appendix A.) In fact, the Lorentz transformation between frames of reference whose axes are not parallel and whose relative velocity is in some arbitrary direction is also represented by a (complex) orthogonal matrix and is therefore also equivalent to a rigid rotation in a four-dimensional space; a portion of this more general conclusion is explored in P15.35.

Typically, physical quantities of interest in a given frame of reference are functions of the spatial and temporal coordinates in that frame of reference, and the way in which these physical quantities transform under a Lorentz transformation provides a useful means to classify them. The simplest such quantities are scalars, which we can define by considering a physical quantity that we believe to have but a single component. Let \( \phi(x) = \phi(x_1, x_2, x_3, x_4) \) express the spatial and temporal dependence of this quantity as measured in the unprimed frame of reference and \( \phi'(x') \) express the spatial and temporal dependence of this same quantity as measured in the primed frame of reference. The quantity \( \phi \) is a scalar if

\[
\phi'(x') = \phi(x)
\]

where \( x' \) and \( x \) are related by the Lorentz transformation. More concretely, a physical quantity is a scalar if it has the same numerical value at corresponding points in space-time in any two frames of reference related by a Lorentz transformation. Usually, Eq. (15.12) is written \( \phi' = \phi \) and explicit indication that \( \phi' \) and \( \phi \) are most directly functions of the primed and unprimed coordinates, respectively, is suppressed.

The next most complicated type of physical quantity has four components that become entangled with one another when the frame of reference is subjected to a Lorentz transformation. The four-component vector \( \mathbf{x} = (x_1, x_2, x_3, x_4) \) is the prototype of these quantities; they are called four-vectors and their components transform under the Lorentz transformation just as the components of \( \mathbf{x} \) transform. That is, the four-component entity \( \mathbf{A} = (A_1, A_2, A_3, A_4) \) is a four-vector if its components \( \mathbf{A}' = (A'_1, A'_2, A'_3, A'_4) \) in the primed frame of reference are determined from those in the unprimed frame by

\[
A'_\mu = \mathcal{L}_{\mu\nu} A_\nu
\]

where here (hereafter) the above-mentioned summation convention is (will be) employed. The first three components of a four-vector are often called the spatial components; the fourth component is the temporal component. Occasionally we shall use the symbol \( \mathbf{A} \) for the three-dimensional vector formed from the spatial components of the four-vector \( \mathbf{A} \).

Among other quantities, the four-dimensional gradient of a scalar \( \phi \), whose four components are \( \partial \phi / \partial x_\mu \), is a four-vector. To prove this contention, we note first that differentiation of Eq. (15.12) with respect to \( x'_\mu \) gives

\[
\frac{\partial \phi'}{\partial x'_\mu} = \frac{\partial \phi}{\partial x_\nu} \frac{\partial x_\nu}{\partial x'_\mu}
\]

(15.14)
From Eqs. (15.10) and (15.11), however, we find that

\[ x_\nu = (\mathcal{L}^{-1})_{\nu\mu}x'_\mu = \mathcal{L}_{\mu\nu}x'_\mu \implies \frac{\partial x_\nu}{\partial x'_\mu} = \mathcal{L}_{\mu\nu} \]

(15.15)

and Eq. (15.14) becomes

\[ \frac{\partial \phi'}{\partial x'_\mu} = \mathcal{L}_{\mu\nu} \frac{\partial \phi}{\partial x_\nu} \]

(15.16)

which states that the components of the (four-dimensional) gradient in two different frames of reference are related by Eq. (15.13) and hence that this gradient is in fact a four-vector. (Q.E.D.) Since \( \phi = \phi' \), we can write Eq. (15.16) as a relationship among differential operators

\[ \frac{\partial}{\partial x'_\mu} = \mathcal{L}_{\mu\nu} \frac{\partial}{\partial x_\nu} \]

(15.17)

and think of the operator whose four components are \( \partial/\partial x_\mu \) as a four-vector. We shall frequently use the notation \( \partial'_\mu \) for \( \partial/\partial x'_\mu \) and \( \partial_\nu \) for \( \partial/\partial x_\nu \), so that Eq. (15.17) may also be written in the more compact form \( \partial'_\mu = \mathcal{L}_{\mu\nu} \partial_\nu \).

A third useful type of physical quantity that is identified by the way it behaves under a Lorentz transformation is a sixteen-component entity called a second-rank tensor. It is usual to arrange these sixteen components into a \( 4 \times 4 \) matrix whose elements are, say, \( F_{\mu\nu} \) in the unprimed frame of reference and \( F'_{\mu\nu} \) in the primed frame of reference. A prototype second-rank tensor is a quantity whose sixteen components are the sixteen possible products of a component of a four-vector \( A \) with a component of another four-vector \( B \) so that \( F_{\mu\nu} = A_\mu B_\nu \), but there are many second-rank tensors not so easily factored. Whatever its origin, a sixteen-component quantity \( F \) is a second-rank tensor by definition if

\[ F'_{\mu\nu} = \mathcal{L}_{\mu\sigma} \mathcal{L}_{\nu\tau} F_{\sigma\tau} \]

(15.18)

where two sums (on \( \sigma \) and \( \tau \)) are implied. Equivalently, this transformation rule can be written as a matrix product by noting that

\[ F'_{\mu\nu} = \mathcal{L}_{\mu\sigma} F_{\sigma\tau} (\mathcal{L}^T)_{\tau\nu} = (\mathcal{L} F \mathcal{L}^T)_{\mu\nu} \implies F' = \mathcal{L} F \mathcal{L}^T \]

(15.19)

The pattern illustrated in Eq. (15.18) can be extended to third- and higher-rank tensors, but we shall have no need for these more complicated entities.

We shall conclude this summary of the mathematical machinery of special relativity by identifying several important scalars (i.e., Lorentz invariants). Many invariants can be constructed directly from known vectors and tensors. For example, let \( A \) and \( B \) be four-vectors and consider the quantity \( A_\mu B_\mu \) (implicit sum on \( \mu \)). We find that

\[ A'_\mu B'_\mu = (\mathcal{L}_{\mu\sigma} A_\sigma)(\mathcal{L}_{\mu\tau} B_\tau) = A_\sigma (\mathcal{L}^T)_{\sigma\mu} \mathcal{L}_{\mu\tau} B_\tau = A_\sigma (\mathcal{L}^{-1})_{\sigma\tau} B_\tau = A_\sigma B_\tau \]

(15.20)

since \( \mathcal{L}^{-1} \mathcal{L} \) is the identity matrix. Equation (15.20), however, states that the quantity \( A_\mu B_\mu \) (which is a four-dimensional “dot product”) not only has the same value in two different frames but is computed in each frame by applying the same prescription to the components of the vector in that frame. This dot product is both invariant in value (i.e., is a scalar) and invariant in form when the constituent vectors are subjected to the same
Lorentz transformation. By similar arguments, we can demonstrate that the trace \( F_{\mu\mu} \) of a second-rank tensor, the four-divergence
\[
\partial_{\mu} A_{\mu} = \nabla \cdot A + \frac{\partial}{\partial t} \left( \frac{A_4}{ic} \right)
\]
(15.21)
of a vector \( A \), and the four-dimensional “Laplacian” (which is called the \textit{d’Alembertian})\(^9,10\)
\[
\Box^2 = \partial_{\mu} \partial_{\mu} = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}
\]
(15.22)
are all Lorentz invariants (P15.10).

Finally, if we assume familiarity with the theorem that multidimensional volume elements transform under a change of variables from \( u_1, u_2, \ldots, u_n \) to \( v_1, v_2, \ldots, v_n \) by\(^11\)
\[
d v_1 \, dv_2 \, \cdots \, dv_n = \left| J \left( \frac{v_1, v_2, \ldots, v_n}{u_1, u_2, \ldots, u_n} \right) \right| \, du_1 \, du_2 \, \cdots \, du_n
\]
(15.23)
where the Jacobian \( J(\cdots) \)\(^12\) is the determinant of a matrix whose \( ij \) element is \( \partial v_i / \partial u_j \), we can demonstrate that the four-dimensional volume element
\[
d^4x = dx_1 \, dx_2 \, dx_3 \, dx_4
\]
(15.24)
is invariant to the Lorentz transformation. From Eq. (15.23), we have that
\[
d^4x' = \left| J \left( \frac{x'_1, x'_2, x'_3, x'_4}{x_1, x_2, x_3, x_4} \right) \right| \, d^4x
\]
(15.25)
The Jacobian appearing here, however, is the determinant of a matrix whose \( \mu \nu \) element \( \partial x'_\mu / \partial x_\nu \) is \( L'_{\mu\nu} \). [See Eq. (15.9).] Thus, the Jacobian in Eq. (15.25) is the determinant of the matrix \( L' \). Since \( L \) is orthogonal, however, its determinant can only be \( \pm 1 \) (PA.12). Equation (15.25) therefore reduces to
\[
dx'_1 \, dx'_2 \, dx'_3 \, dx'_4 = dx_1 \, dx_2 \, dx_3 \, dx_4
\]
(15.26)
and we have established the Lorentz invariance of the (four-dimensional) volume element.

**PROBLEMS**

P15.4. Show that the Lorentz transformation reduces to the Galilean transformation in the nonrelativistic limit \( v/c \ll 1 \).

---

\(^9\)We use the notation \( \Box^2 \) for the d’Alembertian because of its similarity to \( \nabla^2 \); the symbol \( \Box \) (without the exponent) is also used for this operator.

\(^10\)French mathematician, physicist, and music theorist Jean-Baptise le Rond d’Alembert, b. 16 November 1717 in Paris, France; d. 29 October 1783 in Paris, France.


\(^12\)German mathematician Carl Gustav Jacob Jacobi, b. 10 December 1804 in Potsdam, Kingdom of Prussia; d. 18 February 1851 in Berlin, Kingdom of Prussia.
P15.5. Suppose a scalar function \( u(r,t) \) satisfies the wave equation

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u = 0
\]

in the unprimed frame of reference in Eq. (15.2). Show by direct manipulation with the equation that this function also satisfies the wave equation in the primed frame of reference and hence conclude that the scalar wave equation is invariant to the Lorentz transformation. Compare this problem with P15.3.

P15.6. (a) Show that two events occurring at the same point \((\Delta x' = 0)\) in the primed frame of reference but separated by a temporal interval \(\Delta t'\) are separated in the unprimed frame of reference by a temporal interval \(\Delta t = \gamma \Delta t'\). Note that \(\Delta t > \Delta t'\); this phenomenon is referred to as the relativistic time dilation. (b) Show that two events occurring at the same time \((\Delta t' = 0)\) in the primed frame of reference but separated by a spatial interval \(\Delta x'\) are separated in the unprimed frame by a temporal interval \(\Delta t = \gamma \beta \Delta x'/c\). Thus, simultaneous events in the primed frame are simultaneous in the unprimed frame only if \(\Delta x' = 0\). (c) A rod of length \(L'\) lies along the \(x'\) axis and is at rest in the primed frame. An observer in the unprimed frame measures the length of this rod by measuring the time \(\Delta t\) for the rod to move by, calculating the length as \(L = v \Delta t\). Show that \(L = L'/\gamma\). Note that \(L < L'\); this phenomenon is referred to as the Lorentz-Fitzgerald contraction.\(^{13}\) Hint: Consider the two events defined by those two instants when the one or the other end of the rod is opposite the unprimed observer.

P15.7. The position of a moving object is given by \(R(t)\) and \(R'(t')\) in the unprimed and primed frames of reference, respectively. Let the velocity of the object by \(V(t) = dR/dt\) and \(V'(t') = dR'/dt'\) in the two frames. (a) Show from the Lorentz transformation in Eq. (15.2) that

\[
\begin{pmatrix}
V'_x \\
V'_y \\
V'_z
\end{pmatrix} = \begin{pmatrix}
\frac{1}{1-V_x v/c^2} & 0 & 0 \\
0 & \frac{V_y}{\gamma} & 0 \\
0 & 0 & \frac{V_z}{\gamma}
\end{pmatrix}
\begin{pmatrix}
V_x - v \\
V_y \\
V_z
\end{pmatrix}
\]

(b) Now consider a light beam propagating in the \(xy\) plane at speed \(c\) in a direction making an angle \(\theta\) with the \(x\) axis as in Fig. 15.2. Find the speed and the direction \(\theta'\) of propagation of this beam in the primed frame of reference. The phenomenon of aberration of starlight is predicted by the difference between \(\theta\) and \(\theta'\). (c) Use a computer to determine \(\theta'\) as a function of \(\beta\) for selected values of \(\theta\) and plot graphs of these relationships.

P15.8. In the Fizeau experiment (1859),\(^{14}\) light passes through a moving liquid that has index of refraction \(n\) (Fig. 15.3) Using an interferometer, one can measure the velocity of the light relative to a stationary observer. For realizable (i.e. nonrelativistic) speeds \(v\) of the liquid, Fizeau found experimentally that

\[
\text{speed of light relative to laboratory} = \frac{c}{n} + v \left( 1 - \frac{1}{n^2} \right)
\]

Predict this result from the Lorentz transformation. \(H\)ints: (1) What is the speed of light relative to the liquid? (2) See P15.7.

---

\(^{13}\)Irish professor of physics George Francis FitzGerald, b. 3 August 1851 in Dublin, Ireland; d. 22 February 1901 in Dublin, Ireland.

\(^{14}\)French physicist Armand Hippolyte Louis Fizeau, b. 23 September 1819 in Paris, France; d. 18 September 1896 in Venteuil, France.
15.1. A REVIEW OF SPECIAL RELATIVITY

Figure 15.3: Figure for P15.8.

P15.9. Prove the following theorem: If $A_\mu B_\mu$ is a scalar for arbitrary four-vectors $B$, then $A$ is also a four-vector.

P15.10. Verify that the four-divergence and the d’Alembertian are correctly given by the right-hand sides of Eqs. (15.21) and (15.22) and present complete arguments showing that these quantities are invariant to the Lorentz transformation.

P15.11. Let two events be separated by the infinitesimal intervals $dx_\mu$ in the unprimed frame and $dx'_\mu$ in the primed frame. (a) Recognizing that the coordinates of each event are related in the two frames by the Lorentz transformation, show that $dx'_\mu = L_{\mu\nu} dx_\nu$. (b) Show that the so-called proper time interval $d\tau$ defined by
t
\[
c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2
\]
is a Lorentz invariant. (c) Argue that the quantity whose four components are $u_\mu = dx_\mu/d\tau$ is a four-vector and show that these components are $(\Gamma V_x, \Gamma V_y, \Gamma V_z, i\Gamma)$, where $\Gamma = (1 - V^2/c^2)^{-1/2}$, $V_x = dx/dt$, $V_y = dy/dt$, and $V_z = dz/dt$. The four vector $u$ is called the world velocity. (d) Show that $u_\mu u_\mu = -c^2$.

P15.12. In relativistic mechanics, the momentum $p$ and the total energy $E$ of a particle of rest mass $m$ moving with velocity $V$ are defined by $p = \Gamma mV$ and $E = \Gamma mc^2$, where $\Gamma = [1 - V^2/c^2]^{-1/2}$. Further, in a second frame of reference, the momentum $p'$ and energy $E'$ are given by $p' = \Gamma' mV'$ and $E' = \Gamma' mc^2$, where $\Gamma' = [1 - (V')^2/c^2]^{-1/2}$. Let the primed and unprimed frames be related by Eq. (15.2). (a) Show that $\Gamma' = \gamma \Gamma (1 - V_x v/c^2)$. Hint: See P15.7. (b) Show that the four-component quantity $(p_x, p_y, p_z, iE/c)$ is a four vector; i.e., show that
\[
p'_x = \gamma \left( p_x - \frac{v}{c^2} E \right) \quad p'_z = p_z
\]
\[
p'_y = p_y \quad E' = \gamma (E - pv)
\]
(c) Recognizing that $p_\mu p_\mu$ is a scalar, show that $p^2 c^2 - E^2$ is a Lorentz invariant and that it has the value $-m^2 c^4$. 

15.2 MaxweII’s Equations in Covariant Form;  
The Electromagnetic Field Tensor

As we developed them in Chapter 6, MaxweII’s equations for the electromagnetic field in the absence of matter are

\[
\begin{align*}
\nabla \cdot \mathbf{B} &= 0 \quad (15.27) \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ 
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \quad (15.29) \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \\ 
\end{align*}
\]

Further, the fields can be derived from the potentials using the equations

\[
\begin{align*}
\mathbf{B} &= \nabla \times \mathbf{A} \\ 
\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ 
\end{align*}
\]

and, in a Lorentz gauge defined by the requirement (Lorentz condition) that

\[
\nabla \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \quad (15.33)
\]

the potentials satisfy the equations

\[
\begin{align*}
\Box^2 \mathbf{A} &= -\mu_0 \mathbf{J} \quad (15.34) \\
\Box^2 V &= -\frac{\rho}{\epsilon_0} \\
\end{align*}
\]

Finally, the current and charge densities are related by the equation of continuity

\[
\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (15.36)
\]

The object of this section now is to identify suitable four-vectors and tensors and to reexpress the above equations in terms of these quantities, thereby making the behavior of each equation under a Lorentz transformation directly determinable from the (known) behavior of four-vectors and tensors. As we identify each four-vector or tensor, we must be certain that it has the requisite transformation properties. (Mere possession of the proper number of components offers no assurances that those components behave correctly under a Lorentz transformation.)

We introduce first a four-vector that combines the current density and the charge density. Experimentally, observers in different frames of reference measure the same total charge in a uniquely defined region of space, even though each measures a different charge density in the region and assigns a different volume to the region. More mathematically, this empirical observation states that the charge \(dq\) measured in an infinitesimal volume in
the unprimed frame of reference and the charge \( dq' \) measured in the same region of space in the primed frame of reference are equal, \( dq = dq' \), or that

\[
\rho \, dx_1 \, dx_2 \, dx_3 = \rho' \, dx'_1 \, dx'_2 \, dx'_3
\]  

(15.37)

Equivalently the quantity \( \rho \, dx_1 \, dx_2 \, dx_3 \) is a Lorentz invariant. Thus, the four-component entity

\[
\rho dx \, dx_1 \, dx_2 \, dx_3 = \rho' dx' \, dx'_1 \, dx'_2 \, dx'_3
\]  

(15.38)

which is the product of a scalar and a four-vector, is itself a four-vector. We established in the last section, however, that the four-dimensional volume element in parentheses on the right in Eq. (15.38) is a scalar. Consequently, the quantity whose components are

\[
J_\mu = \rho dx_\mu \, dx_4 = \rho \frac{dx_\mu}{dt}
\]  

(15.39)

must be a four-vector. Now, if \( dx_\mu \) in Eq. (15.39) is interpreted as the infinitesimal separation between two successive space-time “positions” of one of the particles in the charge and current distribution,\(^{15} \)

\[
dx_\mu / dt \text{ for } \nu = 1, 2, \text{ and } 3 \text{ yields the components of the (drift) velocity of the particle and } dx_4 / dt = ic.
\]

We have therefore that

\[
J = (\rho v_x, \rho v_y, \rho v_z, icp) = (J_x, J_y, J_z, icp) = (J, icp)
\]  

(15.40)

[See Eqs. (2.10) and (2.18).] In short, experimental evidence on the invariance of electric charge leads to the conclusion that the quantity \( J \) constructed out of the ordinary current and charge densities via Eq. (15.40) is a four-vector; we shall call it the \textit{four-current}. Finally, in terms of the four current, the equation of continuity assumes the form

\[
\nabla \cdot J + \frac{\partial (icp)}{\partial (ict)} = \nabla \cdot J + \frac{\partial J_4}{\partial x_4} \implies \partial_\mu J_\mu = 0
\]  

(15.41)

In this form, the equation of continuity clearly transforms to the \textit{same} equation \( \partial'_\mu J'_\mu = 0 \) under a Lorentz transformation and hence is properly invariant under a change in the frame of reference.

We next introduce a four-vector that combines the potentials \( A \) and \( V \). Essentially, we note that Eq. (15.35) for \( V \) can be written in the form

\[
\Box^2 V = -\mu_0 \left( \frac{1}{\mu_0 \epsilon_0} \right) \rho = -\mu_0 c^2 \rho = -\mu_0 (-ic)(icp) \implies \Box^2 \left( \frac{V}{c} \right) = -\mu_0 J_4
\]  

(15.42)

Thus, if we introduce a four-component quantity

\[
\mathbf{A} = (A_x, A_y, A_z, i \frac{V}{c}) = (\mathbf{A}, i \frac{V}{c})
\]  

(15.43)

we can combine Eqs. (15.34) and (15.42) into the single four-component equation

\[
\Box^2 A_\mu = -\mu_0 J_\mu
\]  

(15.44)

\(^{15}\text{Strictly, we should consider different types of particles separately, obtaining an evaluation of } J_\mu \text{ for each type and then summing the separate results. We elect, however, to simplify the discussion by assuming that only a single type of particle is present.}\)
CHAPTER 15. RELATIVISTIC FORM OF MAXWELL’S EQUATIONS

Accepting the (experimentally supported) correctness of Maxwell’s equations in all frames of reference, we finally conclude from Eq. (15.44) that $\mathbf{A}$ must be a four-vector, because $\Box^2$ is a Lorentz invariant and $\Box^2 A_\mu$ can therefore be equal to a scalar multiple of $J_\mu$ in all frames of reference only if $\mathbf{A}$ and $\mathbf{J}$ have the same transformation properties. We shall call $\mathbf{A}$ the \textit{four-potential}; its four divergence

$$\partial_\mu A_\mu = \nabla \cdot \mathbf{A} + \frac{\partial A_4}{\partial x_4} = \nabla \cdot \mathbf{A} + \frac{\partial(iV/c)}{\partial(ict)} = \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t}$$

(15.45)

is exactly the quantity appearing in the Lorentz condition, Eq. (15.33). Thus, the Lorentz condition may be written in the form

$$\partial_\mu A_\mu = 0$$

(15.46)

and the invariance of this equation under a Lorentz transformation assures that Lorentz gauge potentials in one frame of reference will transform into Lorentz gauge potentials in any other frame of reference.

A third important quantity, which looks \textit{something} like a four-dimensional curl, is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(15.47)

and is manifestly a second-rank tensor (being constructed of the sums of products of two-quantities known to transform as vectors). Clearly, the tensor $F$ is antisymmetric ($F_{\mu\nu} = -F_{\nu\mu}$) and in particular its diagonal elements, $F_{11}$, $F_{22}$, etc., are zero. Of potentially sixteen elements, only six (say $F_{12}$, $F_{13}$, $F_{14}$, $F_{23}$, $F_{24}$, and $F_{34}$) can have arbitrary values. We shall now show that these six elements are related to the six components of the electromagnetic field. If neither index in Eq. (15.47) is a 4, we find that $F_{\mu\nu}$ is a component of the magnetic induction field, for example,

$$F_{12} = -F_{21} = \partial_1 A_2 - \partial_2 A_1 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z$$

[Eq. (15.31)]; similarly $F_{13} = -F_{31} = -B_y$ and $F_{23} = -F_{32} = B_x$. If, on the other hand, one of the indices in Eq. (15.47) is a 4, we find that $F_{\mu\nu}$ is related to a component of the electric field, for example,

$$F_{14} = -F_{41} = \partial_1 A_4 - \partial_4 A_1 = \frac{i}{c} \frac{\partial V}{\partial x} - \frac{1}{i c} \frac{\partial A_x}{\partial t}$$

$$= \frac{i}{c} \left( \frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t} \right) = -\frac{i}{c} E_x$$

(15.49)

[Eq. (15.32)]; similarly, $F_{24} = -F_{42} = -i E_y/c$ and $F_{34} = -F_{43} = -i E_z/c$. In total, we find that the components of the electric and magnetic induction fields are related to those of the \textit{electromagnetic field tensor} $F$ as indicated in the matrix

$$F = \begin{pmatrix}
0 & B_z & -B_y & -\frac{i E_x}{c} \\
-B_z & 0 & B_x & -\frac{i E_y}{c} \\
B_y & -B_x & 0 & -\frac{i E_z}{c} \\
\frac{i E_x}{c} & \frac{i E_y}{c} & \frac{i E_z}{c} & 0
\end{pmatrix}$$

(15.50)
Furthermore, because we accept the correctness of Maxwell’s equations in all frames of reference, the correspondence between field components and elements of \( F \) indicated in Eq. (15.50) applies equally in all frames of reference.

It remains to express the field equations in terms of the electromagnetic field tensor. Note that

\[
\partial_{\mu}F_{\mu 1} = \partial_1 F_{11} + \partial_2 F_{21} + \partial_3 F_{31} + \partial_4 F_{41} = \left[ 0 - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} + \frac{1}{ic} \frac{\partial(iE_x/c)}{\partial t} \right] \\
\Rightarrow \partial_{\mu}F_{\mu 1} = -\mu_0 J_1 \quad (15.51)
\]

which suggests that the equation

\[
\partial_{\mu}F_{\mu \nu} = -\mu_0 J_\nu \quad (15.52)
\]

combines Eqs. (15.29) and (15.30) into a single equation whose behavior under the Lorentz transformation is obvious from the notation; both sides of the equation are four-vectors. The correctness of Eq. (15.52) for all \( \nu \) is demonstrated in P15.14. Note also that

\[
\nabla \cdot B = \frac{\partial F_{23}}{\partial x} + \frac{\partial F_{31}}{\partial y} + \frac{\partial F_{12}}{\partial z} = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 \quad (15.53)
\]

which suggests that the four equations

\[
\partial_{\mu}F_{\nu \sigma} + \partial_{\nu}F_{\sigma \mu} + \partial_{\sigma}F_{\mu \nu} = 0 \quad (15.54)
\]

where \((\mu, \nu, \sigma)\) assumes the values \((1,2,3)\), \((1,2,4)\), \((1,3,4)\), and \((2,3,4)\), might combine the two homogeneous Maxwell equations, Eqs. (15.27) and (15.28); the correctness of this suggestion is demonstrated in P15.17. Note also that Eq. (15.54) is correct when two or more indices are the same but that, in those cases, it is identically correct and provides no further constraints on the field tensor.

In summary, we have shown that we can introduce two four-vectors and a tensor, specifically

\[
\mathbf{J} = (J, i\sigma) \quad ; \quad \mathbf{A} = \left( A, iV \right) \quad ; \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (15.55)
\]

and that the equations of electromagnetism can be written in terms of these quantities as in Table 15.1.

Although the physical content of these equations is no different from that of Eqs. (15.27)–(15.36), their form is now such that their correctness in all frames of reference is apparent from the way in which vectors and tensors transform. Further, by examining these transformation properties, we have gained considerable insight into the relationship between the electric and magnetic induction fields.

### PROBLEMS

**P15.13.** Confirm the correctness of all entries in Eq. (15.50) by working out each element of \( F_{\mu \nu} \) from Eq. (15.47).
Table 15.1: Some of the Equations of Electromagnetism in Relativistic Notation.

Maxwell’s Equations

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \]

\[ \nabla \times B = \mu_0 + \frac{1}{c^2} \frac{\partial E}{\partial t} \] \[=\] \[\partial_\mu F_{\mu\nu} = -\mu_0 J_\nu\]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times E = -\frac{\partial B}{\partial t} \] \[=\] \[\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu}\]

Equation of Continuity

\[ \nabla \cdot J + \frac{\partial \rho}{\partial t} = 0 \] \[\Longrightarrow \] \[\partial_\mu J_\mu = 0\]

Lorentz Condition

\[ \nabla \cdot A + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0 \] \[\Longrightarrow \] \[\partial_\mu A_\mu = 0\]

Equation for the (Lorentz Gauge) Potentials

\[ \Box^2 A = -\mu_0 J \]

\[ \Box^2 V = -\frac{\rho}{\varepsilon_0} \] \[=\] \[\Box^2 A_\mu = -\mu_0 J_\mu\]

P15.14. Verify that \(\partial_\mu F_{\mu\nu} = -\mu_0 J_\nu\) reduces to the three components of Eq. (15.30) and to Eq. (15.29) for \(\nu = 1, 2, 3,\) and 4, respectively.

P15.15. Derive the equation \(\partial_\mu F_{\mu\nu} = -\mu_0 J_\nu\) by formal differentiation of Eq. (15.47) defining \(F_{\mu\nu}\). \textit{Hint:} See Eqs. (15.44) and (15.46).

P15.16. Derive the equation of continuity \(\partial_\nu J_\nu = 0\) directly from Eq. (15.52). \textit{Hint:} \(F\) is an antisymmetric tensor.

P15.17. Verify that Eq. (15.54) correctly reduces to Eq. (15.27) and to the three components of Eq. (15.28) when \((\mu, \nu, \sigma)\) assume the values given in the text.

P15.18. (a) Show that \(E^2 - c^2 B^2\) is a Lorentz invariant. \textit{Hint:} \(F_{\mu\nu}F^{\mu\nu}\) is an invariant. Why? (b) Show that an electromagnetic field that is purely magnetic in one frame of reference cannot be transformed into a purely electric field by changing to a different frame of reference.
15.3 Transformation of the Electromagnetic Field

Now that we have established that the components of the electromagnetic field combine to form a second-rank tensor \( F \) as in Eq. (15.50), determination of the way in which the electromagnetic field transforms under a Lorentz transformation from an unprimed to a primed frame of reference involves merely evaluating the matrix product \( \mathcal{L} F \mathcal{L}^T \) as in Eq. (15.19). In particular, for \( \mathcal{L} \) given by Eq. (15.8) and \( F \) by Eq. (15.50), we find that

\[
F' = \mathcal{L} F \mathcal{L}^T = \begin{pmatrix}
0 & \gamma \left( B_z - \frac{\beta}{c} E_y \right) & -\gamma \left( B_y + \frac{\beta}{c} E_z \right) & -i\frac{E_x}{c} \\
-\gamma \left( B_z - \frac{\beta}{c} E_y \right) & 0 & B_x & -i\frac{i\gamma}{c} (E_y - \beta c B_z) \\
\gamma \left( B_y + \frac{\beta}{c} E_z \right) & -B_x & 0 & -i\frac{i\gamma}{c} (E_z + \beta c B_y) \\
\frac{i}{c} E_x & \frac{i\gamma}{c} (E_y - \beta c B_z) & \frac{i\gamma}{c} (E_z + \beta c B_y) & 0
\end{pmatrix}
\]  

(15.56)

Imagining primes on all entries in Eq. (15.50) and comparing the resulting matrix with Eq. (15.56), we conclude that the field in the primed frame of reference is determined from the field in the unprimed frame of reference by

\[
E'_x = E_x \hspace{2cm} B'_x = B_x \\
E'_y = \gamma (E_y - \beta c B_z) \hspace{1cm} B'_y = \gamma \left( B_y + \frac{\beta}{c} E_z \right) \hspace{2cm} (15.57)
B'_z = \gamma \left( B_z - \frac{\beta}{c} E_y \right)
\]

More conveniently for some purposes (and more generally), we introduce vector components parallel (\( \parallel \)) and perpendicular (\( \perp \)) to the relative velocity \( v \) (compare P0.5) and find that Eq. (15.57) is equivalent to the expressions

\[
\begin{align*}
E'_\parallel &= E_\parallel \\
B'_\parallel &= B_\parallel \\
E'_\perp &= \gamma (E_\perp + \mathbf{v} \times \mathbf{B}) \\
B'_\perp &= \gamma \left( B_\perp - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \right)
\end{align*}
\]  

(15.58)

(See P15.20.) Several applications of these transformations are explored in the problems.

**PROBLEMS**

**P15.19.** Evaluate the matrix product \( \mathcal{L} F \mathcal{L}^T \) and verify Eq. (15.56).

**P15.20.** Show that Eq. (15.58) reduces to Eq. (15.57) when \( v = v \hat{i} \).

**P15.21.** Show that if the electric and magnetic induction fields are perpendicular for one observer, they are perpendicular for all observers. *Hint:* By manipulating with Eq. (15.58) show first that \( \mathbf{E} \cdot \mathbf{B} \) is a Lorentz invariant.
CHAPTER 15. RELATIVISTIC FORM OF MAXWELL’S EQUATIONS

P15.22. Suppose constant fields \( E \) and \( B \) exist in a particular frame of reference. Argue that a second frame of reference in which \( B' = 0 \) can be found only if \( E > cB \) and \( E \) is perpendicular to \( B \), and find a possible velocity of this second frame relative to the first. Is the second frame unique? Hint: Use the invariance of \( E^2 - c^2B^2 \) (P15.18), the invariance of \( E \cdot B \) (P15.21), and Eq. (15.58).

P15.23. A point charge \( q \) moves with a uniform speed \( v \) in the positive \( x \) direction. (a) Use the transformation rules for the electromagnetic field to show that the fields established by this charge are given by

\[
E = \frac{q(1 - \beta^2)}{4\pi\epsilon_0} \frac{(x - vt)\hat{i} + y\hat{j} + z\hat{k}}{[(x - vt)^2 + (1 - \beta^2)(y^2 + z^2)]^{3/2}}
\]

\[
B = \frac{1}{c^2} v \times E
\]

Compare these results with those obtained in P14.15. Hint: Write down first the fields produced in a frame of reference in which the charge is at rest and then transform the fields to the original frame of reference. (b) Show that the expression for \( B \) reduces to the Biot-Savart Law, \( B = \frac{(\mu_0 q v \times r)}{(4\pi r^3)} \) in the nonrelativistic limit. Here, \( r \) is the position of the observation point relative to the position of the particle.

P15.24. (a) Assuming the four-vector character of the four-potential \( A \), show that, under the Lorentz transformation in Eq. (15.6), the potentials \((A, V)\) become \((A', V')\), where

\[
A'_x = \gamma \left( A_x - \frac{v}{c^2} V \right) \quad A'_z = A_z
\]

\[
A'_y = A_y \quad V' = \gamma (V - vA_x)
\]

(b) Use these results to find the potentials \( A \) and \( V \) established by the moving charge in P15.23. Compare your results with those given in P14.11. Hint: First write down the potentials \( A' \) and \( V' \) in a frame of reference in which the charge is at rest.

P15.25. Assuming the four-vector character of the four-current \( J \), show that, under the Lorentz transformation in Eq. (15.6), the current density and charge density \((J, \rho)\) become \((J', \rho')\), where

\[
J'_x = \gamma (J_x - v\rho) \quad J'_z = J_z
\]

\[
J'_y = J_y \quad \rho' = \gamma \left( \rho - \frac{v}{c^2} J_x \right)
\]

15.4 The Stress-Energy-Momentum Tensor

In Chapter 6, we concluded our discussion of the electromagnetic field by finding expressions for energy and momentum in the fields. Relativistically, the energy and momentum in an electromagnetic field turn out to be elements of a (four-dimensional) second-rank tensor that also contains the elements of the (three-dimensional) stress tensor explored in P6.37. We identify this new tensor by noting first that the components of the force density, \( \rho E + J \times B \), are the spatial components of the four-vector whose components are \( F_{\mu\nu}J_{\nu} \). For example,

\[
(\rho E + J \times B)_x = \rho E_x + J_yB_z - J_zB_y = (icF_{14}) \left( \frac{J_4}{ic} \right) + F_{12}J_2 + F_{13}J_3 = F_{1\nu}J_\nu \quad (15.59)
\]
15.4. THE STRESS-ENERGY-MOMENTUM TENSOR

The fourth component of this vector is related to the power input per unit volume to the particles composing the charge and current distribution that is the source of the fields, i.e.,

\[ F_{4\nu} J_\nu = F_{41} J_1 + F_{42} J_2 + F_{43} J_3 = \frac{i}{c} \mathbf{E} \cdot \mathbf{J} \] (15.60)

[See Eq. (3.39).]

We next show that the quantities \( F_{\mu\nu} J_\nu \) can be written as the (four-)divergence of a (symmetric) second-rank tensor, i.e., that

\[ F_{\mu\nu} J_\nu = \partial_\alpha T_{\mu\alpha} \] (15.61)

with \( T_{\mu\alpha} = T_{\alpha\mu} \). A direct substitution from Eq. (15.52) for \( J_\nu \) yields that

\[ \mu_0 F_{\mu\nu} J_\nu = F_{\mu\nu} \partial_\alpha F_{\alpha\nu} = \partial_\alpha (F_{\mu\nu} F_{\alpha\nu}) - (\partial_\alpha F_{\mu\nu}) F_{\nu\alpha} \] (15.62)

(Remember that \( F \) is anti-symmetric.) Now, since it involved merely a renaming of summation indices, we find that

\[ (\partial_\alpha F_{\mu\nu}) F_{\nu\alpha} = (\partial_\nu F_{\mu\alpha}) F_{\alpha\nu} \] (15.63)

whence

\[ (\partial_\alpha F_{\mu\nu}) F_{\nu\alpha} = \frac{1}{2} [(\partial_\alpha F_{\mu\nu}) F_{\nu\alpha} + (\partial_\nu F_{\mu\alpha}) F_{\alpha\nu}] = \frac{1}{2} [\partial_\alpha (F_{\mu\nu} + F_{\nu\mu})] F_{\nu\alpha} \]

\[ = -\frac{1}{2} (\partial_\mu F_{\nu\alpha}) F_{\nu\alpha} = -\frac{1}{2} \partial_\mu (F_{\sigma\tau} F_{\sigma\tau}) \] (15.64)

where the next to the last form follows after using Eq. (15.54) and, in the final form, the summation indices \( \nu \) and \( \alpha \) have been changed to \( \sigma \) and \( \tau \). Finally, we substitute Eq. (15.64) into Eq. (15.62) to find that

\[ \mu_0 F_{\mu\nu} J_\nu = \partial_\alpha [F_{\mu\nu} F_{\nu\alpha} + \frac{1}{4} \delta_{\mu\alpha} F_{\sigma\tau} F_{\sigma\tau}] \] (15.65)

where the Kronecker \( \delta_{\mu\alpha} \) has the value 1 if \( \mu = \alpha \) and the value 0 otherwise. Thus, by comparison with Eq. (15.61), we conclude that

\[ T_{\mu\alpha} = \frac{1}{\mu_0} \left[ F_{\mu\nu} F_{\nu\alpha} + \frac{1}{4} \delta_{\mu\alpha} F_{\sigma\tau} F_{\sigma\tau} \right] \] (15.66)

which by its construction is demonstrably a tensor of the second rank (P15.26) and is also symmetric. Its elements can be worked out from knowledge of \( F \) (P15.27); one finds that

\[ T = \begin{pmatrix}
    T_{11} & T_{12} & T_{13} & -icG_x \\
    T_{21} & T_{22} & T_{23} & -icG_y \\
    T_{31} & T_{32} & T_{33} & -icG_z \\
    -icS_x & -icS_y & i\frac{c}{\epsilon}S_z & u_{EM}
\end{pmatrix} \] (15.67)
where

\[ T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \delta_{ij} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \]  

(15.68)

for \( i, j \leq 3 \) are the elements of the (three-dimensional) stress tensor explored in P6.37, \( S = \frac{1}{\mu_0} (E \times B) \) (15.69) is the Poynting vector [Eq. (6.57)], \( \mathcal{G} = \epsilon_0 \mathbf{E} \times \mathbf{B} = \frac{1}{c^2} S \) (15.70) is the momentum density [Eq. (6.63)], and \( u_{\text{EM}} = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \) (15.71) is the electromagnetic energy density [Eq. (6.54)]. The tensor \( T_{\mu\alpha} \) is called the electromagnetic stress-energy-momentum tensor, and it combines these several attributes of the electromagnetic field into a single sixteen-component entity whose components become entangled with one another under a Lorentz transformation of the frame of reference.

The conservation laws of energy and momentum, which we have already treated in Sections 6.4 and 6.5, are contained with Eq. (15.61). For example, when \( \mu = 4 \), Eq. (15.61) becomes

\[ F_{\nu\lambda} J_\nu = \partial_\alpha T_{4\alpha} \]

\[ \implies \frac{i}{c} \mathbf{E} \cdot \mathbf{J} = \partial_1 T_{41} + \partial_2 T_{42} + \partial_3 T_{43} + \partial_4 T_{44} \]

\[ = \nabla \cdot \left( -\frac{i}{c} S \right) + \frac{1}{ic} \frac{\partial u_{\text{EM}}}{\partial t} \]

\[ \implies \mathbf{E} \cdot \mathbf{J} = -\nabla \cdot S - \frac{\partial u_{\text{EM}}}{\partial t} \]  

(15.72)

which is Eq. (6.52)—an expression we have already interpreted as an equation of energy balance. In like fashion (P15.29), Eq. (15.61) for \( \mu = 1, 2, \text{ and } 3 \) yields statements equivalent to Eq. (6.61), which we have interpreted as an equation of momentum balance.

PROBLEMS

P15.26. Assuming that \( F_{\mu\nu} \) is a second-rank tensor, show that \( T_{\mu\alpha} \) as defined by Eq. (15.66) is a second-rank tensor. \textit{Hint:} Show first that \( \delta_{\mu\alpha} \) is a second rank tensor.

P15.27. Demonstrate that the elements of \( T_{\mu\alpha} \) given in Eq. (15.67) follow from \( F \) as given in Eq. (15.50). \textit{Hints:} (1) Show that \( F_{\sigma\tau} F_{\sigma\tau} = 2(B^2 - E^2/c^2) \). (2) Show that \( F_{\mu\nu} F_{\nu\alpha} \) is the \( \mu\alpha \) element of the matrix product \( F^2 \).

P15.28. Show that the trace \( T_{\mu\mu} \) of the stress-energy-momentum tensor is zero.

P15.29. Show that Eq. (15.61) with \( \mu = 1, 2, \text{ and } 3 \) is equivalent to Eq. (6.61).
15.5 A New Viewpoint: The Law of Biot-Savart Revisited

Until Chapter 15, we considered electric and magnetic phenomena as observed in a single frame of reference, and we started the study of each topic by quoting the results of a few pertinent experiments. In Chapter 15, we have discovered that the division of a particular electromagnetic field into separate electric and magnetic induction fields is not intrinsic to the fields themselves but depends as well on the frame of reference within which that division is accomplished. Purely electric fields in one frame of reference, for example, will give rise to both magnetic and electric effects when viewed from some other frame of reference. This observation suggests that Coulomb’s Law and the theory of relativity should be sufficient to permit prediction of at least some magnetic phenomena without introducing the experimental evidence used in our previous development. In this section, we shall illustrate the accuracy of this suggestion by showing how one might infer the law of Biot-Savart from application of special relativity to Coulomb’s Law.\(^{16}\)

As a preliminary to the main argument, we must develop the rule for transforming forces from one frame of reference to another. The starting point is contained in the results of P15.11 and P15.12, where it was shown (1) that the momentum \(p\) and energy \(E\) defined for a particle of (rest) mass \(m\) and velocity \(V\) by

\[
p = \frac{mV}{\sqrt{1 - \left(\frac{V^2}{c^2}\right)}} = \Gamma mV \quad ; \quad E = \Gamma mc^2
\]

combine to form a four-vector (the four-momentum)

\[
p = \left(p_x, p_y, p_z, \frac{iE}{c}\right)
\]

(15.74)

(2) that \(\Gamma\) and \(V\) in the unprimed frame of reference in Fig. 15.2 are related to \(\Gamma'\) and \(V'\) in the primed frame of reference by

\[
\Gamma' = \gamma \Gamma \left(1 - \frac{V_x v}{c^2}\right)
\]

or, equivalently (exchanging primes and “unprimes”) and replacing \(v\) with \(-v\), by

\[
\Gamma = \gamma \Gamma' \left(1 + \frac{V'_x v}{c^2}\right)
\]

(15.75)

and (3) that \(d\tau\)—the proper time interval—defined by

\[
c^2 \, d\tau^2 = c^2 \, dt^2 - dx^2 - dy^2 - dz^2 = c^2 \, dt^2 \left(1 - \frac{V^2}{c^2}\right) \implies d\tau = \frac{1}{\Gamma} \, dt
\]

(15.77)

is a scalar. Finally, we note that the relativistic force remains by definition the time derivative of momentum,

\[
F = \frac{dp}{dt} \quad ; \quad F' = \frac{dp'}{dt'}
\]

(15.78)

\(^{16}\) We shall present only a very limited development from the point of view of special relativity. More detailed treatments may be found in E. M. Purcell, Electricity and Magnetism, Second Edition, (McGraw-Hill Book Company, New York, 1985), Chapter 5; in P. Lorrain and D. R. Corson, Electromagnetic Fields and Waves (W. H. Freeman and Company, San Francisco, 1970), Chapter 6; and in numerous other sources.
and that the quantity \( dp_\mu /d\tau \)—a vector divided by a scalar—is a four-vector. Thus, we find that
\[
\frac{dp_\mu}{d\tau} = (\mathcal{L}^{-1})_{\mu\nu} \frac{dp'_\nu}{d\tau}
\]  
(15.79)
or, by virtue of Eqs. (15.77) and (15.76), that
\[
\frac{dp_\mu}{d\tau} = \Gamma \frac{dp'_\mu}{dt' = (\mathcal{L}^{-1})_{\mu\nu} \frac{dp'_\nu}{d\tau}}
\]  
(15.80)
\[\Rightarrow \frac{dp_\mu}{dt} = \frac{1}{\gamma(1 + V'_x v/c^2)} (\mathcal{L}^{-1})_{\mu\nu} \frac{dp'_\nu}{dt'}
\]
\[\Rightarrow \frac{dp_\mu}{dt} = \frac{1}{\gamma(1 + V'_x v/c^2)} (\mathcal{L}^{-1})_{\mu\nu} \frac{dp'_\nu}{dt'}
\]
(15.80)
Since \( \mathcal{L}^{-1} \) is the transpose of the matrix in Eq. (15.8) [see Eq. (15.11)], we find from Eq. (15.80) that
\[F_x = \frac{F'_x - i\beta dp'_4/dt'}{(1 + V'_x v/c^2)}
\]  
(15.81)
\[F_y = \frac{F'_y}{\gamma(1 + V'_z v/c^2)}
\]  
(15.82)
\[F_z = \frac{F'_z}{\gamma(1 + V'_x v/c^2)}
\]  
(15.83)
Equation (15.81), however, has a more useful form obtained by noting that
\[
\frac{dp'_4}{dt'} = \frac{i}{c} \frac{dE'}{dt'} = \frac{i}{c} F' \cdot V'
\]  
(15.84)
since \( dE'/dt' \) is the rate at which the force \( F' \) does work on the particle. Substituting Eq. (15.84) into Eq. (15.81), we find finally that
\[F_x = \frac{F'_x + (v/c^2) F'_c \cdot V'}{(1 + V'_x v/c^2)} = \frac{v(V'_y F'_y + V'_z F'_z)}{c^2 + V'_x v}
\]  
(15.85)
A derivation of the inverse transformation is the subject of P15.30.

We shall now derive the law of Biot-Savart by considering the force between two charges \( q_1 \) and \( q_2 \) at rest at the origin and at \( (x', y', z') \) in the primed frame of reference (Fig. 15.4). Using Coulomb’s Law in the primed frame, we find that the force on \( q_2 \) as measured in that frame is given by
\[
\{F'_x, F'_y, F'_z\} = \frac{q_1 q_2}{4\pi \epsilon_0} \left\{x', y', z'\right\} \frac{1}{\left[(x')^2 + (y')^2 + (z')^2\right]^{3/2}}
\]  
(15.86)
Written in terms of the unprimed coordinates (assuming the measurement is made at time zero for simplicity), this force is also given by
\[\{F'_x, F'_y, F'_z\} = \frac{q_1 q_2}{4\pi \epsilon_0 R^3} \{\gamma x, y, z\}
\]  
(15.87)
where
\[R = [\gamma^2 x^2 + y^2 + z^2]^{1/2}
\]  
(15.88)
Figure 15.4: Two point charges observed from two frames of reference. The charges are at rest in the primed frame.

Equations (15.82), (15.83), and (15.85) with $V' = 0$ then give the expression

$$\{F_x, F_y, F_z\} = q_1 q_2 \gamma \frac{r}{4\pi \epsilon_0 R^3} \left\{ \frac{x}{\gamma^2}, \frac{y}{\gamma^2}, \frac{z}{\gamma^2} \right\} \quad \Rightarrow \quad F = \frac{q_1 q_2 \gamma}{4\pi \epsilon_0 R^3} \left[ r - \frac{v^2}{c^2} (y \hat{j} + z \hat{k}) \right]$$

where we have used again the invariance of the charges $q_1$ and $q_2$. Now, since the velocity of both charges in the unprimed frame is $v = v \hat{i}$, we find that

$$v \times (v \times r) = (v \cdot r)v - v^2r = v^2x \hat{i} - v^2r = -v^2(y \hat{j} + z \hat{k})$$

and Eq. (15.89) can be written in the form

$$F = q_2 \left[ \frac{q_1 \gamma r}{4\pi \epsilon_0 R^3} + v \times \left\{ \frac{q_1 \gamma (v \times r)}{4\pi \epsilon_0 c^2 R^3} \right\} \right]$$

Thus, solely on the basis of a relativistic transformation of Coulomb’s Law, we are led to introduce two fields

$$E = \frac{q_1 \gamma r}{4\pi \epsilon_0 R^3}$$  \hspace{1cm} (15.92)

$$B = \frac{q_1 \gamma (v \times r)}{4\pi \epsilon_0 c^2 R^3}$$  \hspace{1cm} (15.93)

established in the unprimed frame by the moving charge $q_1$. Further, we determine the force on the moving charge $q_2$ in this frame from the expression

$$F = q_2 (E + v \times B)$$  \hspace{1cm} (15.94)

Starting from Coulomb’s Law, the invariance of charge, and special relativity, we have thus proved that magnetic induction fields must exist. More specifically, writing $\mu_0$ for $1/\epsilon_0 c^2$ and assuming $v/c \ll 1$ so that $\gamma \to 1$, we find from Eq. (15.93) that the $B$-field established by a single charge moving non-relativistically through the origin with velocity $v$ is given by

$$B = \frac{\mu_0 q_1 v \times r}{4\pi r^3}$$  \hspace{1cm} (15.95)
where \( r^2 = x^2 + y^2 + z^2 \). More generally, if a source charge \( q' \) is moving through the point \( r' \) rather than through the origin, the resulting \( B \)-field at point \( r \) is obtained from Eq. (15.95) by interpreting \( r \) as the vector from the source point to the observation point; we find that

\[
B(r) = \frac{\mu_0 q' \mathbf{v} \times (r - r')}{4\pi |r - r'|^3}
\]

(15.96)

Finally, we determine the \( B \)-field produced by a general current distribution by using Eq. (15.96) and the principle of superposition. We divide the general distribution of interest into volume elements, with the element \( dv' \) centered at \( r' \). Within \( dv' \), there may be particles of several different types, those of type \( a \) carrying charge \( q_a \) and having density \( n_a(r') \) and (drift) velocity \( v_a(r') \). Summing Eq. (15.96) over all particles yields

\[
B(r) = \frac{\mu_0}{4\pi} \int \mathbf{J}(r') \times \frac{(r - r')}{|r - r'|^3} \, dv'
\]

(15.97)

where, as in Eq. (2.18),

\[
\mathbf{J}(r') = \sum_a q_a n_a(r') v_a(r')
\]

(15.98)

is the current density describing the distribution. Equation (15.97) is the law of Biot-Savart, and we have achieved the objective of this paragraph by deriving this law from Coulomb’s Law, special relativity, the invariance of charge, and the principle of superposition.

This demonstration that the law of Biot-Savart can in a sense be viewed as an aspect of Coulomb’s Law provides a deep insight into the relationship between electric and magnetic induction fields. These fields are not only tied together within a given frame of reference by their simultaneous occurrence in some of Maxwell’s equations; they are also connected across frames of reference by the requirements of the theory of special relativity. Some magnetic induction fields can even be thought of as arising from a relativistic transformation of a purely electric field, and one might argue that such magnetic induction fields exist only because the observer has (unwisely) picked the “wrong” frame of reference. Not every magnetic induction field, however, can be transformed to a purely electric field (see P15.18), so we cannot eliminate magnetic induction fields altogether by thinking of them solely as relativistic consequences of electric fields. Nonetheless, starting with Coulomb’s Law and the theory of special relativity, it is possible to demonstrate the existence of the magnetic induction field and to obtain the Lorentz force without drawing on the results of any experiments specifically involving magnetic forces. It is also possible to deduce the transformation equations for the fields (P15.32) and to deduce Maxwell’s equations from the beginnings discussed in this section. (See footnote 16.) Thus, beginning with a pair of fields originally conceived to be independent, we have been led first to Maxwell’s equations in a single frame of reference and then—partly by apparent contradictions between the predictions of Maxwell’s equations and the behavior of the physical world—to the theory of special relativity. Finally, from the vantage point provided by special relativity, we have in this section looked back on our starting points and discovered very significant but originally unsuspected connections between electric and magnetic induction fields.

PROBLEMS

**P15.30.** Derive a transformation rule giving the primed components of a force in terms of the unprimed components of the force and the unprimed components of the velocity of the particle; i.e., derive the inverse of Eqs. (15.82), (15.83), and (15.85).
15.5. A NEW VIEWPOINT: THE LAW OF BIOT-SAVART REVISITED

P15.31. Suppose the charge \( q_2 \) in Fig. 15.4 is moving with some velocity \( \mathbf{V}' \) in the primed reference frame. Show that the force on this charge in the unprimed frame is given by \( q_2 \mathbf{E} + q_2 \mathbf{V} \times \mathbf{B} \) with \( \mathbf{E} \) and \( \mathbf{B} \) given by Eqs. (15.92) and (15.93). Hints: (1) Since \( q_1 \) is still at rest in the primed frame, the force on \( q_2 \) in that frame remains solely an electrostatic force. (2) See P15.7.

P15.32. Given the transformation rules for force and for velocity (P15.7) and knowing that \( \mathbf{F}' = q(\mathbf{E}' + \mathbf{V}' \times \mathbf{B}') \) while \( \mathbf{F} = q(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \), find the transformation rules for the fields.

P15.33. (a) Show from Eqs. (15.92) and (15.93) that the fields of a moving point charge are given by

\[
\mathbf{E} = \frac{q_1}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \left( 1 - \frac{\beta^2}{(1 - \beta^2 \sin^2 \alpha)^{3/2}} \right) ; \quad \mathbf{B} = \frac{q_1}{4\pi\epsilon_0 c} \frac{\beta(1 - \beta^2) \sin \alpha}{r^2 (1 - \beta^2 \sin^2 \alpha)^{3/2}}
\]

where \( r^2 = x^2 + y^2 + z^2 \), \( \alpha \) is the angle between \( \mathbf{r} \) and the (positive) \( x \) axis (= direction of \( \mathbf{v} \)), and \( \hat{n} \) is a unit vector in the direction of \( \hat{i} \times \hat{r} \). (b) Sketch graphs of \( |\mathbf{E}| \) and \( |\mathbf{B}| \) as functions of \( \alpha \) for fixed \( r \) and various values of \( \beta \).

SUPPLEMENTARY PROBLEMS

P15.34. Add to Fig. 15.2 a double-primed frame of reference moving with speed \( v_1 \) relative to the primed frame along the common positive \( x'-x'' \) direction. Find the single transformation relating double-primed coordinates directly to unprimed coordinates and show that the velocity of the double-primed origin relative to the unprimed origin is given by \( (v_1 + v)/(1 + vv_1/c^2) \). Hints: (1) \( x'' = \mathcal{L}_{\mu\nu}(v_1/c) \mathcal{L}_{\rho\sigma}(v/c) x_\sigma \). Why? (2) Note that \( \mathcal{L}_{\mu\nu}(v_1/c) \mathcal{L}_{\rho\sigma}(v/c) \) is the \( \mu \sigma \) element of a matrix product.

P15.35. Let \( \mathbf{A} \) by the three-dimensional-vector whose components are the spatial components of the four-vector \( \mathbf{A} \). (a) Show that the Lorentz transformation of the four-vector \( \mathbf{A} \) can be expressed in the form

\[
\mathbf{A}' = \gamma(\mathbf{A} + i\beta^2 \mathbf{A}_4)
\]

\[
\beta \times (\mathbf{A}' \times \mathbf{\beta}) = \beta \times (\mathbf{A} \times \mathbf{\beta})
\]

\[
\mathbf{A}'_4 = \gamma(\mathbf{A}_4 - i\mathbf{A} \cdot \mathbf{\beta})
\]

where \( \mathbf{\beta} \) is a vector of magnitude \( \beta = v/c \) in the direction of the velocity of the primed frame relative to the unprimed frame. (b) Find a \( 4 \times 4 \) matrix expressing the Lorentz transformation between two frames of reference whose axes are parallel but whose relative motion is described by \( \mathbf{\beta} = \beta_x \hat{i} + \beta_y \hat{j} + \beta_z \hat{k} \). (c) Show that your result reduces to Eq. (15.8) when \( \beta_y = 0 \) and \( \beta_x = \beta \). (d) Verify that your result is a (complex) orthogonal matrix.

P15.36. Show that the gauge transformation [Eqs. (6.68) and (6.70)] between two equivalent sets of potentials \( (A_1, V_1) \) and \( (A_2, V_2) \) in a single frame of reference can be written in terms of the four-potential in the form \( A_{2\mu} = A_{1\mu} + \partial_\mu \Lambda \), where \( \Lambda \) is an arbitrary function of the spatial and temporal coordinates.

P15.37. The homogeneous Maxwell equations, Eq. (15.54), can be written more compactly if we introduce the notion of a dual tensor. Let the quantity \( \epsilon_{\alpha\beta\gamma\delta} \) be defined to be 1 when \( (\alpha, \beta, \gamma, \delta) \) is an even permutation of \( (1, 2, 3, 4) \), \(-1 \) when \( (\alpha, \beta, \gamma, \delta) \) is an odd permutation of \( (1, 2, 3, 4) \), and \( 0 \) when \( (\alpha, \beta, \gamma, \delta) \) has any other combination of allowed values. (a) Show that \( \epsilon_{\alpha\beta\gamma\delta} \) is a fourth-rank tensor under the Lorentz transformation in Eq. (15.8). (b) The tensor \( \mathbf{F} \) defined by \( F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} F_{\mu\nu} \) is called the tensor dual to \( \mathbf{F} \). Use the results of part (a) to argue that \( F_{\alpha\beta} \) is a second-rank tensor under the Lorentz transformation in Eq. (15.8), find the matrix similar to Eq. (15.50) expressing the relationship between \( \mathbf{F} \) and the components of the electromagnetic field, and show that Eq. (15.54) has the alternative expression \( \partial_\mu F_{\alpha\beta} = 0 \). (c) Show that \( \mathbf{E} \cdot \mathbf{B} \) is a Lorentz invariant. Hint: \( F_{\mu\nu} F_{\nu\sigma} \) is a scalar. Why? Note: In part tacitly, the conclusions of this
problem are confined to behavior under the so-called proper Lorentz transformations, which do not involve reflections of the coordinates. If the improper transformations, which involve reflection of one or of three of the coordinates, are allowed, then it is necessary to distinguish two kinds of tensors, the first of which transforms, for example, as in Eq. (15.18) under both types of transformation and the second of which transforms as in Eq. (15.18) only under the proper transformations. Members of the second group are called pseudotensors. Strictly, the tensor \( \epsilon_{\alpha\beta\gamma\delta} \) is a fourth-rank pseudotensor and consequently \( \mathbf{F} \) is a second-rank pseudotensor and \( F_{\mu\nu}F^{\mu\nu} \) is a pseudoscalar.

P15.38. (a) Transform the electrostatic field of a long uniformly charged line (P4.6) to a frame of reference moving with speed \( v \) parallel to the wire. Suggestion: Let the wire lie along the \( x \) axis in Fig. 15.2 and be at rest in the unprimed frame. (b) Compare the resulting fields with those produced by an infinitely long current [see Eq. (5.11)] and discuss the difference between a current and a moving charged rod.

P15.39. Consider a region of space devoid of charges and currents. (a) Show that Eq. (15.44) for the potentials is satisfied by the four-component plane wave

\[ A_\mu(r, t) = \alpha_\mu e^{i(\kappa \cdot r - \omega t)} \]

where \( \alpha_\mu \) is a component of a constant four-vector, provided \( \kappa^2 c^2 = \omega^2 \). (b) Argue that a plane wave in one frame of reference must transform to a plane wave in another frame of reference. Thus

\[ A'_\mu(r', t') = \alpha'_\mu e^{i(\kappa' \cdot r' - \omega' t')} = L_{\mu\nu}\alpha_\nu e^{i(\kappa \cdot r - \omega t)} \]

(c) Show that Eq. (2) cannot be true for arbitrary \( r \) and \( t \) unless \( \kappa \cdot r - \omega t = \kappa' \cdot r' - \omega' t' \) and, hence, show that \( \kappa_\mu = (\kappa_x, \kappa_y, \kappa_z, i\omega/c) \) is a four-vector. (See P15.9.) (d) Write

\[ F_{\mu\nu}(r, t) = F^0_{\mu\nu} e^{i\kappa_\sigma x_\sigma} \]

and find the matrix \( F^0_{\mu\nu} \). (e) Suppose a wave with frequency \( \omega \) is propagating in the unprimed frame of Fig. 15.2 in a direction making an angle \( \alpha \) with the \( x \) axis. Use the transformation rules for a four-vector to show that

\[ \tan \alpha' = \frac{\sqrt{1 - \beta^2} \sin \alpha}{(\cos \alpha) - \beta} ; \quad \omega' = \omega \frac{1 - \beta \cos \alpha}{\sqrt{1 - \beta^2}} \]

where, in the primed frame, \( \omega' \) is the frequency of the wave and \( \alpha' \) is the angle between the direction of propagation and the \( x' \) axis. The first of these expressions predicts the phenomenon of aberration, as explored in P15.7; the second predicts the relativistic Doppler effect.\(^{17}\) (f) The longitudinal Doppler effect is analogous to the classical Doppler effect and occurs when the relative motion and the direction of propagation are the same (\( \alpha = 0 \)). This effect is responsible for the red shift in the spectrum of light from distant galaxies. Astronomers define the percent red shift of a galaxy fixed in the unprimed frame and observed in the primed frame in terms of the wavelength \( \lambda \) by \( 100(\lambda' - \lambda)/\lambda \). Show that the percent red shift is given by

\[ P = 100 \left[ \sqrt{\frac{1 + \beta}{1 - \beta}} - 1 \right] \]

Obtain a graph of \( P \) versus \( \beta \) over the range \(-1 < \beta < 1\), and, in particular calculate the velocity of a galaxy for which \( P = 50\% \). (g) The transverse Doppler effect occurs

---

\(^{17}\) Austrian mathematician and physicist Christian Doppler, b. 29 November 1803 in Salzburg, Austria; d. 17 March 1853 in Venice, Italy.
when $\alpha = 90^\circ$ and has no classical analog. Obtain also a graph of $\lambda'/\lambda$ versus $\beta$ for the transverse effect. Note that the longitudinal effect is first order in $\beta$ while the transverse effect is second order in $\beta$.

**P15.40.** Let $\mathbf{p} = (p, iE/c)$ be the momentum four-vector of a particle (P15.12). (a) Show that the expression

$$P = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3} \frac{dp_\mu}{d\tau} \cdot \frac{dp_\mu}{d\tau}$$

reduces to Eq. (14.94) in the limit of $v/c \ll 1$. Thus, this invariant expression is a natural relativistic generalization of Larmor’s formula, Eq. (14.94), for the total power radiated by an accelerated particle. (b) Show that this expression can be written in the alternative form given in Liénard’s formula, Eq. (14.95).

**P15.41.** A particle starts from rest at the origin and moves relativistically in a uniform electric field $\mathbf{E} = E \hat{i}$. Solve the equation of motion

$$\frac{d}{dt} \left( \frac{mv}{\sqrt{1 - (v/c)^2}} \right) = qE ; \quad v = \frac{dx}{dt}$$

to find the position of the particle as a function of time. Sketch graphs of $x$ and of $v$ versus $t$ and compare these graphs with the corresponding non-relativistic results.

**P15.42.** A charged particle moves relativistically in a constant magnetic induction field $\mathbf{B} = B \hat{k}$. There is no electric field. The relativistic equation of motion therefore is $d\mathbf{p}/dt = q\mathbf{v} \times \mathbf{B}$, where $\mathbf{p} = mv/\sqrt{1 - (v/c)^2}$ is the relativistic momentum of the particle (P15.12). (a) Show that the energy $E$ of this particle, given by $E^2 = p^2 c^2 + m^2 c^4$, is constant and hence that $\gamma = \left[1 - v^2/c^2\right]^{-1/2}$ is constant. (b) Show that, even relativistically, a charged particle in a uniform magnetic induction field moves in a helical path. (c) Show that the frequency of circulation about the helical path is given by $\omega = qB/m\gamma$. Show also that, when the velocity is perpendicular to the $\mathbf{B}$-field and the path is a circle, the radius of that circle is given by $R = mv\gamma/qB$. (d) Sketch graphs of $\omega$ and $R$ versus $v/c$ and compare these graphs with the corresponding nonrelativistic results.
Appendix A

Linear Equations, Determinants, and Matrices

This appendix contains a summary of some properties of determinants and matrices. More detailed discussions may be found in the references at the end of the appendix.

A.1 Simultaneous Linear Equations and Determinants

Consider the general system of two linear equations

\[
\begin{align*}
  a_{11} x_1 + a_{12} x_2 &= b_1 \\
  a_{21} x_1 + a_{22} x_2 &= b_2
\end{align*}
\]  

(A.1)

with coefficients \(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\), inhomogeneties \(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\), and unknowns \(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\). Systematic elimination first of \(x_2\) and then of \(x_1\) yields the solution

\[
\begin{align*}
  x_1 &= \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}} \\
  x_2 &= \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{21} a_{12}}
\end{align*}
\]  

(A.2)

If we now introduce what is called a second-order (or \(2 \times 2\)) determinant, defined for an array with elements \(q_{ij}\) by

\[
\begin{vmatrix}
  q_{11} & q_{12} \\
  q_{21} & q_{22}
\end{vmatrix} = q_{11} q_{22} - q_{12} q_{21}
\]  

(A.3)

we can write Eq. (A.2) in the form

\[
\begin{align*}
  x_1 &= \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \\
  x_2 &= \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}
\end{align*}
\]  

(A.4)

Each member of the solution to two simultaneous linear equations can therefore be immediately obtained as the ratio of two determinants. The denominator is the determinant of
the coefficients; the numerator is obtained from the denominator by replacing one column with the column of inhomogenieties—the first column for the first variable and the second column for the second variable.

Although the algebra is more tedious for the general system of three linear equations of the form

\[
\begin{align*}
  a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\
  a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\
  a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3
\end{align*}
\]

systematic elimination of any two unknowns nevertheless leads to a solution for the remaining unknown that can be expressed as a ratio of two third-order (or 3 × 3) determinants, provided that the 3 × 3 determinant of an array with elements \( q_{ij} \) is defined by

\[
\begin{vmatrix}
  q_{11} & q_{12} & q_{13} \\
  q_{21} & q_{22} & q_{23} \\
  q_{31} & q_{32} & q_{33}
\end{vmatrix}
= q_{11} \begin{vmatrix}
  q_{22} & q_{23} \\
  q_{32} & q_{33}
\end{vmatrix} - q_{12} \begin{vmatrix}
  q_{21} & q_{23} \\
  q_{31} & q_{33}
\end{vmatrix} + q_{13} \begin{vmatrix}
  q_{21} & q_{22} \\
  q_{31} & q_{32}
\end{vmatrix}
\]

Here each 2 × 2 determinant is obtained from the original 3 × 3 array by deleting the row and column containing its pre-multiplier. (Warning: The minus sign before the second term is easily forgotten.) With this definition the solution to Eq. (A.5) can be immediately obtained by the same rule as was described for the system of two equations. For example, the solution for \( x_2 \) is

\[
x_2 = \frac{\begin{vmatrix}
  a_{11} & b_1 & a_{13} \\
  a_{21} & b_2 & a_{23} \\
  a_{31} & b_3 & a_{33}
\end{vmatrix}}{\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}}
\]

the denominator being the determinant of the coefficients and the numerator being obtained from the denominator by replacing the second column with the column of inhomogenieties. The rules for reducing the solution of a set of simultaneous linear equations to ratios of determinants are known as Cramer’s rules.\(^1\) Equations (A.4) and (A.7) are examples of these rules, but \( n \)th order \((N \times N)\) determinants can be defined so that these rules apply to systems of equations of any size.

Determinants of all orders share many interesting properties that can be exploited to facilitate their evaluation. Among these properties are the following:

1. The value of a determinant is unaltered by the exchange of rows for columns.
2. The value of a determinant is changed in sign of any two rows (or columns) are exchanged.
3. The value of a determinant is zero if every element on any row (or column) is zero.
4. The value of a determinant is unaltered if every element in one row (or column) is multiplied by a constant and the result is added element by element to another row (or column).

\(^1\)Swiss mathematician Gabriel Cramer, b. 31 July 1704 in Geneva, Republic of Benua; d. 4 January 1752 in Bagnois-sur-Cèze, France.
5. The value of a determinant is multiplied by the constant \( c \) if every element in a single row (or column) is replaced by \( c \) times that element.

6. The determinant \( D \) of an \( N \times N \) array can be reduced to a sum of determinants of \( N - 1 \times N - 1 \) arrays by an expansion similar to Eq. (A.6) in which the pre-multipliers are the elements of any row (or column). Let \( q_{ij} \) be the element in the \( i \)th row and \( j \)th column of an \( N \times N \) array and let \( M_{ij} \)—called the minor of \( q_{ij} \)—be the determinant of of the \( N - 1 \times N - 1 \) array obtained by deleting the \( i \)th row and \( j \)th column from the original \( N \times N \) array. Then, the Laplace development of \( D \) states that

\[
D = \sum_{j=1}^{N} (-1)^{i+j} q_{ij} M_{ij} \quad \text{(expansion on the } i \text{th row)} \quad (A.8)
\]

\[
D = \sum_{i=1}^{N} (-1)^{i+j} q_{ij} M_{ij} \quad \text{(expansion on the } j \text{th column)} \quad (A.9)
\]

Equation (A.3) can be viewed as a special case of Eq. (A.8) for \( N = 2 \) and \( i = 1 \). A bit more obviously, Eq. (A.6) is the special case of Eq. (A.8) for \( N = 3 \) and \( i = 1 \).

Note that the determinant is defined only for arrays with the same number of rows as columns, i.e., for square arrays.

A system of linear equations is said to be homogeneous if all inhomogenieties are zero \([b_1 = b_2 = b_3 = 0 \text{ in the } 3 \times 3 \text{ case of Eq. (A.5)}]\). If a system is homogeneous, the numerator in the solution given by Cramer’s rules for every unknown contains a column of zeros. By property (3) above, all of the unknowns in the most general homogeneous system will then be zero and the solution is said to be trivial. For some special systems, however, it may happen that the denominator of the solutions given by Cramer’s rules also vanishes. In such cases, these solutions are indeterminate and nontrivial solutions can in fact be found. Since the critical denominator is the determinant of the coefficients, we have made plausible the theorem that nontrivial solutions of a homogeneous system of \( n \) linear (algebraic) equations in \( n \) unknowns exist if and only if the determinant of the coefficients vanishes. When this condition is satisfied, at least one of the equations in the system can be obtained as a linear combination of the other equations. Any such equation contains superfluous information and can be discarded.

PROBLEMS

PA.1. (a) What would you say about a pair of equations for which the denominator in Eq. (A.4) is zero but the numerators are different from zero? (b) What would you say if the numerators and the denominator in Eq. (A.4) were all zero? (c) Invent an example of each type of system.

PA.2. Using property (2) in Section A.1, show that the value of a determinant is zero if each element of some row (or column) is equal to the corresponding element of some other row (or column).

PA.3. Use Cramer’s rules to solve the equations

\[
\begin{align*}
3x_1 + x_2 - x_3 & = -6 \\
-x_1 + 2x_2 + x_3 & = 10 \\
5x_1 - x_2 + x_3 & = -2
\end{align*}
\]
Use a different method (expansion on row 1, expansion on column 2, adding rows, etc.) to evaluate each of the four determinants arising.

**PA.4.** Show that the system

\[
\begin{align*}
2x_1 - 3x_2 + x_3 &= 0 \\
x_1 + x_2 - x_3 &= 0 \\
9x_1 - x_2 - 3x_3 &= 0
\end{align*}
\]

possesses nontrivial solutions and find three different sets of numbers \((x_1, x_2, x_3)\) not including \((0,0,0)\) satisfying these equations.

**PA.5.** Write a general computer program that uses Cramer’s rules to solve Eq. (A.5). In brief, your program might accept specific values of \(a_{ij}\) and \(b_i\) as input from the user, and then calculate and print out the values of \(x_1, x_2,\) and \(x_3.\) Test your program with the system in problem **PA.3** and on other systems of your choosing. Optional: Modify your program so that it tests the determinant of the coefficients and prints a suitable message if this determinant is zero. Important note: Cramer’s rules become computationally very inefficient as the size of the system increases. Further, computer evaluation of large determinants directly from primary definitions such as Eq. (A.6) may be subject to disastrous roundoff errors. Better methods for solving linear equations involve systematic elimination and are described in books on numerical analysis, e.g., S. D. Conte, *Elementary Numerical Analysis* (McGraw-Hill Book Company, New York, 1965).

**PA.6.** Given the three points \((x_i, y_i), i = 1, 2, 3,\) develop a set of three simultaneous linear equations to determine the coefficients \(a, b\) and \(c\) of the parabola \(y = ax^2 + bx + c\) that passes through these three points, then use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to find symbolic expressions for the coefficients in terms of \((x_i, y_i),\) and finally find a symbolic expression for the value of \(x\) at which the extremum point of the parabola occurs.

**PA.7.** Use an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica to find \(x\) and \(y\) when these two variables are related by the equations

\[
\begin{align*}
x + y &= z \\
a(x - y) \cos \theta &= bz \cos \phi
\end{align*}
\]

in which \(z, a, b, \theta,\) and \(\phi\) are known quantities. Note that these equations are a simplified version of Eqs. (13.109) and (13.112) determining the Fresnel equations when the incident wave is polarized perpendicular to the plane of incidence.
A.2 Matrix Algebra

Let $Q$ symbolize a rectangular array of (possibly complex) numbers having $m$ rows and $n$ columns, and let $q_{ij}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, be the entry in the $i$th row and $j$th column. Such an array is normally presented in the form

$$
\begin{pmatrix}
q_{11} & q_{12} & \cdots & q_{1j} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2j} & \cdots & q_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{i1} & q_{i2} & \cdots & q_{ij} & \cdots & q_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{m1} & q_{m2} & \cdots & q_{mj} & \cdots & q_{mn}
\end{pmatrix}
$$

and is called a matrix. The individual numbers $q_{ij}$ are called the elements (or in some cases the components) of the matrix, and the number of rows $m$ and columns $n$ are called its dimensions.

Since we are in effect inventing matrices, we may assign to them whatever algebraic properties we wish. Without attempting to motivate our particular choice here, we shall adopt the following properties:

1. **Equality.** Two matrices $Q$ and $R$ having the same dimensions are defined to be equal if and only if every element of $Q$ is equal to the corresponding element of $R$; i.e., $Q = R$ if and only if $q_{ij} = r_{ij}$ for all $i$ and $j$. Equality is not defined for matrices having different dimensions.

2. **Addition.** A matrix $S$ is defined to be the sum of two other matrices $Q$ and $R$ having the same dimensions if and only if every element of $S$ is the sum of the corresponding elements of $Q$ and $R$; i.e., $S = Q + R$ if and only if $s_{ij} = q_{ij} + r_{ij}$ for all $i$ and $j$. It then follows from the corresponding properties of ordinary addition that matrix addition is commutative, $Q + R = R + Q$, and associative, $(Q + R) + T = Q + (R + T)$. The sum of two matrices having different dimensions is not defined.

3. **Multiplication of a matrix by a scalar.** A matrix $R$ is defined to be the product of a scalar $s$ and another matrix $Q$ if and only if every element of $R$ is equal to $s$ times the corresponding element of $Q$; i.e., $R = sQ$ if and only if $r_{ij} = sq_{ij}$ for all $i$ and $j$; the product $Qs$ is defined to be equal to $sQ$. It then follows from the properties of ordinary multiplication that multiplication of a matrix by a scalar is distributive, $(s + t)Q = sQ + tQ$ and $s(Q + R) = sQ + sR$. The difference between two matrices having the same dimensions may now be defined by $Q - R = Q + (-R)$.

4. **Multiplication of a matrix by a matrix.** Let $m_q$ and $n_q$ be the number of rows and columns in the matrix $Q$ and let $m_r$ and $n_r$ be the number of rows and columns in the matrix $R$. If $n_q = m_r$ (number of columns in $Q =$ number of rows in $R$), the matrix $S$ whose elements are given by

$$s_{ij} = \sum_{k=1}^{n_q} q_{ik} r_{kj}; \quad 1 \leq i \leq m_q, \quad 1 \leq j \leq n_r \quad (A.11)$$
Symbolic manipulating tools like MAXIMA, MAPLE, and Mathematica conjugates of the elements of $Q$ vector (which is equal to the complex conjugate of the transpose (of rows processing matrices in the ways described above and in many other ways as well. Processing tools like IDL, MATLAB, OCTAVE, and PYTHON typically provide routines for 436

APPENDIX A. LINEAR EQUATIONS, DETERMINANTS, AND MATRICES

is defined to be the matrix product $QR$ of the matrices $Q$ and $R$; i.e., $S = QR$ if Eq. (A.11) holds. Less formally, Eq. (A.11) simply states that the element appearing in the $i$th row and $j$th column of $S$ is obtained by multiplying each element in the $i$th row of $Q$ by the corresponding element in the $j$th column of $R$ (first by first, second by second, etc.) and adding these products. Even more specifically, we have the product,

$$
\begin{pmatrix}
1 & 4 & 3 \\
-2 & 1 & 6
\end{pmatrix}
\begin{pmatrix}
-1 & 1 & 0 & -3 \\
4 & 3 & 1 & -1 \\
-2 & -2 & 2 & 1
\end{pmatrix}
= 
\begin{pmatrix}
9 & 7 & 10 & -4 \\
-6 & -11 & 13 & 11
\end{pmatrix}
$$

(A.12)

in which, for example, the $(2,3)$ element (shown in boldface) is obtained by multiplying corresponding elements of the boldface row in the first matrix and the boldface column in the second matrix and adding the products, $13 = (-2)(0)+(1)(1)+(6)(2)$. Provided rows and columns match appropriately so that the necessary products are defined, it then follows from this definition that matrix multiplication is distributive, $Q(R+T) = QR+QT$, and associative, $(QR)T = Q(RT)$. Even when both products $QR$ and $RQ$ are defined (i.e., when $n_q = m_r$ and $n_r = m_q$), matrix multiplication is not in general commutative, $QR \neq RQ$. The product of two matrices is not defined if the number of columns in the first factor differs from the number of rows in the second factor.

Symbolic manipulating tools like MAXIMA, MAPLE, and Mathematica and numeric processing tools like IDL, MATLAB, OCTAVE, and PYTHON typically provide routines for processing matrices in the ways described above and in many other ways as well.

Several additional terms are used in discussing matrices. Let $Q$ again be a matrix with $m$ rows and $n$ columns. Particularly important special forms for $Q$ include the square matrix $(m=n)$, the column matrix or column vector $(n=1)$, and the row matrix or row vector $(m=1)$. The complex conjugate $Q^*$ is that matrix whose elements are the complex conjugates of the elements of $Q$, the transpose $Q^T$ is that matrix whose columns are the rows of $Q$, and the adjoint $Q^\dagger$ is the transpose of the complex conjugate, $Q^\dagger = (Q^*)^T$, which is equal to the complex conjugate of the transpose $(Q^T)^*$. A square matrix is said to be symmetric if $Q^T = Q$, antisymmetric (or skew symmetric) if $Q^T = -Q$, Hermitian if $Q^\dagger = Q$, and antiHermitian (or skew Hermitian) if $Q^\dagger = -Q$. A diagonal matrix is a square matrix in which only the diagonal elements—those elements $q_{ij}$ for which $i = j$—are different from zero, and a unit matrix, often symbolized by $I$, is a diagonal matrix all of whose diagonal elements are unity. It is readily verified that $IQ = QI = Q$ for any square matrix $Q$. Finally, the inverse $Q^{-1}$ of a square matrix $Q$ is that matrix having the property $Q^{-1}Q = QQ^{-1} = I$, and a matrix is said to be orthogonal if $Q^{-1} = Q^T$ and unitary if $Q^{-1} = Q^\dagger$.

PROBLEMS

PA.8. Verify all the elements in the right-hand side of Eq. (A.12).

PA.9. Let

$$
A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 1 \\ 5 & -1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}
$$

and evaluate $x^T y$, $y x^T$, $B x$, and $A (A + B)$. 

PA.10. The trace $\text{tr}(Q)$ of a square matrix $Q$ is defined to be the sum of the diagonal elements of $Q$. Let $A$ and $B$ be square matrices of the same dimension and show that $\text{tr}(AB) = \text{tr}(BA)$, even if $AB \neq BA$.

PA.11. Show that the inverse of a diagonal matrix $D$ is itself a diagonal matrix whose diagonal elements are the reciprocals of the diagonal elements of $D$.

PA.12. Given that $\det(QR) = [\det(Q)][\det(R)]$, where $Q$ and $R$ are $n \times n$ matrices and $\det(\ldots)$ denotes the determinant, show that (a) $\det(sQ) = s^n \det(Q)$, (b) $\det(Q^T) = \det(Q)$, (c) $\det(Q^{-1}) = 1/\det(Q)$, and (d) $\det(Q) = \pm 1$ if $Q$ is orthogonal.

PA.13. (a) Show that Eq. (A.1) can be written in the form $Ax = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ; \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ; \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

(b) Multiplication of this form by $A^{-1}$ from the left gives the solution $x = A^{-1}b$. Infer the $2 \times 2$ matrix $A^{-1}$ from Eq. (A.2). (c) Can you infer a general rule for finding the inverse of a larger (square) matrix?

PA.14. Let a primed Cartesian coordinate system be obtained from an unprimed system by rigid rotation through an angle $\phi$ about the $z$ axis (Fig. A.1). Show that the coordinates of a (fixed) point in space in these two systems are related by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and verify that the matrix expressing this rigid rotation is orthogonal.

PA.15. (a) Convince yourself that the result of multiplying an $n$-component column vector $x$ by an $n \times n$ matrix $Q$ is another $n$-component column vector $y$, $Qx = y$. (b) For many matrices $Q$ there exist characteristic vectors, called eigenvectors of $Q$. For the $i$th eigenvector $x_i$, $Qx_i$ is merely a (particular) multiple $\lambda_i$ of $x_i$, i.e., $Qx_i = \lambda_i x_i$; $\lambda_i$ is called an eigenvalue of $Q$. Show that an eigenvector can be found only if $\lambda_i$ is one of the values of $\lambda$ satisfying
det(Q – λI) = 0, where det(…) denotes a determinant. (c) Find the three eigenvalues and the corresponding eigenvectors of the matrix A in PA.9.
Appendix B

Binomial and Taylor Expansions

Very often in physics we encounter a complicated function of some variable, say \(x\), but are really interested only in the behavior of this function for small values of \(x\) or perhaps for values of \(x\) deviating by some small amount from a specific value \(a\). In such cases, an expansion of the function in powers of \(x\) or of \(x - a\) coupled with neglect of all but the earliest few terms in this expansion may simplify the complicated function without sacrificing any of its essential characteristics. This appendix contains a brief statement of two techniques for obtaining such expansions. More detailed discussions may be found in the references at the end of the appendix.

Perhaps the most frequently used approach invokes the binomial theorem, which states that the binomial \((1 + x)^n\) has the expansion

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}x^m + \cdots \tag{B.1}
\]

where \(m!\), read \(m\) factorial, stands for the product of all positive integers up to and including \(m\). Thus, we have that

\[
\frac{1}{1+f} = (1 + f)^{-1} = 1 - f + \frac{(-1)(-2)}{2!}f^2 + O(f^3)
\]

\[
= 1 - f + f^2 + O(f^3) \tag{B.2}
\]

\[
(1 - y^2)^{-3/2} = 1 + \left(\frac{-3}{2}\right)(-y^2) + \frac{1}{2!}\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(-y^2)^2 + O(y^6)
\]

\[
= 1 + \frac{3}{2}y^2 + \frac{15}{8}y^4 + O(y^6) \tag{B.3}
\]

and so on. Here, the notation \(O(f^3)\), for example, indicates that the omitted terms contain powers of \(f\) with exponents no smaller than 3. In these examples, infinite series have resulted and the convergence of the binomial series should therefore be examined. If \(n\) is a positive integer, the expansion in Eq. (B.1) ends with the term \(x^n\) (why?), and the series certainly converges for all \(x\). More generally, when this expansion does not terminate, it
converges absolutely if $|x| < 1$, diverges if $|x| > 1$, and may either converge or diverge when $x = \pm 1$ depending on the value of $n$. In the present context, the variable $x$ is restricted by the condition $|x| \ll 1$ and the series in Eq. (B.1) can usually be truncated after the first few terms without significant loss of accuracy. The number of terms one must retain to provide a given accuracy must be determined individually for each case that arises.

The Taylor expansion\(^1\) of a function $f(x)$ of a single variable $x$ about the point $a$ is given by

$$
f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=a} (x-a)^n
$$

(with the understanding that $0! = 1$); the Taylor expansion of a function $f(x, y)$ of two variables $x$ and $y$ about the point $(a, b)$ begins with the terms

$$
f(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{1}{2} f_{xx}(a, b)(x-a)^2 + f_{xy}(a, b)(x-a)(y-b) + \frac{1}{2} f_{yy}(a, b)(y-b)^2 + \cdots
$$

In these equations, $f'(a)$ and $f''(a)$ stand for the first and second derivatives of $f(x)$ evaluated at $x = a$, $f_x(a, b)$ stands for $\partial f(x, y)/\partial x$ evaluated at $(x, y) = (a, b)$, etc. When $|x-a|$ and $|y-b|$ are sufficiently small, these series can often be truncated after a very few terms without significant loss of accuracy. As with the binomial series, however, the number of terms to be retained in a given Taylor expansion must be separately determined for each case that arises.

**PROBLEMS**

**PB.1.** Generalize Eq. (B.1) to obtain an expansion for $(a + b)^n$.

**PB.2.** Use mathematical induction to prove the binomial theorem for $n$ a positive integer. That is, (1) show that Eq. (B.1) is true for $n = 1$ and (2) assume that Eq. (B.1) is correct for $n = N$ and prove that it is correct for $n = N + 1$.

**PB.3.** Use the binomial theorem to obtain the first three nonvanishing terms in the expansion of the quantity

$$
m c^2 \left( \frac{1}{\sqrt{1-(v/c)^2}} - 1 \right)
$$

in powers of $(v/c)^2$. This quantity, in which $c$ is the speed of light, is the relativistic expression for the kinetic energy of a particle of rest mass $m$ moving with speed $v$.

**PB.4.** Within what range must $f$ be confined if the approximation $(1 + f)^{-1} \approx (1 - f)$ is to have an error no larger than 1%?

---

\(^1\)English mathematician Brook Taylor, b. 18 August 1685 in Edmonton, Middlesex, England; d. 29 December 1731 in London, England. While what is known as the Taylor expansion was introduced by Brook Taylor, its concept was discovered by Scottish mathematician James Gregory, b. November 1638 in Drumoak, Aberdeenshire, Scotland; d. October 1675 in Edinburgh, Scotland.
PB.5. The coefficients $P_n(t)$ in the expansion

$$\frac{1}{\sqrt{1 - 2ut + u^2}} = \sum_{n=0}^{\infty} P_n(t) u^n$$

are called the Legendre polynomials. Identify $x$ in Eq. (B.1) with $(-2ut + u^2)$ and expand the quantity on the left to find the first four Legendre polynomials.

PB.6. Obtain the first three nonvanishing terms in the Taylor expansion about the origin ($a = 0$) of $\sin x$, $e^x$, and $(1 + x)^q$. The last result is, of course, the binomial expansion. When done rigorously, this approach proves the binomial theorem even if $q$ is not a positive integer. Work these expansions out first by hand but then explore the features of an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica for evaluating Taylor expansions.
Appendix C

Vector Identities and Relationships

In this appendix, \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{Q}, \) and \( \mathbf{R} \) represent vector fields, \( \Phi \) and \( \Psi \) represent scalar fields, \( \mathbf{r} \) represents the position vector, and \( \hat{e}_i \) is a Cartesian unit vector. Note also the following:

(a) In identities (C.20), (C.22), and (C.24), each of which involves a line integral about a closed path, \( d\mathbf{S} \) by convention has the direction determined by the thumb of the right hand when the fingers point in the direction of \( d\mathbf{l} \) and the palm faces the area enclosed by the path.

(b) In identities (C.21), (C.23), (C.25), (C.26), and (C.27), each of which involves a surface integral over a closed surface, \( d\mathbf{S} \) by convention has the direction of the outward normal.

\[
\begin{align*}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \quad \text{(C.1)} \\
(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} \quad \text{(C.2)} \\
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \quad \text{(C.3)} \\
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix} \quad \text{(C.4)} \\
\nabla(\Phi + \Psi) &= \nabla\Phi + \nabla\Psi \quad \text{(C.5)} \\
\nabla(\Psi \Phi) &= \Phi \nabla\Psi + \Psi \nabla\Phi \quad \text{(C.6)} \\
\nabla \times (\mathbf{Q} + \mathbf{R}) &= \nabla \times \mathbf{Q} + \nabla \times \mathbf{R} \quad \text{(C.7)} \\
\nabla \times (\Phi \mathbf{Q}) &= \nabla\Phi \times \mathbf{Q} + \Phi \nabla \times \mathbf{Q} \quad \text{(C.8)} \\
\n\nabla \times \mathbf{r} &= \mathbf{0} \quad \text{(C.9)} \\
\n\nabla \cdot (\mathbf{Q} + \mathbf{R}) &= \nabla \cdot \mathbf{Q} + \nabla \cdot \mathbf{R} \quad \text{(C.10)} \\
\n\nabla \cdot (\Phi \mathbf{Q}) &= \nabla\Phi \cdot \mathbf{Q} + \phi \nabla \cdot \mathbf{Q} \quad \text{(C.11)}
\end{align*}
\]
\[ \nabla \cdot (Q \times R) = R \cdot (\nabla \times Q) - Q \cdot (\nabla \times R) \quad \text{(C.12)} \]
\[ \nabla \cdot \mathbf{r} = 3 \quad \text{(C.13)} \]
\[ \nabla \cdot \frac{\mathbf{r}}{r^3} = 0, \quad r \neq 0 \quad \text{(C.14)} \]
\[ \nabla (Q \cdot R) = (Q \cdot \nabla)R + (R \cdot \nabla)Q + Q \times (\nabla \times R) + R \times (\nabla \times Q) \quad \text{(C.15)} \]
\[ \nabla \times (Q \times R) = Q(\nabla \cdot R) - R(\nabla \cdot Q) + (R \cdot \nabla)Q - (Q \cdot \nabla)R \quad \text{(C.16)} \]
\[ \nabla \cdot (\nabla \times Q) = 0 \quad \text{(C.17)} \]
\[ \nabla \times (\nabla \times Q) = 0 \quad \text{(C.18)} \]
\[ \nabla \times (\nabla \times Q) = \nabla (\nabla \cdot Q) - \nabla^2 Q \quad \text{(C.19)} \]

(Cartesian coordinates only)
\[ \oint \Phi \, dl = \int dS \times \nabla \Phi \quad \text{(C.20)} \]
\[ \oint \Phi \, dS = \int \nabla \Phi \, dv \quad \text{(C.21)} \]
\[ \oint Q \cdot dS = \int \nabla \times Q \cdot dS \quad \text{(C.22)} \]
\[ \oint Q \cdot dS = \int \nabla \cdot Q \, dv \quad \text{(C.23)} \]
\[ \oint Q \times dI = \int (\nabla \cdot Q) \, dS - \sum_i \hat{e}_i \int [(\hat{e}_i \cdot \nabla)Q] \cdot dS \quad \text{(C.24)} \]

(Cartesian coordinates only)
\[ \oint dS \times Q = \int \nabla \times Q \, dv \quad \text{(C.25)} \]
\[ \oint Q(R \cdot dS) = \int [Q(\nabla \cdot R) + (R \cdot \nabla)Q] \, dv \quad \text{(C.26)} \]
\[ \oint (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot dS = \int (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) \, dv \quad \text{(Green’s theorem)} \quad \text{(C.27)} \]
\[ (A \cdot \nabla) \mathbf{r} = A \quad \text{(C.28)} \]
Appendix D

Complex Numbers and Fourier Analysis

A complex number $z$—not to be confused with the $z$ coordinate of a point in three-dimensional space—is an ordered pair of real numbers $(x, y)$, $x$ being called the real part of $z$, denoted $\text{Re}(z)$ or $\Re\{z\}$, and $y$ being called the imaginary part of $z$, denoted $\text{Im}(z)$ or $\Im\{z\}$, i.e.,

$$z = (x, y) \iff \text{Re}(z) = \Re\{z\} = x, \quad \text{Im}(z) = \Im\{z\} = y \quad (D.1)$$

In this appendix, we shall summarize the algebraic properties that define complex numbers more fully and then very briefly develop those few applications that we need in the main part of this book. More detailed discussions may be found in the references at the end of the appendix.

D.1 The Algebra of Complex Numbers

We adopt the point of view that we are in effect inventing complex numbers. Thus, we may assign to them whatever algebraic properties we wish. Without attempting here to motivate our particular choices, we shall assign the following properties:

1. **Equality.** Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined to be equal if and only if the real and imaginary parts are separately equal; i.e., $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

2. **Addition.** A complex number $z$ is defined to be the sum of two other complex numbers $z_1$ and $z_2$ if and only if the real and imaginary parts of $z$ are separately the sums of the real and imaginary parts of $z_1$ and $z_2$; i.e., $z = z_1 + z_2$ if and only if $x = x_1 + x_2$ and $y = y_1 + y_2$. It then follows from the corresponding properties of real numbers that addition of complex numbers is commutative, $z_1 + z_2 = z_2 + z_1$, and associative, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$. By this definition, the complex number $(0, 0)$ has the property that $(0, 0) + z_1 = z_1$ and is therefore the identity element for addition. Finally, if we define $-z$ by requiring that $z + (-z) = (0, 0)$, we find that $-z = (-x, -y)$ and the definition $z_1 - z_2 = z_1 + (-z_2)$ for the difference between two complex numbers is unambiguous.
3. **Multiplication.** A complex number \( z \) is defined to be the product of two other complex numbers \( z_1 \) and \( z_2 \), \( z = z_1 z_2 \), if and only if \( x = x_1 x_2 - y_1 y_2 \) and \( y = y_1 x_2 + x_1 y_2 \). It then follows (PD.1) that multiplication of complex numbers is commutative, \( z_1 z_2 = z_2 z_1 \); associative, \( z_1 (z_2 z_3) = (z_1 z_2) z_3 \); and distributive with respect to addition, \( z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3 \). By this definition, the complex number \((1,0)\) has the property that \((1,0) \cdot z = z \cdot (1,0) = z \) and is therefore the identity element for multiplication. Finally, if we define the reciprocal \( z_r = 1/z = (x_r, y_r) \) of the complex number \( z \) by requiring that \( zz_r = (x, y) \cdot (x_r, y_r) = (1,0) \), we find by applying the definition of multiplication that \( xx_r - yy_r = 1 \) and \( xy_r + yx_r = 0 \) or, on solution for \( x_r \) and \( y_r \), that \( x_r = x/(x^2 + y^2) \) and \( y_r = -y/(x^2 + y^2) \). Thus, the definition \( z_1/z_2 = z_1 \cdot (1/z_2) \) for the quotient of two complex numbers is now unambiguous.

It follows from the definitions of the previous paragraphs that complex numbers of the form \((x,0)\) with zero imaginary part have the same algebraic properties as the real numbers \( x \). We are therefore justified in putting \((x,0)\) into correspondence with \( x \) and indeed in denoting \((x,0)\) more simply by \( x \). The complex numbers thus include the real numbers, but they also include more general numbers. For example, the above rule for multiplication gives \((0,1)^2 = (-1,0) = -1\), a property possessed by no real number. The number \((0,1)\) is usually denoted by \( i \) and is called the imaginary unit; it has the curious property that \( i^2 = -1 \). Further, by the above rules, we can write

\[
z = (x, y) = (x, 0)(1, 0) + (0, 1)(y, 0) = x + iy \tag{D.2}
\]

and the complex number can be viewed as a (complex) sum of two parts, the second containing the imaginary unit \( i \) as a factor. Further, all of the above rules for algebraic manipulation of complex numbers can be summarized in the statement that complex numbers written in the form of Eq. (D.2) can be treated as ordinary binomials except that \( i^2 \) can be replaced by \(-1\) and equality between two such binomials implies separate equality of the real and imaginary parts. Thus, for example,

\[
(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \tag{D.3}
\]

\[
(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2) \tag{D.4}
\]

\[
x_1 + iy_1 = x_2 + iy_2 \iff x_1 = x_2 \text{ and } y_1 = y_2 \tag{D.5}
\]

Division of two numbers written in the form of Eq. (D.2) is facilitated by introducing the **complex conjugate** \( z^* \) of \( z \), obtained by making the replacement \( i \rightarrow -i \), i.e.,

\[
z = x + iy \implies z^* = x - iy \tag{D.6}
\]

and then noting that

\[
zz^* = x^2 + y^2 \tag{D.7}
\]

is a **real** number. Thus, the sequence of operations

\[
\frac{z_1}{z_2} = z_1 \cdot z_2^* = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} = \frac{x_1 x_2 - y_1 y_2}{x_2^2 + y_2^2} + i \frac{y_1 x_2 + x_1 y_2}{x_2^2 + y_2^2} \tag{D.8}
\]

reduces the quotient \( z_1/z_2 \) to the form \( a + ib \) with \( a \) and \( b \) real and thus verifies that the quotient of two complex numbers is itself a complex number. Finally, in terms of \( z \) and \( z^* \), we can now derive the relationships

\[
x = \text{Re}(z) = \frac{1}{2} (z + z^*) \quad ; \quad y = \text{Im}(z) = \frac{1}{2i} (z - z^*) \tag{D.9}
\]
D.1. THE ALGEBRA OF COMPLEX NUMBERS

Figure D.1: A graphical representation of the complex number $z$.

for the real and imaginary parts (PD.7).\footnote{Note, incidentally, that the imaginary part of $z$ is the coefficient of $i$; it does not include the $i$.}

A graphical meaning for the complex number $z = x + iy$ emerges if we represent $z$ as the point $(x, y)$ in the complex $z$ plane defined by a horizontal real axis, along which $x$ is plotted, and a vertical imaginary axis, along which $y$ is plotted (Fig. D.1). Equivalently, we can think of $z$ as a vector in this plane, this vector having horizontal and vertical components $x$ and $y$, having length $|z|$, usually called the magnitude or the absolute value of $z$, and making an angle $\theta$, usually called the argument or phase of $z$, with the positive real axis. Analytically,

\[ |z| = \sqrt{x^2 + y^2} \quad ; \quad |z|^2 = zz^* \quad \text{(D.10)} \]

and

\[ \theta = \arg(z) = \tan^{-1} \frac{y}{x} \quad \text{(D.11)} \]

A complex number can thus be written either in the Cartesian form of Eq. (D.2) or in the polar form,

\[ z = |z|(\cos \theta + i \sin \theta) \quad \text{(D.12)} \]

where the complex number, $\cos \theta + i \sin \theta$, represents a vector of unit magnitude at an angle $\theta$ to the real axis. In this geometric representation, addition of complex numbers is equivalent to vector addition in the complex plane.

To express Eq. (D.12) in a still more compact form, we digress briefly to discuss how functions of $z$, now regarded as a complex variable, might be defined. Since multiplication is well defined, the integer power $z^n$ is defined, and polynomial functions of $z$, such as

\[ f(z) = (1 + 2i)z^5 - 3iz^2 + 6 + \pi i \quad \text{(D.13)} \]

are meaningful. Without raising here the question of convergence, we also know at least intuitively what is meant by an infinite power series, and it is natural to define such functions
as \( \sin(z) \), \( \cos(z) \), and \( e^z \) by replacing the real variable in the usual power series (PB.6) by the complex variable \( z \), e.g.,
\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}
\]

With this definition, let us now examine \( e^{i\theta} \); we find that
\[
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \cdots
\]

The series emerging in square brackets, however, are the series for \( \cos \theta \) and \( \sin \theta \), respectively. Thus, we arrive at the Euler formula,
\[
e^{i\theta} = \cos \theta + i \sin \theta
\]
which in particular makes the real and imaginary parts of \( e^{i\theta} \) explicit when \( \theta \) is real. The polar form of the complex number \( z \) [Eq. (D.12)] can therefore also be written as
\[
z = |z| e^{i\theta}
\]
which is the compact expression sought in this paragraph.

**PROBLEMS**

**PD.1.** Working directly with the definition in item (3) above, show that \( z_1 z_2 = z_2 z_1 \) and that \( z_1 (z_2 z_3) = (z_1 z_2) z_3 \).

**PD.2.** Let \( z_1 = 3 + 4i \) and \( z_2 = 1 - i \). (a) Reduce \( z_1 z_2 \) and \( z_2 / z_1 \) to the form \( a + bi \). (b) Write \( z_2 \) in polar form.

**PD.3.** Show that \( z = 1 + i \) satisfies \( z^2 - 2z + 2 = 0 \).

**PD.4.** Show that \( (z_1 + z_2)^* = z_1^* + z_2^* \) and \( (z_1 z_2)^* = z_1^* z_2^* \).

**PD.5.** Show that if \( z_1 z_2 = 0 \), either \( z_1 \) or \( z_2 \) must be zero. (Both, of course, may be zero.)

**PD.6.** Prove the triangle inequality, \(|z_1 + z_2| \leq |z_1| + |z_2| \). Under what conditions does the equality hold?

**PD.7.** Prove Eq. (D.9).

**PD.8.** Geometrically, how is \( z^* \) positioned in Fig. D.1?

**PD.9.** Show that \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \) and \( |z_1 z_2| = |z_1| |z_2| \).

**PD.10.** If all appearances of \( i \) in a complex expression are explicit, evaluating the complex conjugate of the expression involves no more than replacing \( i \) everywhere with \( -i \). If, however, some of the appearances of \( i \) are hidden, as in \( A = a + ib \), where \( a \) and \( b \) are themselves complex, evaluating the complex conjugate must be done more carefully. Show that \( A^* = a^* - ib^* \).

**PD.11.** Since \( e^{i\theta} = (e^{i\theta})^n \), Eq. (D.16) implies that
\[
\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n
\]
for any \( n \). Let \( n = 3 \) and extract expressions for \( \cos 3\theta \) and \( \sin 3\theta \) in terms of \( \cos \theta \) and \( \sin \theta \).
PD.12. By its definition in Eq. (D.14), the exponential function of a complex argument has all the algebraic properties of the exponential function of a real argument. In particular, \( e^{z_1}e^{z_2} = e^{z_1+z_2} \). (a) Show that \( e^z = e^x (\cos y + i \sin y) \). (b) Find the real and imaginary parts of \( \sinh(z) \), which is defined by \( \frac{1}{2}(e^z - e^{-z}) \).

PD.13. (a) Given the Euler formula, Eq. (D.16), show that

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad ; \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\]

(b) Given the definitions

\[
cosh z = \frac{e^z + e^{-z}}{2} \quad ; \quad \sinh z = \frac{e^z - e^{-z}}{2}
\]

show that

\[
cosh(i\theta) = \cos \theta \quad \sinh(i\theta) = i \sin \theta \quad \cos(iz) = \cosh z \quad \sin(iz) = i \sinh z
\]

and that

\[
\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y \quad \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y
\]

thereby displaying the real and imaginary parts of \( \cos(x + iy) \) and \( \sin(x + iy) \) when \( x \) and \( y \) are real.

D.2 Fourier Series

Let \( f(x) \) be a periodic function with period \( 2L \). Since \( \cos(n\pi x/L) \) and \( \sin(n\pi x/L) \) are also periodic with period \( 2L \) for any integer \( n \), it is reasonable to expect that \( f(x) \) can be expanded in a trigonometric series. We choose to separate the term for \( n = 0 \) and to write a trial series, called a Fourier series, in the form

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \quad \text{(D.18)}
\]

To determine the constant coefficients in this series, we first note the identities

\[
\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = 0 \\
\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = L \delta_{nm} \quad \text{(D.19)}
\]

where \( \delta_{nm} = 1 \) if \( n = m \) and \( \delta_{nm} = 0 \) if \( n \neq m \). These identities apply when \( n > 0 \), \( m > 0 \) and can be verified by direct integration (PD.14). Now, assume that term-by-term multiplication and integration of the series in Eq. (D.18) is possible, multiply every term, for example, by \( \sin(m\pi x/L) \) and integrate the result term by term over the interval \(-L \leq x \leq L\). Because of the identities in Eq. (D.19), only one term—that involving \( b_m \)—of the entire series survives, and that term gives

\[
b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx, \quad m = 1, 2, \ldots \quad \text{(D.20)}
\]
By similar arguments, we find that
\[ a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{m\pi x}{L} \right) dx, \quad m = 0, 1, 2, \ldots \]  
(D.21)

Although \( a_0 \) must be treated separately from the rest of the \( a \)'s, Eq. (D.21) nonetheless applies to all of the \( a \)'s [which explains why the factor of \( \frac{1}{2} \) was explicitly inserted with \( a_0 \) in Eq. (D.18)]. At least formally, the series in Eq. (D.18) with coefficients given by Eqs. (D.20) and (D.21) ought to converge to \( f(x) \). Such is in fact the case, as is stated by the following theorem, whose detailed proof we leave to the references:

In \(-L \leq x < L\), let \( f(x) \) be defined and bounded. Further let \( f(x) \) have no more than a finite number of maxima and minima and only a finite number of discontinuities. Finally, let \( f(x) \) be defined outside the interval \(-L \leq x < L\) by the requirement of periodicity, \( f(x + 2L) = f(x) \). Then the Fourier series, Eq. (D.18), with coefficients given by Eqs. (D.20) and (D.21) converges at every \( x \) to \( \frac{1}{2} \left[ f(x + \epsilon) + f(x - \epsilon) \right] \), where \( \epsilon \) is positive and arbitrarily small, and in particular converges to \( f(x) \) at points where \( f(x) \) is continuous. Essentially every periodic function of interest in physical problems satisfies these conditions.

Equation (D.18) can be written in a complex form if the trigonometric functions in it are reexpressed using the results of PD.13. After combining terms, we find that
\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{a_n - ib_n}{2} e^{i(n\pi x/L)} + \frac{a_n + ib_n}{2} e^{-i(n\pi x/L)} \right] \]  
(D.22)

Thus, if we introduce a new set of coefficients defined by
\[ c_0 = \frac{1}{2}a_0 \]
\[ c_n = \frac{1}{2}(a_n - ib_n) \quad n > 0 \]  
(D.23)
\[ c_n = \frac{1}{2}(a_n + ib_n) \quad n < 0 \]

Eq. (D.22) can be expressed more compactly as
\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(n\pi x/L)} \]  
(D.24)

Further, we find by combining Eqs. (D.20) and (D.21) with Eq. (D.23) that
\[ c_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i(m\pi x/L)} dx \]  
(D.25)

for all \( m \), positive, negative, and zero. Although our development does not show it, this exponential form of the Fourier series represents \( f(x) \) correctly even if \( f(x) \) happens to be a complex function of \( x \).

**PROBLEMS**

**PD.14.** Verify the identities in Eq. (D.19). Do not hesitate to use the capabilities of an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica. Hint: Express the sines and cosines in exponential form using the results in PD.13.
PD.15. Find the coefficients in the Fourier series Eq. (D.18) for the function $f(x)$ defined by $f(x) = x$ in $-\pi < x < \pi$. Sketch a graph of the function to which the series converges in $-5\pi < x < 5\pi$. To what value does the series converge at $x = \pi$? Do not hesitate to use the capabilities of an available symbolic manipulating program like MAXIMA, MAPLE, or Mathematica.

PD.16. Suppose that $f(x)$ is an even function of $x$ in the interval $-L < x < L$, i.e. suppose that $f(-x) = f(x)$ in that interval. Show that $b_n = 0$ for all $n$ and hence that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad ; \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

Comment: If $f(x)$ is an odd function $[f(-x) = -f(x)]$ in the interval $-L < x < L$, then $a_n = 0$ for all $n$ and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad ; \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

More usefully, if a series representing $f(x)$ only in the half-interval $0 < x < L$ is needed, $f(x)$ can be extended into the interval $-L < x < 0$ so as to be either even or odd. Thus, when only the half-interval is of interest, any function can be expanded either in a Fourier cosine series by the first equation in this problem or in a Fourier sine series by the second equation in this problem.

D.3 Fourier Transforms

To represent functions that are not periodic, we might imagine letting the interval $-L < x < L$ of Section D.2 expand to encompass all $x$; i.e., we let $L \to \infty$. To obtain expressions valid in that limit, we combine Eqs. (D.24) and (D.25) into the single expression

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} \left[ \int_{-L}^{L} f(x') e^{-i(n\pi x'/L)} \, dx' \right] e^{i(n\pi x/L)} \quad (D.26)$$

where $x'$ has been used for the integration variable to avoid confusion with $x$. We now introduce the notation, $\kappa_n = \pi n/L$, and $\Delta \kappa = \kappa_{n+1} - \kappa_n = \pi/L$. Then, Eq. (D.26) becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left[ \int_{-L}^{L} f(x') e^{-i\kappa_n x'} \, dx' \right] e^{i\kappa n x} \Delta \kappa \quad (D.27)$$

If we think of the quantity in square brackets temporarily simply as some function of $\kappa$, say $g(\kappa)$, evaluated at $\kappa_n$, we can interpret the sum on $n$ in Eq. (D.27) as an approximation to the integral of $g(\kappa)$. Thus, as $L$ approaches infinity and $\Delta \kappa$ simultaneously approaches zero, Eq. (D.27) becomes

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x') e^{-i\kappa x'} \, dx' \right] e^{i\kappa x} \, d\kappa \quad (D.28)$$

which is one form of the Fourier integral theorem. It can be split into two pieces by viewing the integral in square brackets as an integral transform $\tilde{f}(\kappa)$, called the Fourier transform, of $f(x)$,

$$\tilde{f}(\kappa) = \int_{-\infty}^{\infty} f(x') e^{-i\kappa x'} \, dx' \quad (D.29)$$
The rest of Eq. (D.28) is then identified as the Fourier inversion formula,

\[ f(x) = \int_{-\infty}^{\infty} \tilde{f}(\kappa) e^{i\kappa x} \frac{d\kappa}{2\pi} \]  

(D.30)

In effect, Eq. (D.29) defines a transform of some function and Eq. (D.30) permits recovery of the original function if its transform is known. We can also view Eq. (D.30) to express \( f(x) \) as a superposition of elementary functions \( e^{i\kappa x} \); in that view, \( \tilde{f}(\kappa) \) determines how much of each elementary function is present. Note finally that in separating Eq. (D.28) into two pieces, different authors may elect to distribute the factor of \( 2\pi \) differently between Eqs. (D.29) and (D.30); the symmetry between the transform and its inversion is, for example, enhanced if the factor \( 1/\sqrt{2\pi} \) multiplies each equation. We nonetheless elect the forms given above.

Two more points: (1) Fourier transforms and inverses in two or more dimensions also occur. By direct analogy with Eqs. (D.29) and (D.30), we infer, for example, in three dimensions that

\[ \tilde{f}(\kappa) = \int \int \int f(r') e^{-i\kappa \cdot r'} dr' dy' dz' \]  

(D.31)

and then that

\[ f(r) = \int \int \int \tilde{f}(\kappa) e^{i\kappa \cdot r} d\kappa_x d\kappa_y d\kappa_z \frac{(2\pi)^3}{(2\pi)^3} \]  

(D.32)

where \( \kappa = \kappa_x \hat{i} + \kappa_y \hat{j} + \kappa_z \hat{k} \) and \( \kappa \cdot r = \kappa_x x + \kappa_y y + \kappa_z z \). (2) Parseval’s theorem, which relates the function and its transform by

\[ \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(\kappa)|^2 d\kappa \]  

(D.33)

is proved in PD.17.

PROBLEMS

PD.17. Prove Parseval’s theorem, Eq. (D.33). *Hint:* Write \( \int |f(x)|^2 dx \) as \( \int f(x) f^*(x) dx \), substitute the complex conjugate of Eq. (D.30) for \( f^*(x) \), interchange orders of integration, and interpret the resulting inner integral by Eq. (D.29).

PD.18. Let \( f(x) = 1, -a \leq x \leq a \), and \( f(x) = 0 \) outside this range. (a) Find the Fourier transform \( \tilde{f}(\kappa) \) of this function and sketch a graph of \( \tilde{f}(\kappa) \) versus \( \kappa \). (b) Use Parseval’s theorem to show that \( \int_{0}^{\infty} [(\sin^2 \kappa a)/\kappa^2] d\kappa = \pi a/2 \).
Appendix E

Reference Tables

E.1 Selected Physical Constants

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elementary charge $e$</td>
<td>$1.602 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>Electron rest mass $m_e$</td>
<td>$9.109 \times 10^{-31}$ kg</td>
</tr>
<tr>
<td>Proton rest mass $m_p$</td>
<td>$1.672 \times 10^{27}$ kg</td>
</tr>
<tr>
<td>Neutron rest mass $m_n$</td>
<td>$1.675 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Bohr radius $a_0$</td>
<td>$5.292 \times 10^{-11}$ m</td>
</tr>
<tr>
<td>Planck’s constant $\hbar$</td>
<td>$6.626 \times 10^{-34}$ J-sec</td>
</tr>
<tr>
<td>Gravitational constant $G$</td>
<td>$6.670 \times 10^{-11}$ N-m$^2$/kg$^2$</td>
</tr>
<tr>
<td>Avogadro’s number $N_A$</td>
<td>$6.023 \times 10^{23}$ molecules/mole</td>
</tr>
<tr>
<td>Boltzmann’s constant $k$</td>
<td>$1.380 \times 10^{-23}$ J/$^\circ$K</td>
</tr>
<tr>
<td>Speed of light $c$</td>
<td>$2.998 \times 10^8$ m/sec</td>
</tr>
<tr>
<td>Permittivity of free space $\varepsilon_0$</td>
<td>$8.854 \times 10^{-12}$ C$^2$/N-m$^2$</td>
</tr>
<tr>
<td></td>
<td>$8.998 \times 10^9$ N-m$^2$/C$^2$</td>
</tr>
<tr>
<td>Permeability of free space $\mu_0$</td>
<td>$4\pi \times 10^{-7}$ N/A$^2$</td>
</tr>
<tr>
<td></td>
<td>$(1.257 \times 10^{-6}$ N/A$^2$)</td>
</tr>
</tbody>
</table>

---

1 Danish physicist Niels Henrik David Bohr, b. 7 October 1885 in Copenhagen, Denmark; d. 18 November 1962 in Copenhagen, Denmark.
2 German physicist Max Karl Ernst Ludwig Planck, b. 23 April 1858 in Kiel, Duchy of Holstein; d. 4 October 1947 in Göttingen, Lower Saxony, Germany.
3 Italian scientist Amadeo Carlo Avogadro, b. 9 August 1776 in Turin, Piedmont-Sardinia; d. 9 July 1856 in Turin, Piedmont-Sardinia.
E.2 Abbreviations of Unit Names

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit Name</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>ampere</td>
<td>J</td>
</tr>
<tr>
<td>Å</td>
<td>angstrom (10^{-10} m)</td>
<td>K</td>
</tr>
<tr>
<td>abA</td>
<td>abampere</td>
<td>kg</td>
</tr>
<tr>
<td>abC</td>
<td>abcoulomb</td>
<td>m</td>
</tr>
<tr>
<td>C</td>
<td>coulomb</td>
<td>MeV</td>
</tr>
<tr>
<td>°C</td>
<td>degrees Celsius</td>
<td>mm</td>
</tr>
<tr>
<td>cm</td>
<td>centimeter</td>
<td>s</td>
</tr>
<tr>
<td>dyn</td>
<td>dyne</td>
<td>statA</td>
</tr>
<tr>
<td>eV</td>
<td>electron volt</td>
<td>statC</td>
</tr>
<tr>
<td>F</td>
<td>farad</td>
<td>T</td>
</tr>
<tr>
<td>g</td>
<td>gram</td>
<td>V</td>
</tr>
<tr>
<td>G</td>
<td>gauss</td>
<td>W</td>
</tr>
<tr>
<td>GeV</td>
<td>10^9 eV</td>
<td>wb</td>
</tr>
<tr>
<td>H</td>
<td>henry</td>
<td>Ω</td>
</tr>
<tr>
<td>Hz</td>
<td>hertz</td>
<td></td>
</tr>
</tbody>
</table>

E.3 Dot Products between Unit Vectors

\[ \downarrow \cdot \hat{e}_1 \hat{e}_2 \hat{e}_3 \]

\[
\begin{pmatrix}
\hat{e}_1 & 1 & 0 & 0 \\
\hat{e}_2 & 0 & 1 & 0 \\
\hat{e}_3 & 0 & 0 & 1 \\
\end{pmatrix}
\]

E.4 Cross Products between Unit Vectors

\[ \downarrow \times \hat{e}_1 \hat{e}_2 \hat{e}_3 \]

\[
\begin{pmatrix}
\hat{e}_1 & 0 & \hat{e}_3 & -\hat{e}_2 \\
\hat{e}_2 & -\hat{e}_3 & 0 & \hat{e}_1 \\
\hat{e}_3 & \hat{e}_2 & -\hat{e}_1 & 0 \\
\end{pmatrix}
\]
Appendix F

Listings of Lawrence-Generated Computer Procedures

This appendix contains listings of the more significant Lawrence-generated procedures invoked in this book. In any given assemblage, the appendix includes only those subappendices whose procedures use tools that are invoked in the main text. There may be gaps in the numbering of these subappendices.

F.1  IDL functions/procedures

All files defining functions and procedures listed in this section have file type .pro and can be downloaded from the directory $HEADEM/idl.¹

F.1.1  IDL Procedure lugen_grid

pro lugen_grid, x, xrange=xrange, nx=nx, $  
y, yrange=yrange, ny=ny, $  
z, zrange=zrange, nz=nz

;+  
; The procedure lugen_grid generates 2D and 3D arrays of evenly  
; spaced x, y, and z points. Information about the third  
; coordinate is optional and can be omitted, in which case two-  
; dimensional arrays will be produced.  
;  
; In the call to this procedure, x, y, z are arrays in which the  
; coordinates will be stored, xrange, yrange, zrange are two-component  
; vectors giving the range of values on each axis, and nx, ny, nz  
; are the number of subdivisions in each range.  

¹The symbol $HEADEM identifies the head of the directory in which, on your system, files associated with this book are stored. At Lawrence $HEADEM translates to /apps/EandM. Consult your Local Guide to find that translation in your system.
; Set nz appropriately if not supplied
if not keyword_set(nz) then nz = 0

; Guard against entry of floating values
nx=fix(nx) & ny=fix(ny) & nz=fix(nz)

; Set spacing between grid points; create arrays for x and y
dx = ( xrange[1] - xrange[0] )/float(nx)
dy = ( yrange[1] - yrange[0] )/float(ny)
x = fltarr( nx+1, ny+1, nz+1 )
y = fltarr( nx+1, ny+1, nz+1 )

; Set spacing and create array for z (if necessary)
if nz ne 0 then begin
dz = ( zrange[1] - zrange[0] )/float(nz)
z = fltarr( nx+1, ny+1, nz+1 )
endif

; Generate lists of values of x, y, and (if necessary) z
xtmp = xrange[0] + dx*findgen(nx+1)
ytmp = yrange[0] + dy*findgen(ny+1)
if nz ne 0 then ztmp = zrange[0] + dz*findgen(nz+1)

; Write values into arrays for x, y, and (if necessary) z
for i = 0, nx do begin
  for j = 0, ny do begin
    for k = 0, nz do begin
      x[i,j,k] = xtmp[i]
y[i,j,k] = ytmp[j]
      if nz ne 0 then z[i,j,k] = ztmp[k]
    endfor
  endfor
endfor
end

F.1.2 IDL Procedure ludiffeq_23

pro ludiffeq_23, funct, times, yvalues, t0=start, tf=finish, init=init, $
  tol=tol, Params= params, report = report, $
  Listname = listname, Depvar = depvar, maxsteps = maxsteps

;+ 
;NAME:
;        LUDIFFEQ_23
;PURPOSE:
IDL Procedure ludiffeq_23

Solve a system of first order ordinary differential equations

\[ y'_i = f_i(t, y_1(t), \ldots y_n(t)), \quad i = 1, \ldots, n \]
\[ a_i = y_i(t_{\text{start}}), \quad i = 1, \ldots, n \]

using Runge-Kutta method of order 2 and 3. Stepsize is selected automatically and hence is variable.

; CATEGORY:
Mathematical Functions, General

; CALLING SEQUENCE:

ludiffeq_23, funct, times, yvalues, $
\quad \text{init} = \text{init}, \quad t_0 = \text{start}, \quad t_f = \text{finish}, \quad$
\quad \text{tol} = \text{tol}, \quad \text{Params} = \text{params}, \quad \text{report} = \text{report}, \quad$
\quad \text{Listname} = \text{listname}, \quad \text{Depvar} = \text{depvar}. \quad$
\quad \text{maxsteps} = \text{maxsteps}

; INPUTS:

; funct = character string containing the name of the user supplied
\quad \text{function implementing} \quad f = [f_1, \ldots, f_n]. \quad \text{This function}
\quad \text{should be written in IDL and have two arguments, namely}
\quad \text{the scalar valued time argument} \quad t \quad \text{and the vector argument}
\quad [y_1(t), \ldots, y_n(t)]. \quad \text{Additional constant parameters may be}
\quad \text{supplied through the keyword} \quad \text{Params} \quad \text{or---better---by using}
\quad \text{common storage.}

; KEYWORD PARAMETERS:

; init = the vector \([a_1, \ldots, a_n]\) of initial values.

; start = the initial value of \(t\).

; finish = the final value of \(t\).

; tol = the error tolerance. Default is .001.

; params = a keyword to be passed to the function \(f\). Originally,
\quad \text{this keyword was provided to facilitate passing}
\quad \text{parameters from the calling program to the function}
\quad \text{funct when the ODE(s) to be solved contained constant}
\quad \text{parameters that were to be varied from run to run.}
\quad \text{If the IDL function to compute} \quad f \quad \text{does not have a}
\quad \text{keyword} \quad \text{Params}, \quad \text{then} \quad \text{Params} \quad \text{should not be set in the}
\quad \text{call to} \quad \text{ludiffeq_23}. \quad \text{(This keyword is present for}
\quad \text{backward compatibility; use of common storage is now}
\quad \text{the recommended mechanism for passing parameters.)}

; report = a flag, if set, to signal that at each step the time,
\quad \text{the step size, and the values of the dependent variable}
\quad \text{should be written to the screen or to a file specified}
\quad \text{by keyword} \quad \text{Listname}.

; listname = the name of the file to receive any output. Default is
\quad \text{to the screen.}

; depvar = a string array of the names of the dependent variables to
\quad \text{be used in the output.} \quad \text{Depvar}(i) = \text{name of variable} \quad i.
IDL Procedure `ludiffeq_23`

; maxsteps = maximum number of steps into which total interval from
; t0 to tf will be divided. Default is 20000.

; OUTPUT PARAMETERS:
;
; times = the name of the variable in whose elements the times
; at which the solution is generated are to be stored.
; yvalues = the name of the array in whose rows the solution
; generated at each time is to be stored.

; COMMON BLOCKS: None, except when used to pass parameters in the ODEs.
; SIDE EFFECTS: None
; RESTRICTIONS: None

; EXAMPLE:
;
; Solve
; y1' = -.1 * y1,
; y2' = .1*y1 - .05*y2,
; y3' = .05*y2
;
; y1[0] = 1000, y2[0] = 0, y3[0] = 0
;
; on the interval [0, 5].
;
; First, we define the function radio by
;
; function radio, t, y, params = params
; k = params[0] & kp = params[1]
; return, [-k*y[0], k*y[0] - kp*y[1], kp* y[1]]
; end
;
; Next, we call
; `ludiffeq_23`, "radio", init=[1000, 0, 0], t0=0, tf=5., $ 
; times, yvalues, params = [.1, .05], /report
;
; The result may be plotted by
;
; plot, times, yvalues[0,*]
; for i = 1,2 do oplot, times, yvalues[i,*]

; PROGRAM ORIGIN
;
; This program was written in September, 1991, by David M. Cook with
; assistance from CAB, then an employee of Research Systems, Inc. Its
; algorithm is patterned after the algorithm then used in the
; MATLAB routine ode23, which algorithm is used here and distributed
; by permission of The MathWorks, Inc., Natick, MA.

; MODIFICATION HISTORY:
On_Error, 2

; Check parameters
if n_params(0) lt 3 THEN $
   message, "Missing parameters."
if KEYWORD_SET(finish) eq 0 THEN $
   message, "Missing end time."
if KEYWORD_SET(init) eq 0 THEN $
   message, "Missing initial conditions."
if KEYWORD_SET(listname) THEN $
   openw, unit, /get, listname $
else unit = -1
if KEYWORD_SET(tol) eq 0 THEN tol = .001
if KEYWORD_SET(maxsteps) eq 0 THEN maxsteps = 20000
if KEYWORD_SET(Params) eq 0 THEN $
   Paramset = 0 $
else Paramset = 1

if KEYWORD_SET(report) eq 0 THEN report = 0 $
ELSE BEGIN
   SN = size( depvar )
   n = n_elements(init)
   if (SN[1] eq 0) THEN BEGIN
      I = indgen(n)
      Names = ['Var' + StrTrim(I, 2)]
   ENDIF ELSE $
      if SN[1] lt n THEN BEGIN
         I = Indgen(N)
         Names = [DepVar, 'Var' + StrTrim(I(SN[1] : N-1), 2)]
      ENDIF else Names = depvar

   printf, unit, format = '(A13, 2x, A13, 2x, $)', "Times", "Stepsize"
   for i = 0, n-2 do printf, unit, format = '(A13, 2x, $)', Names(i)
   printf, unit, format = '(A13)', Names(n-1)
   Printf, unit, " "
IDL Procedure ludiffeq

; Initialize
h = (finish - start) ; h = stepsize
minh = h/float(maxsteps)
maxh = h/5.
times = start
h = h/100.
t = start
v = init
yvalues = v

if report ne 0 THEN BEGIN
  printf, unit, format = '(G13.6, 2x, G13.6, 2x, $)', t, h
  for i = 0, n-2 do printf, unit, format = '(G13.6, 2x, $)', v[i]
  printf, unit, format = '(G13.6)', v(n-1)
ENDIF

errbound = tol * max([sqrt(total(v^2)), 1])

; compute yvalues for variable step sizes

while t lt finish and h ge minh DO BEGIN
  if t+h gt finish THEN h = finish - t

  if Paramset eq 0 THEN BEGIN
    k1 = CALL_FUNCTION(funct, t, v)
    k2 = CALL_FUNCTION(funct, t+h, v + h * k1)
    k3 = CALL_FUNCTION(funct, t + h/2, v + h*(k1 + k2)/4)
  ENDIF ELSE BEGIN
    k1 = CALL_FUNCTION(funct, t, v, Params = Params)
    k2 = CALL_FUNCTION(funct, t+h, v + h * k1, Params = Params)
    k3 = CALL_FUNCTION(funct, t + h/2, v + h*(k1 + k2)/4, Params = Params)
  ENDELSE

  err = (h*(k1 - 2*k3 + k2)/3)^2
  err = sqrt(total(err))
  err_bound = tol * max([sqrt(total(v^2)), 1])

  if err le err_bound THEN BEGIN
    t = t + h
    v = v + h*(k1 + 4*k3 + k2)/6
    times = [times, t]
yvalues = [[yvalues], [v]]
    if report ne 0 THEN BEGIN
F.1.3  IDL Procedure ludiffeq_45

pro ludiffeq_45, funct, times, yvalues, init=init, t0=start, tf=finish, tol=tol,$
  report = report, Params = params, Listname = listname,$
  Depvar = depvar, maxsteps = maxsteps

;+  ;PURPOSE:
;   Solve system of first order ordinary differential equations
;   yi' = fi(t, y1(t),...yn(y)), i = 1,..., n
;   ai = yi(start),    i = 1,..., n
;   using Runge-Kutta method of order 4 and 5. Stepsize is selected
;   automatically and hence is variable.
;
;CATEGORY:
;   Mathematical Functions, General
;
;CALLING SEQUENCE:
;     ludiffeq_45, funct, times, yvalues, init=init, t0=start, tf=finish, $
;         tol = tol, Params = params, report = report, $
;         Listname = listname, Depvar = depvar, $
;         maxsteps = maxsteps
;
;INPUTS:
;
;    funct    = character string containing the name of the user supplied
;              function implementing f = [f1, ...., fn]. This function
;              should be written in IDL and have two arguments, namely
IDL Procedure ludiffeq.45

; the scalar valued time argument t and the vector argument [y1(t),..., yn(t)]. Additional constant parameters may be supplied through the keyword Params or---better---by using common storage.

; KEYWORD PARAMETERS:

;  init     = the vector [a1, ..., an] of initial values.
;  start    = the initial value of t.
;  finish   = the final value of t.
;  tol      = the error tolerance. Default is 1.e-6.
;  params   = a keyword to be passed to the function f. Originally, this keyword was provided to facilitate passing parameters from the calling program to the function funct when the ODE(s) to be solved contained constant parameters that were to be varied from run to run.
;  If the IDL function to compute f does not have a keyword Params, then Params should not be set in the call to ludiffeq.45. (This keyword is present for backward compatibility; use of common storage is now the recommended mechanism for passing parameters.)
;  report   = a flag, if set, to signal that at each step the time, the step size, and the values of the dependent variable should be written to the screen or to a file specified by keyword Listname.
;  listname = the name of the file to receive any output. Default is to the screen.
;  depvar   = a string array of the names of the dependent variables to be used in the output. Depvar(i) = name of variable i.
;  maxsteps = maximum number of steps into which total interval from t0 to tf will be divided.

; OUTPUT PARAMETERS:

;  times    = the name of the variable in whose elements the times at which the solution is generated are to be stored.
;  yvalues  = the name of the array in whose rows the solution generated at each time is to be stored.

; COMMON BLOCKS: None
; SIDE EFFECTS: None
; RESTRICTIONS: None
; EXAMPLE:

   See example for ludiffeq.23

; PROGRAM ORIGIN

; This program was written in September, 1991, by David M. Cook with
IDL Procedure ludiffeq_45

; assistance from CAB, then an employee of Research Systems, Inc. Its
; algorithm is patterned after the algorithm then used in the
; MATLAB routine ode45, which algorithm is used here and distributed
; by permission of The MathWorks, Inc., Natick, MA.
;
; MODIFICATION HISTORY:
; Mark F. Gehrke, LU '94; July, 1992 - Add keyword Params
; David M. Cook, December, 1998 - Add keyword maxsteps
; David M. Cook, January, 2020 - Shift arrays from a() to a[

; ON_ERROR, 2

; PROCEDURE:
; We use the constants of Fehlberg to obtain Runge-Kutta
; formula. See "Klassische Runge-Kutta Formeln vierter und
; niedrigerer Ordnung mit Schrittweiten-Kontrolle und ihre
; Anwendung auf Wärmeleitungsprobleme", Computing 6, 1970,
; pp. 61 - 71.

On_Error, 2

; Check parameters

if n_params(0) lt 3 THEN $
  message, "Missing parameters."

if KEYWORD_SET(finish) eq 0 THEN $
  message, "Missing end time."

if KEYWORD_SET(init) eq 0 THEN $
  message, "Missing initial conditions."

if KEYWORD_SET(listname) THEN $
  openw, unit, /get, listname$
else unit = -1

if KEYWORD_SET(tol) eq 0 THEN tol = 1.e-6

if KEYWORD_SET(maxsteps) eq 0 THEN maxsteps = 20000

if KEYWORD_SET(Params) eq 0 THEN $
  Paramset = 0$
else Paramset = 1

if KEYWORD_SET(report) eq 0 THEN report = 0 $
ELSE BEGIN
  SN = size( depvar)
  n = n_elements(init)
  if (SN[1] eq 0) THEN BEGIN


I = indgen(n)
Names = ['Var' + StrTrim(I, 2)]
ENDIF ELSE $
if SN[1] lt n THEN BEGIN
I = Indgen(N)
Names = [DepVar, 'Var' + StrTrim(I(SN[1] : N-1),2)]
ENDIF else Names = depvar
printf, unit, format = '(A13, 2x, A13, 2x, $)', "Times", "Stepsize"
for i = 0, n-2 do printf,unit,format = '(A13,2x,$)',Names(i)
printf,unit, format = '(A13)',Names(n-1)
Printf,unit, " 
ENDElse

; Constants of Fehlberg
a = [1/4., 3/8., 12/13., 1., 1/2.]
b = [ [ 1./4, 0, 0, 0, 0], $ [ 3./32., 9/32., 0, 0, 0], $ [1932./2197, -7200/2197., 7296/2197., 0, 0], $ [439/216., -8., 3680/513., -845./4104., 0], $ [-8/27., 2., -3544/2565., 1859./4104, $ -11/40.]]
slope_weight = [16./135., 0, 6656./12825., 28561/56430., -9./50., 2./55.]
err_weight = [-1./360., 0, 128./4275., 2197./75240., -1./50., $ -2./55.]

; Initialize
h = (finish - start)
minh = h/float(maxsteps)
maxh = h/5.
times = start
h = h/100.
t = start
v = init
yvalues = v
if report ne 0 THEN BEGIN
printf,unit, format ='(G13.6, 2x, G13.6, 2x, $)', t, h
for i = 0, n-2 do printf,unit, format ='(G13.6, 2x,$)', v[i]
printf,unit, format = '(G13.6)', v[n-1]
ENDIF
slopes = fltarr(N_ELEMENTS(v), 6)
errbound = tol * max([sqrt(total(v^2)), 1])

; compute yvalues for smaller (as needed) step sizes

while t lt finish and h ge minh DO BEGIN

    if t+h gt finish THEN h = finish - t

    if Paramset eq 0 THEN BEGIN
        slopes(0, 0) = CALL_FUNCTION( funct, t, v)
        for i = 1,5 do $
            slopes(0, i) = CALL_FUNCTION( funct, t + a(i-1) * h, v + h * (slopes(*,0:4) # b(*,i-1)))
        ENDF
    ELSE BEGIN
        slopes(0, 0) = CALL_FUNCTION( funct, t, v, Params = Params)
        for i = 1,5 do $
            slopes[0, i] = CALL_FUNCTION( funct, t + a[i-1] * h, v + h * (slopes[*:0;4] # b[*;i-1]), Params =Params)
        ENDE

    ENDBEGIN

err = ( h * (slopes # err_weight))^2

err = sqrt(total(err))
err_bound = tol * max([sqrt(total(v^2)), 1])

if err le err_bound THEN BEGIN
    t = t + h
    v = v + h*(slopes # slope_weight)
    times = [times,t]
    yvalues = [[yvalues], [v]]
    if report ne 0 THEN BEGIN
        printf,unit, format = '(G13.6, 2x, G13.6, 2x, $)', t, h
        for i = 0, n-2 do printf,unit, format = '(G13.6, 2x,$)', v[i]
        printf,unit,format = '(G13.6)', v[n-1]
    ENDF
    ENDF

if KEYWORD_SET(do_print) ne 0 THEN $
    printf,unit, " Beware of singularity"
if unit ne -1 THEN Free_lun,unit
return
end

F.1.4 IDL Procedures electfield, calcefield, arrowhead, getpointE

PRO electfield

; electfield draws the field of an array of up to 10 point charges.

; Get number of charges and charge on and location of each charge,
; storing each in a named common area for access in various
; subprocedures

common nqxy, n, q, x, y

read, n, prompt='Number of point charges (<=10): '
q = fltarr(n) & x = fltarr(n) & y = fltarr(n)
for i = 0, n-1 do begin
   number = string(i+1, format='(I3)')
   pmt = 'Q, X, Y for charge'+number+': '
   read, q1, x1, y1, prompt=pmt
   q[i]=q1 & x[i]=x1 & y[i]=y1
endfor

; Initialize plotting field and mark charges

plot, [-10.0,10.0], [-10.0,10.0], /isotropic, /nodata, $
   xtitle='x', ytitle='y', charsize=1.5
oplot, [0.0,0.0], [-10.0,10.0]
oplot, [-10.0,10.0], [0.0,0.0]
theta = findgen(12)/11.0
thetarad = 2.0*pi*theta
dxcircle = cos(thetarad)
dycircle = sin(thetarad)
for i = 0, n-1 do begin
   if q[i] lt 0 then begin
      oplot, x[i]+ 0.3*dxcircle, y[i]+0.3*dycircle
   endif else begin
      polyfill, x[i]+0.3*dxcircle, y[i]+0.3*dycircle
   endelse
endfor

; Select type of input; strip spaces,
; convert to uppercase; reject bad entry
source = ''
tst = -1
while tst eq -1 do begin
  read, source, prompt = 'Numeric (N) or graphic (G) input: '
  source = strtrim(strupcase(source), 2)
  source = strupcase(strmid(source,0,1))
  tst =strpos( 'NG', source )
  if tst eq -1 then print, 'Invalid character; try again.'
endwhile

; Prime starting point for trace

type='' getpointE, source, x0, y0, stepsize, type
timetostop=0
if x0 eq 9999 then timetostop=1

; Plot one or more field lines and/or equipotentials

while ~timetostop do begin
  if strmid(type, 0, 1) eq ',' then type = strmid(type, 1)
  ; peel off any leading comma
  type = strtrim(type, 2) ; remove leading and trailing blanks
  type = strupcase(type) ; force upper case

  xt = [x0] & yt = [y0] ; initialize vectors for points on trace
  if type eq 'FIELD' then w=3 else w=2 ; set line width (3 = field, 2 = pot)

  ; Calculate field (components and magnitude) at starting point
  ; Calculate second point on trace
  ; Append to vector of points on trace

  xc = x0 & yc = y0
calcefield, xc,yc, EXc, EYc, Ec
  if type eq 'FIELD' then begin
    xn = xc + stepsize*EXc/Ec & yn = yc + stepsize*EYc/Ec
  endif else begin
    xn = xc - stepsize*EYc/Ec & yn = yc + stepsize*EXc/Ec
  endelse
calcefield, xn, yn, EXn, EYN, En
  EX = (EXc+EXn)/2.0 & EY = (EYc+EYN)/2.0
  E = sqrt(EX^2+EY^2)
  if type eq 'FIELD' then begin
    xc = xc + stepsize*EX/E & yc = yc + stepsize*EY/E
  endif else begin
    xc = xc - stepsize*EY/E & yc = yc + stepsize*EX/E
Calculate and add remaining points on trace, stopping when
trace goes out of bounds (t1, t2), trace returns to starting
point (t3), trace comes close to one of charges (t4), or
field near zero (trace reverses direction; t5)

```idl
xt = [[xt], xc] & yt = [[yt], yc]

if type eq 'FIELD' then begin
  xn = xc + stepsize*EXc/Ec & yn = yc + stepsize*EyC/Ec
endif else begin
  xn = xc - stepsize*EyC/Ec & yn = yc + stepsize*EXc/Ec
endif

if type eq 'FIELD' then begin
  xc = xc + stepsize*EX/E & yc = yc + stepsize*Ey/E
endif else begin
  xc = xc - stepsize*Ey/E & yc = yc + stepsize*EX/E
endif

if (t1||t2||t3||t4||t5) then begin
  done=1
endif else begin
  xt = [[xt], xc] & yt = [[yt], yc]
endif
endwhile

if t3 then begin xt = [[xt], x0] & yt = [[yt], y0] & endif

; plot accumulated trace and add arrow if type is FIELD
```

```idl
oplot, xt, yt, thick=w
```

```
```
IDL Procedures electfield, calcefield, arrowhead, getpointE

endelse
endif
if t5 then $
  oplot, 0.5*(xt[nn[3]-1]+xc)+0.45*dxcircle, $
  0.5*(yt[nn[3]-1]+yc)+0.45*dycircle

; get specification of next trace
getpointE, source, x0, y0, stepsize, type

; terminate loop if terminating flag entered
if x0 eq 9999 then timetostop=1
endwhile

RETURN
END

PRO calcefield, x1,y1, Ex, Ey, E

; Calculates x and y components of field and magnitude at (x1,y1).
; Common storage is used to communicate number of charges, q on each,
; and x, y location of each from electfield, where the values are
; entered.

common nqxy, n, q, x, y
EX = 0.0 & EY = 0.0
for i = 0, n-1 do begin
  xd = x1-x[i] & yd = y1-y[i] & r = sqrt(xd^2+yd^2)
  rd = r^1.5
  EX = EX + q[i]*xd/rd & EY = EY + q[i]*yd/rd
endfor
E = sqrt(EX^2+EY^2)
RETURN
END

PRO arrowhead, x1,x2,y1,y2,d1

; Draws arrowhead with point at x2,y2, base width 2*d1, and
; pointing in the direction of the vector [ x2-x1, y2-y1 ].

Ex1=x2-x1 & Ey1=y2-y1
E1=sqrt(Ex1^2+Ey1^2)
x3 = x1 - d1*Ey1/E1 & y3 = y1 + d1*Ex1/E1
x4 = x1 + d1*Ey1/E1 & y4 = y1 - d1*Ex1/E1
polyfilll, [x3,x2,x4,x3], [y3,y2,y4,y3]
RETURN
END
PRO getpointE, source, x0, y0, stepsize, type

; Seeks input of starting point for desired trace, step
; size, and type of trace to be drawn.

st = ''
if source eq 'N' then begin ; Obtain numeric input
  pmt = 'X0, Y0, STPSZ, TYPE: '
  read, x0, y0, stepsize, type, prompt=pmt
endif else begin ; Obtain graphic input
  tst = -1
  while tst eq -1 do begin
    read, st, prompt = 'Character for stepsize and type: '
    st = strtrim(st,2)
    st = strupcase(strmid(st,0,1))
    tst = strpos( 'QWERTYUIOPZXCVBNMJKL9', st )
    if tst eq -1 then print, 'Invalid character; try again.'
  endwhile
  if st eq '9' then begin
    x0=9999 & y0=0 & stepsize=0 & type=0
  endif else begin
    case st of
      'Q': begin stepsize = 0.50 & type = 'FIELD' & end
      'W': begin stepsize = 0.20 & type = 'FIELD' & end
      'E': begin stepsize = 0.10 & type = 'FIELD' & end
      'R': begin stepsize = 0.05 & type = 'FIELD' & end
      'T': begin stepsize = 0.02 & type = 'FIELD' & end
      'Y': begin stepsize = -0.02 & type = 'FIELD' & end
      'U': begin stepsize = -0.05 & type = 'FIELD' & end
      'I': begin stepsize = -0.10 & type = 'FIELD' & end
      'O': begin stepsize = -0.20 & type = 'FIELD' & end
      'P': begin stepsize = -0.50 & type = 'FIELD' & end
      'Z': begin stepsize = 0.50 & type = 'POT' & end
      'X': begin stepsize = 0.20 & type = 'POT' & end
      'C': begin stepsize = 0.10 & type = 'POT' & end
      'V': begin stepsize = 0.05 & type = 'POT' & end
      'B': begin stepsize = 0.02 & type = 'POT' & end
      'N': begin stepsize = -0.02 & type = 'POT' & end
      'M': begin stepsize = -0.05 & type = 'POT' & end
      'J': begin stepsize = -0.10 & type = 'POT' & end
      'K': begin stepsize = -0.20 & type = 'POT' & end
      'L': begin stepsize = -0.50 & type = 'POT' & end
    endcase
    cursor, x0, y0, /data, /down
  endelse
endelse

RETURN
F.1.5  IDL Procedures magnetfield, calcbfield, arrowhead, getpointB

PRO magnetfield

; magnetfield draws the field of an array of up to 10 infinitely
; long parallel wires carrying steady currents in a plane
; perpendicular to the wires.

; Get number of wires and current on and the location of each wire,
; storing each in a named common area for access in various
; subprocedures

common nixy, n, cur, x, y

read, n, prompt='Number of wires (<=10): '
cur = fltarr(n) & x = fltarr(n) & y = fltarr(n)
for i = 0, n-1 do begin
  number = string(i+1, format='(I3)')
pmt = 'I, X, Y for current'+number+'
read, cur1, x1, y1, prompt=pmt
cur[i]=cur1 & x[i]=x1 & y[i]=y1
endfor

; Initialize plotting field and mark wires

plot, [-10.0,10.0], [-10.0,10.0], /isotropic, /nodata, $
xtitle='x', ytitle='y', charsize=1.5
oplot, [0.0,0.0], [-10.0,10.0]
oplot, [-10.0,10.0], [0.0,0.0]
theta = findgen(12)/11.0
thetarad = 2.0*pi*theta
dxcircle = cos(thetarad)
dycircle = sin(thetarad)
for i = 0, n-1 do begin
  if cur[i] lt 0 then begin
    oplot, [x(i)-0.3,x(i)+0.3], [y(i)+0.3,y(i)-0.3]
    oplot, [x(i)+0.3,x(i)-0.3], [y(i)+0.3,y(i)-0.3]
  endif else begin
    polyfill, x[i]+0.3*dxcircle, y[i]+0.3*dycircle
  endelse
endfor

; Select type of input; strip spaces, convert to uppercase
source = ''
tst = -1
while tst eq -1 do begin
    read, source, prompt = 'Numeric (N) or graphic (G) input: '
    source = strtrim(strupcase(source), 2)
    source = strupcase(strmid(source,0,1))
    tst = strpos( 'NG', source )
    if tst eq -1 then print, 'Invalid character; try again.'
endwhile

; Prime starting point for trace

timetostop=0
getpointB, source, x0, y0, stepsize
if x0 eq 9999 then timetostop=1

; Plot one or more field lines and/or equipotentials

while ~timetostop do begin
    xt = [x0] & yt = [y0] ; initialize vectors for points on trace
    ; Calculate field (components and magnitude) at starting point
    ; Calculate second point on trace
    ; Append to vector of points on trace
    xc = x0 & yc = y0
    calcbfield, xc,yc, BXc, BYc, Bc
    xn = xc + stepsize*BXc/Bc & yn = yc + stepsize*BYc/Bc
    calcbfield, xn, yn, BXn, BYn, Bn
    BX = (BXc+BXn)/2.0 & BY = (BYc+BYn)/2.0
    B = sqrt(BX^2+BY^2)
    xc = xc + stepsize*BX/B & yc = yc + stepsize*BY/B
    xt = [ [xt], xc ] & yt = [ [yt], yc ]

    ; Calculate and add remaining points on trace, stopping when
    ; trace goes out of bounds (t1, t2), trace returns to starting
    ; point (t3), trace comes close to one of wires (t4), or
    ; field near zero (trace reverses direction; t5)
    done = 0
    while ~done do begin
        calcbfield, xc,yc, BXc, BYc, Bc
        xn = xc + stepsize*BXc/Bc & yn = yc + stepsize*BYc/Bc
        calcbfield, xn, yn, BXn, BYn, Bn
        BX = (BXc+BXn)/2.0 & BY = (BYc+BYn)/2.0
        B = sqrt(BX^2+BY^2)
        xc = xc + stepsize*BX/B & yc = yc + stepsize*BY/B
        t1 = abs(xc) gt 10.0 & t2 = abs(yc) gt 10.0
t3 = sqrt((xc-x0)^2 + (yc-y0)^2) lt 0.25*stepsize

t4 = 0
for i = 0, n-1 do begin
    if sqrt((xc-x[i])^2 + (yc-y[i])^2) lt stepsize then t4 = 1
endfor

t5 = EXC*EXn+BYc*BYn lt 0.0

if (t1||t2||t3||t4||t5) then begin
    done=1
endif else begin
    xt = [ [xt], xc ] & yt = [ [yt], yc ]
endelse

if t3 then begin xt = [ [xt], x0 ] & yt = [ [yt], y0 ] & endif

endwhile

; plot accumulated trace and add arrow

oplot, xt, yt, thick=2

nn = size(xt) & nnn=nn[3] & nnn = fix(nnn/2)

d= abs(fix(0.3/stepsize))

if stepsize gt 0 then arrowhead, $:
    xt[nnn-d], xt[nnn+d], yt[nnn-d], yt[nnn+d], 0.25 $
else arrowhead, $:
    xt[nnn+d], xt[nnn-d], yt[nnn+d], yt[nnn-d], 0.25 $
if t5 then $
    oplot, 0.5*(xt[nn[3]-1]+xc)+0.3*dxcircle, $
    0.5*(yt[nn[3]-1]+yc)+0.3*ycircle $

; get specification of next trace
getpointB, source, x0, y0, stepsize

; terminate loop if terminating flag entered
if x0 eq 9999 then timetostop=1

endwhile

RETURN
END

PRO calcbfield, x1,y1, Bx, By, B

; Calculates x and y components of field and magnitude at (x1,y1).
; Common storage is used to communicate number of currents, the current
; on each, and the x, y location of each wire.

common nixy, n, cur, x, y

Bx = 0.0 & By = 0.0
for i = 0, n-1 do begin
```
xd = x1-x[i] & yd = y1-y[i] & rd = xd^2+yd^2
Bx = Bx - cur[i]*yd/rd & By = By + cur[i]*xd/rd
endfor
B = sqrt(Bx^2+By^2)
RETURN
END

PRO arrowhead, x1,x2,y1,y2,d1
; Draws arrowhead with point at x2,y2, base width 2*d1, and
; pointing in the direction of the vector [ x2-x1, y2-y1 ].
Ex1=x2-x1 & Ey1=y2-y1
E1=sqrt(Ex1^2+Ey1^2)
x3 = x1 - d1*Ey1/E1 & y3 = y1 + d1*Ex1/E1
x4 = x1 + d1*Ey1/E1 & y4 = y1 - d1*Ex1/E1
polyfill, [x3,x2,x4,x3], [y3,y2,y4,y3]
RETURN
END

PRO getpointB, source, x0, y0, stepsize
; Seeks input of starting point for desired trace and step size.
if source eq 'N' then begin ; Obtain numeric input
  pmf = 'X0, Y0, STPSZ: '
  read, x0, y0, stepsize, prompt=pmf
endif else begin ; Obtain graphic input
  st = ''
tst = -1
while tst eq -1 do begin
  read, st, prompt = 'Character for stepsize: '
  st = strtrim(st,2)
  st = strupcase(strmid(st,0,1))
  tst = strpos( 'ZXCVBNMJKL9', st )
  if tst eq -1 then print, 'Invalid character; try again.'
  endwhile
if st eq '9' then begin
  x0=9999 & y0=0 & stepsize=0
endif else begin
  case st of
    'Z': stepsize = 0.50
    'X': stepsize = 0.20
    'C': stepsize = 0.10
    'V': stepsize = 0.05
    'B': stepsize = 0.02
    'N': stepsize = -0.02
```
IDL Procedures magnetloop, calcbloop, arrowhead, getpointB, looprad, loopax

F.1.6 IDL Procedures magnetloop, calcbloop, arrowhead, getpointB, looprad, loopax

PRO magnetloop

; magnetloop draws the field of an array of up to 10 circular current loops (current I, radius r) whose planes are parallel to the xy plane, and whose centers lie in the yz plane. Field lines are drawn in the yz plane.

; Get number of loops, the current in each loop, and the location of the center of and the radius of each loop, storing each in a named common area for access in various subprocedures.

common niyz, n, cur, rad, x, y

read, n, prompt='Number of loops (<=10): '
cur = fltarr(n) & x = fltarr(n) & y = fltarr(n) & rad = fltarr(n)
for i = 0, n-1 do begin
    number = string(i+1, format='(I3)')
pmt = 'I, X, Y, R for loop'+number+': '
read, cur1, x1, y1, rad1, prompt=pmt
cur[i]=cur1 & x[i]=x1 & y[i]=y1 & rad[i] = rad1
endfor

; Initialize plotting field and mark loops

plot, [-10.0,10.0], [-10.0,10.0], /isotropic, /nodata, $
xtitle='x', ytitle='y', charszie=1.5
oplot, [0.0,0.0], [-10.0,10.0] oplot, [-10.0,10.0], [0.0,0.0] theta = findgen(12)/11.0 thetarad = 2.0*pi*theta

'\textsuperscript{M}': \text{stepsize} = -0.05
'\textsuperscript{J}': \text{stepsize} = -0.10
'\textsuperscript{K}': \text{stepsize} = -0.20
'\textsuperscript{L}': \text{stepsize} = -0.50
dxcircle = cos(thetarad)
dycircle = sin(thetarad)
for i = 0, n-1 do begin
    if cur[i] lt 0 then begin
        polyfill, x[i]+rad[i]+0.2*dxcircle, y[i]+0.2*dycircle
        oplot, [x[i]-rad[i]-0.2, x[i]-rad[i]+0.2], [y[i]+0.2, y[i]-0.2]
        oplot, [x[i]-rad[i]+0.2, x[i]-rad[i]-0.2], [y[i]+0.2, y[i]-0.2]
        ; oplot, x[i]-rad[i]+0.2*dxcircle, y[i]+0.2*dycircle
    endif else begin
        oplot, [x[i]+rad[i]-0.2, x[i]+rad[i]+0.2], [y[i]+0.2, y[i]-0.2]
        oplot, [x[i]+rad[i]+0.2, x[i]+rad[i]-0.2], [y[i]+0.2, y[i]-0.2]
        ; oplot, x[i]+rad[i]+0.2*dxcircle, y[i]+0.2*dycircle
        polyfill, x[i]-rad[i]+0.2*dxcircle, y[i]+0.2*dycircle
    endelse
endfor

; Select type of input; strip spaces, convert to uppercase
source = ''
tst = -1
while tst eq -1 do begin
    read, source, prompt = 'Numeric (N) or graphic (G) input:'
    source = strtrim(strupcase(source), 2)
    source = strupcase(strmid(source,0,1))
    tst = strpos( 'NG', source )
    if tst eq -1 then print, 'Invalid character; try again.'
endwhile

; Prime starting point for trace

type=''
getpointB, source, x0, y0, stepsize

; Plot one or more field lines and/or equipotentials

timetostop=0
while ~timetostop do begin
    xt = [x0] & yt = [y0] ; initialize vectors for points on trace
    ; Calculate field (components and magnitude) at starting point
    ; Calculate second point on trace
    ; Append to vector of points on trace

    xc = x0 & yc = y0
    calcbloop, xc, yc, BXc, BYc, Bc
    xn = xc + stepsize*BXc/Bc & yn = yc + stepsize*BYc/Bc
    calcbloop, xn, yn, BXn, BYn, Bn
    BX = (BXc+BXn)/2.0 & BY = (BYc+BYn)/2.0
IDL Procedures magnetloop, calcbloop, arrowhead, getpointB, looprad, loopax

\[
B = \sqrt{BX^2 + BY^2}
\]
\[
xc = xc + \text{stepsize} \times BX/B \quad \text{and} \quad yc = yc + \text{stepsize} \times BY/B
\]
\[
xt = [ [xt], xc ] \quad \text{and} \quad yt = [ [yt], yc ]
\]

; Calculate and add remaining points on trace, stopping when
; trace goes out of bounds (t1, t2), trace returns to starting
; point (t3), or field near zero (trace reverses direction; t5)

done = 0
while ~done do begin
  calcbloop, xc, yc, BXc, BYc, Bc
  \[
  xn = xc + \text{stepsize} \times BXc/Bc \quad \text{and} \quad yn = yc + \text{stepsize} \times BYc/Bc
  \]
  calcbloop, xn, yn, BXn, BYn, Bn
  \[
  BX = (BXc + BXn)/2.0 \quad \text{and} \quad BY = (BYc + BYn)/2.0
  \]
  \[
  B = \sqrt{BX^2 + BY^2}
  \]
  xc = xc + \text{stepsize} \times BXc/Bc \quad \text{and} \quad yc = yc + \text{stepsize} \times BYc/Bc
  \[
  t1 = \text{abs}(xc) > 10.0 \quad \text{and} \quad t2 = \text{abs}(yc) > 10.0
  \]
  \[
  t3 = \text{sqrt}((xc-x0)^2 + (yc-y0)^2) < 0.25 \times \text{stepsize}
  \]
  \[
  t5 = BXc \times BXn + BYc \times BYn < 0.0
  \]
  if (t1 || t2 || t3 || t5) then begin
    done = 1
  endif else begin
    xt = [ [xt], xc ] \quad \text{and} \quad yt = [ [yt], yc ]
  endelse
  if t3 then begin
    xt = [ [xt], x0 ] \quad \text{and} \quad yt = [ [yt], y0 ]
  endif
endwhile

; plot accumulated trace and add arrow

oplot, xt, yt, thick=2
\[
nn = \text{size}(xt) \quad \text{and} \quad nnn = \text{nn}[3] \quad \text{and} \quad nnn = \text{fix}(nn/2)
\]
\[
d = \text{abs}(\text{fix}(0.3/\text{stepsize}))
\]
if \(\text{stepsize} > 0\) then
  arrowhead, $
  \quad \text{xt}[\text{nnn}-d], \text{xt}[\text{nnn}+d], \text{yt}[\text{nnn}-d], \text{yt}[\text{nnn}+d], 0.25$
else
  arrowhead, $
  \quad \text{xt}[\text{nnn}+d], \text{xt}[\text{nnn}-d], \text{yt}[\text{nnn}+d], \text{yt}[\text{nnn}-d], 0.25$
if t5 then
  oplot, \(0.5 \times (\text{xt}[\text{nn}[3]-1]+xc)+0.3\times\text{dxcircle}, 0.5\times(\text{yt}[\text{nn}[3]-1]+yc)+0.3\times\text{dycircle}

; get specification of next trace
getpointB, source, x0, y0, stepsize

; terminate loop if terminating flag entered
if x0 eq 9999 then timetostop = 1

endwhile
IDL Procedures

magnetloop, calcbloop, arrowhead, getpointB, looprad, loopax

RETURN
END

PRO calcbloop, x1, y1, BX, BY, B

; Calculates x and y components of field and magnitude at (x1, y1).
; Common storage is used to communicate number of loops, the current
; in and radius of each, and the x, y location of each. Common
; storage is also used to communicate the location of the
; observation point relative to the center of the loop.

common niyz, n, cur, rad, x, y
common param, xd, yd
BX = 0.0 & BY = 0.0
for i = 0, n-1 do begin
  xd = (x1-x[i])/rad[i] & yd = (y1-y[i])/rad[i]
  BX = BX + cur[i]*qromb( 'looprad', 0.0, !Pi )/rad[i]
  BY = BY + cur[i]*qromb( 'loopax', 0.0, !Pi )/rad[i]
endfor
B = sqrt(BX^2+BY^2)
RETURN
END

PRO arrowhead, x1, x2, y1, y2, d1

; Draws arrowhead with point at x2, y2, base width 2*d1, and
; pointing in the direction of the vector [ x2-x1, y2-y1 ].

Ex1 = x2-x1 & Ey1 = y2-y1
E1 = sqrt(Ex1^2+Ey1^2)
x3 = x1 - d1*Ey1/E1 & y3 = y1 + d1*Ex1/E1
x4 = x1 + d1*Ey1/E1 & y4 = y1 - d1*Ex1/E1
polyfill, [x3, x2, x4, x3], [y3, y2, y4, y3]
RETURN
END

PRO getpointB, source, x0, y0, stepsize

; Seeks input of starting point for desired trace and step
; size.

st = ''
if source eq 'N' then begin
  pmt = 'X0, Y0, STPSZ: '
  read, x0, y0, stepsize, prompt=pmt
endif else begin
  tst = -1
endif
while tst eq -1 do begin
    read, st, prompt = 'Character for stepsize: '
    st = strtrim(st,2)
    st = strupcase(strmid(st,0,1))
    tst = strpos('ZXCVBNMJKL9', st)
    if tst eq -1 then print, 'Invalid character; try again.'
endwhile
if st eq '9' then begin
    x0=9999 & y0=0 & stepsize=0
endif else begin
    case st of
        'Z': stepsize = 0.50
        'X': stepsize = 0.20
        'C': stepsize = 0.10
        'V': stepsize = 0.05
        'B': stepsize = 0.02
        'N': stepsize = -0.02
        'M': stepsize = -0.05
        'J': stepsize = -0.10
        'K': stepsize = -0.20
        'L': stepsize = -0.50
    endcase
    cursor, x0, y0, /data, /down
    endelse
endelse

RETURN
END

function looprad, phi
;
; LOOPRAD defines the integrand for computing the radial
; component of the magnetic field produced at a point in
; the RZ plane by a circular current loop.
;
common param, xd, yd
tmp1 = cos(phi)
tmp2 = xd^2 + yd^2 - 2.0*xd*tmp1 + 1.0
tmp3 = yd*tmp1/tmp2^1.5
return, tmp3/!Pi
end

function loopax, phi
;
; LOOPAX defines the integrand for computing the axial
; component of the magnetic field produced at a point in
; the RZ plane by a circular current loop.
;
IDL Procedure 

lisjus

common param, xd, yd
tmp1 = cos(phi)
tmp2 = xd^2 + yd^2 - 2.0*xd*tmp1 + 1.0
tmp3 = ( 1.0 - xd*tmp1 )/tmp2^-1.5
return, tmp3/!Pi
end

F.1.7 IDL Procedure lisjus

; lisjus.pro

; lisjus.pro draws the patterns that result when two sine waves
; are superimposed at right angles. The relative frequencies and
; phases of the two waves are specified by the user.

; ***** INITIALIZE *****

read, col, row, prompt='# columns, # rows: '
!p.multi=[0,fix(col),fix(row)]

; ***** ENTER PARAMETERS *****

read, Xf, Yf, PP, N, prompt= 'f_x, f_y, Phase, # cycles: '
while (fix(Xf) ne 9999) do begin
    P=!Pi*PP/180.0 &
    R = float(Yf)/float(Xf) &
    ; ***** PLOT GRAPH *****
    Q = 200*N &
    XX=fltarr(Q+1) & YY=fltarr(Q+1) &
    step=0.01*!Pi &
    for i=0,fix(Q) do begin
        T=i*step & XX[i]=SIN(T) & YY[i]=SIN(R*T+P) &
    endfor &
    lab = 'f_x ='+string(fix(Xf), FORMAT='(I2)')+ $
        , f_y ='+string(fix(Yf), FORMAT='(I2)')+ $
        , Phase ='+ string(fix(PP), FORMAT='(I4)')+''U0!N' &
    plot, [-1.2,1.2], [-1.2,1.2], /nodata, title=lab, charsize=2 &
    oplot, XX, YY, thick=4 &
    read, Xf, Yf, PP, N, prompt= 'f_x, f_y, Phase, # cycles: '
endwhile
F.1.8 IDL Procedure antena

; antena.pro

; antena.pro calculates and displays intensity as a function of
; angle for an array of point sources in a plane.

; ***** IDENTIFY SOURCES *****

read, n, lambda, prompt='Number of sources, wavelength: ’
K = 2.0*!Pi/lambda ; Calculate wave number
A = fltarr(n) ; Amplitudes of sources
X = fltarr(n) ; X coordinate of sources
Y = fltarr(n) ; Y coordinate of sources
P = fltarr(n) ; Phase of sources in degrees
C = !Pi/180.0
for i = 0, n-1 do begin$
    number = string(i+1, format='(I3)') &$
    pmt = 'X, Y, AMP, Phase for source'+number+': ' &$
    read, XX, YY, AA, PP, prompt=pmt &$
end

; ***** SCALE AND MARK SOURCES *****

plot, [-10.0,10.0], [-10.0,10.0], /nodata, /isotropic
D1 = 0.1 ; Set size of dot
D = [ [1.0, .000],$
     [0.5,0.866],[-0.5,0.866],[-1.0,0.000],[-0.5,-0.866],$
     [0.5,-0.866],[1.0, 0.000],[-1.0, 0.000],[-0.5, 0.866],$
     [0.5,-0.866], [-0.5,-0.866],[0.5,0.866] ]
D = D1*D
for i = 0, n-1 do begin$
    X2=X[i]/K & Y2=Y[i]/K &$
    oplot, X2+D[0,*], Y2+D[1,*] &$
end

; ***** INPUT CONTROLS, INITIALIZE *****

read, A1, A2, n8, prompt='Start(degrees), Stop(degrees), # steps: ’
C9=COS(A3) & S9=SIN(A3) & C8=COS(A1) & S8=SIN(A1)

; ***** CALCULATE AND PLOT GRAPH *****
xxx = fltarr(n8) & yyy = fltarr(n8)
for i=0, n8-1 do begin $
\quad U1=0 & U2=0 &$
\quad for R=0, n-1 do begin $
\quad \quad P9=P[R]-X[R]*C8-Y[R]*S8 &$
\quad \quad U1=U1+A(R)*COS(P9) & U2=U2+A(R)*SIN(P9) &$
\quad \quad end &$
\quad U3=U1*U1+U2*U2 & xxx[i]=U3*C8 & yyy[i]=U3*S8 &$
\quad C7=C8*C9-S8*S9 & S8=S8*C9+C8*S9 & C8=C7 &$
\quad end$

oplot, xxx, yyy
Appendix G

Instructions for use of Lawrence-Generated Computer Procedures

This appendix contains instructions for the use of the programs listed in Appendix F. In any given assemblage, the appendix includes instructions for only those programs that are referred to in the main text. There may be gaps in the numbering of these subappendices.

G.1 Programs for IDL

A PDF file containing the instructions in this section is named EandMProgInst-IDL.pdf and can be downloaded from the directory $HEADM/idl. (The symbol $HEADM identifies the head of the directory in which, on your system, files associated with this book are stored. At Lawrence $HEADM translates to /apps/EandM. Consult your Local Guide to find that translation in your system.)

G.1.1 Instructions for electfield for IDL

G.1.1.1 Function

Using Coulomb’s law to calculate the electric field, electfield traces user-specified field lines and/or equipotentials representing the electric field established in the X-Y plane by up to 10 point charges arbitrarily positioned in that plane.

G.1.1.2 Instructions

1. Copy electfield.pro from $HEADM/idl to a directory in your account, make that directory your default directory, launch IDL, and then execute the statements

   IDL> .compile electfield
   IDL> electfield
Before the second prompt appears, IDL will report successful compilation not only of \texttt{electfield} but also of three internal routines \texttt{arrowhead}, \texttt{calcefield}, and \texttt{getpointE}.

2. The program requests the number of point charges (A single number not exceeding 10 is entered.) and then requests the strength and location of each charge. For each request, three numbers are entered and separated by commas or spaces. If, for example, you wished to explore an electric dipole, you might enter

\begin{verbatim}
IDL> Number of point charges (<=10): 2
IDL> Q,X,Y FOR CHARGE 1: 1, 2, 0
IDL> Q,X,Y FOR CHARGE 2: -1, -2, 0
\end{verbatim}

Charges may be located at any coordinates. The field lines or equipotentials will be plotted only in the region $-10 < x, y < 10$, so you should think a bit about scaling before entering any numbers.

The program erases the screen, then draws the axes and marks the position of each charge lying in the region $-10 < x, y < 10$. Positive charges are marked with a filled circle; negative charges are marked with an open circle.

3. The program asks whether you wish to provide the starting point for each field line or equipotential by typing numbers or by invoking graphic input:

\begin{verbatim}
IDL> Numeric (N) or graphic (G) INPUT: G
\end{verbatim}

The entry \texttt{N} or \texttt{n} will result in a request for numeric entry of the starting point for each trace, the desired step size, and the type of trace to be generated; the entry \texttt{G} or \texttt{g} will generate a request for a code for the step size and type of trace and then invoke graphic input so you can select the starting point by using the cursor. Any other entry will cause the program to print an error message and invite a corrected entry. This selection is made once and cannot be changed without exiting from \texttt{electfield} and starting over.

If you selected numeric input, go to Step 4; if you selected graphic input, go to Step 5.

4. The program requests the following data:

\begin{verbatim}
X0 = X coordinate of starting point for trace ($-10 \leq X0 \leq 10$)
Y0 = Y coordinate of starting point for trace ($-10 \leq Y0 \leq 10$)
STPSZ = step size by which trace is advanced; may be negative
TYPE = specification of field line or equipotential
\end{verbatim}

The first three of these items are numbers; the fourth is either the word \texttt{FIELD} or the word \texttt{POT} (for equipotential).\footnote{The program embodies no error trapping. The entry 'FIELD', all upper case, will draw a field line; any other entry will draw an equipotential.} These items are entered in a single line and separated by commas or spaces, e.g.,

\begin{verbatim}
IDL> X0, Y0, STPSZ, TYPE: 2.2,0.2,0.1,FIELD
\end{verbatim}
Table G.1: Codes for specifying step size and type of trace to `electfield`.

<table>
<thead>
<tr>
<th>Code</th>
<th>STPSZ</th>
<th>TYPE</th>
<th>Code</th>
<th>STPSZ</th>
<th>TYPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>+.50</td>
<td>FIELD</td>
<td>Z</td>
<td>+.50</td>
<td>POT</td>
</tr>
<tr>
<td>W</td>
<td>+.20</td>
<td>FIELD</td>
<td>X</td>
<td>+.20</td>
<td>POT</td>
</tr>
<tr>
<td>E</td>
<td>+.10</td>
<td>FIELD</td>
<td>C</td>
<td>+.10</td>
<td>POT</td>
</tr>
<tr>
<td>R</td>
<td>+.05</td>
<td>FIELD</td>
<td>V</td>
<td>+.05</td>
<td>POT</td>
</tr>
<tr>
<td>T</td>
<td>+.02</td>
<td>FIELD</td>
<td>B</td>
<td>+.02</td>
<td>POT</td>
</tr>
<tr>
<td>Y</td>
<td>−.02</td>
<td>FIELD</td>
<td>N</td>
<td>−.02</td>
<td>POT</td>
</tr>
<tr>
<td>U</td>
<td>−.05</td>
<td>FIELD</td>
<td>M</td>
<td>−.05</td>
<td>POT</td>
</tr>
<tr>
<td>I</td>
<td>−.10</td>
<td>FIELD</td>
<td>J</td>
<td>−.10</td>
<td>POT</td>
</tr>
<tr>
<td>O</td>
<td>−.20</td>
<td>FIELD</td>
<td>K</td>
<td>−.20</td>
<td>POT</td>
</tr>
<tr>
<td>P</td>
<td>−.50</td>
<td>FIELD</td>
<td>L</td>
<td>−.50</td>
<td>POT</td>
</tr>
</tbody>
</table>

Some experimentation may be required with the value of STPSZ, since the accuracy of the results depends particularly on this quantity.

The program plots the specified field line or equipotential curve, stopping when the trace returns to its starting point, when the trace approaches one of the point charges, when the trace goes outside the region defined by $-10 \leq X \leq 10$, $-10 \leq Y \leq 10$, or when the trace passes into a region very close to a zero of the field. If the trace stops because the field is zero, the point at which the trace stops is marked with an open circle slightly larger than the circle used to mark negative charges. On completion of a trace, the program returns to Step 4.

To terminate execution of the program and return to the IDL prompt, enter the values 9999,0,0,0.

5. The program will request input of a character to specify the step size and type of trace as described in Table G.1 and then turn on the cursor, which you should move to the desired starting point for the trace and then click any mouse button. The program plots the specified line, stopping under the same conditions as described in Step 4, and then returns to Step 5. Entry of an invalid character will cause the program to print an error message and invite a corrected entry.

To terminate execution of the program and return to the IDL prompt, enter the character '9'. 
G.1.2 Instructions for magnetfield for IDL

G.1.2.1 Function

Using the basic expression

\[ B(x, y) = \frac{\mu_0 I \hat{\phi}}{2\pi \sqrt{x^2 + y^2}} = \frac{\mu_0 I (-\sin \phi \hat{i} + \cos \phi \hat{j})}{2\pi \sqrt{x^2 + y^2}} \] (G.1)

for the magnetic field of a long, straight wire perpendicular to the \( xy \) plane and carrying current \( I \) out of the page at the point \( (x, y) = (0, 0) \), magnetfield traces user-specified field lines representing the magnetic induction field established in the \( xy \) plane (\( x \) left on page, \( y \) up on page, \( z \) out of page) by up to 10 such wires perpendicular to but otherwise arbitrarily positioned in that plane.

G.1.2.2 Instructions

1. Copy magnetfield.pro from \$THEDEM/idl to a directory in your account, make that directory your default directory, launch IDL, and then execute the statements

   IDL> .compile magnetfield
   IDL> magnetfield

   Before the second prompt appears, IDL will report successful compilation of magnetfield and of three internal routines arrowhead, calcbfield, and getpointB.

2. The program requests the number of wires (A single number not exceeding 10 is entered.) and then requests the strength and location of each current. For each request, three numbers are entered, separated by commas, e.g.,

   IDL> Number of wires (<=10): 2
   IDL> I, X, Y for current 1: -1.0, -2.0, 0.0
   IDL> I, X, Y for current 2: 1.0, 2.0, 0.0

   Wires may be located at any coordinates. The field lines will be plotted only in the region \(-10 < x, y < 10\), so think a bit about scaling before entering any numbers.

   The program erases the screen, then draws the axes and marks the position of each wire. Wires carrying currents out of the page are marked with solid dots; wires carrying currents into the page are marked with the symbol \( \times \).

3. The program asks whether you wish to provide the starting point for each field line by typing numbers or by invoking graphic input:

   IDL> Numeric (N) or graphic (G) input: G

   The entry \( N \) or \( n \) will result in a request for numeric entry of the starting point for each trace and the desired step size; the entry \( G \) or \( g \) will generate a request for a code for the step size and then invoke graphic input so you can select the starting point by using the cursor. Any other entry will cause the program to print an error message and invite a corrected entry. This selection is made once and cannot be changed without exiting from magnetfield and starting over.
Table G.2: Codes for specifying step size for magnetfield.

<table>
<thead>
<tr>
<th>Code</th>
<th>STPSZ</th>
<th>Code</th>
<th>STPSZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>+.50</td>
<td>N</td>
<td>−.02</td>
</tr>
<tr>
<td>X</td>
<td>+.20</td>
<td>M</td>
<td>−.05</td>
</tr>
<tr>
<td>C</td>
<td>+.10</td>
<td>J</td>
<td>−.10</td>
</tr>
<tr>
<td>V</td>
<td>+.05</td>
<td>K</td>
<td>−.20</td>
</tr>
<tr>
<td>B</td>
<td>+.02</td>
<td>L</td>
<td>−.50</td>
</tr>
</tbody>
</table>

If you selected numeric input, go to Step 4; if you selected graphic input, go to Step 5.

4. The program requests the following data:

\[ X0 = \text{X coordinate of starting point for trace } (-10 \leq X0 \leq 10) \]
\[ Y0 = \text{Y coordinate of starting point for trace } (-10 \leq Y0 \leq 10) \]
\[ \text{STPSZ} = \text{step size by which trace is advanced; may be negative} \]

These items are entered in a single line and separated by commas, e.g.,

IDL> X0, Y0, STPSZ: 4.0, 0.0, 0.1

Some experimentation may be required with the value of STPSZ, since the accuracy of the results depends particularly on this quantity.

The program plots the specified field line, stopping when the trace returns to its starting point, when the trace approaches one of the wires, when the trace goes outside the region defined by \(-10 \leq x, y \leq 10\), or when the trace passes into a region very close to a zero of the field. If the trace stops because the field is zero, the point at which the trace stops is marked with an open circle. On completion of a trace, the program returns to Step 4.

To terminate execution of the program and return to the IDL prompt, enter the values 9999,0,0.

5. The program will request input of a character to specify the step size as described in Table G.2 and then turn on the cursor, which you should move to the desired starting point for the trace and then click any mouse button. The program plots the specified line, stopping under the same conditions as described in Step 4, and then returns to step 5. Entry of an invalid character will cause the program to print an error message and invite a corrected entry.

To terminate execution of the program and return to the IDL prompt, enter the character 9.
G.1.3 Instructions for magnetloop for IDL

G.1.3.1 Function

Using the basic expression

\[
B(x,y) = \frac{\mu_0 I a}{2\pi} \int_0^\pi \frac{(y-y_c) \sin \phi' \hat{j} + [a - (x-x_c) \sin \phi'] \hat{j}}{[a^2 + (y-y_c)^2 + (x-x_c)^2]^2 - 2a(x-x_c) \sin \phi'^{3/2}} \, d\phi' \tag{G.2}
\]

for the magnetic field at the point \((x,y)\) of a circular current loop of radius \(a\) with its plane parallel to the \(xz\) plane, its center at \((x_c, y_c)\), and its current \(I\) counterclockwise when viewed from a point above the loop, magnetloop traces user-specified field lines representing the magnetic induction field established in the \(xy\) plane \((x\) left on page, \(y\) up on page, \(z\) out of page) by up to 10 loops.

G.1.3.2 Instructions

1. Copy magnetloop.pro from $HEADEM/idl to a directory in your account, make that directory your default directory, launch IDL, and then execute the statements

   IDL> .compile magnetloop
   IDL> magnetloop

   Before the second prompt appears, IDL will report compilation of magnetloop and of five internal routines arrowhead, calcbloop, getpointB, looprad, and loopax.

2. The program requests the number of loops (A single number not exceeding 10 is entered.) and then requests the current, location, and radius of each loop. For each request, four numbers are entered and separated by commas, e.g.,

   IDL> Number of loops (<=10): 2
   IDL> I, X, Y, R for loop 1: 1.0, 3.0, 0.0, 1.0
   IDL> I, X, Y, R for loop 2: 1.0, -3.0, 0.0, 1.0

   Loops may be located at any coordinates. The field lines will be plotted only in the region \(-10 < x, y < 10\), so think a bit about scaling before entering any numbers.

   The program erases the screen, then draws the axes and marks the position of each loop, using a solid dot at the point where the current comes out of the page and the symbol \(\times\) where the current goes into the page.

3. The program asks whether you wish to provide the starting point for each field line by typing numbers or by invoking graphic input:

   IDL> Numeric (N) or graphic (G) input: G

   The entry N or n will result in a request for numeric entry of the starting point for each trace and the desired step size; the entry G or g will generate a request for a code for the step size and type of trace and then invoke graphic input so you can select the starting point by using the cursor. Any other entry will cause the program to print an error message and invite a corrected entry. This selection is made once and cannot be changed without exiting from magnetloop and starting over.
Table G.3: Codes for specifying step size for magnetloop.

<table>
<thead>
<tr>
<th>Code</th>
<th>STPSZ</th>
<th>Code</th>
<th>STPSZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>+.50</td>
<td>N</td>
<td>−.02</td>
</tr>
<tr>
<td>X</td>
<td>+.20</td>
<td>M</td>
<td>−.05</td>
</tr>
<tr>
<td>C</td>
<td>+.10</td>
<td>J</td>
<td>−.10</td>
</tr>
<tr>
<td>V</td>
<td>+.05</td>
<td>K</td>
<td>−.20</td>
</tr>
<tr>
<td>B</td>
<td>+.02</td>
<td>L</td>
<td>−.50</td>
</tr>
</tbody>
</table>

If you selected numeric input, go to Step 4; if you selected graphic input, go to Step 5.

4. The program requests the following data:

- \( X_0 \) = X coordinate of starting point for trace \((-10 \leq X_0 \leq 10)\)
- \( Y_0 \) = Y coordinate of starting point for trace \((-10 \leq Y_0 \leq 10)\)
- \( \text{STPSZ} \) = step size by which trace is advanced; may be negative

These items are entered in a single line and separated by commas, e.g.,

IDL> \( X_0, Y_0, \text{STPSZ}: 0.0, -10.0, 0.1 \)

Some experimentation may be required with the value of \( \text{STPSZ} \), since the accuracy of the results depends particularly on this quantity.

The program plots the specified field line, stopping when the trace returns to its starting point, when the trace goes outside the region defined by \(-10 \leq x,y \leq 10\), or when the trace passes into a region very close to a zero of the field. If the trace stops because the field is zero, the point at which the trace stops is marked with an open circle. On completion of a trace, the program returns to Step 4.

To terminate execution of the program and return to the IDL prompt, enter the values \( 9999,0,0,0 \).

5. The program will request input of a character to specify the step size and type of trace as described in Table G.3 and then turn on the cursor, which you should move to the desired starting point for the trace and then click any mouse button. The program plots the specified line, stopping under the same conditions as described in Step 4, and then returns to Step 5. Entry of an invalid character will cause the program to print an error message and invite a corrected entry.

To terminate execution of the program and return to the IDL prompt, enter the character 9.
G.1.4 Instructions for lisjus for IDL

G.1.4.1 Function

LISJUS draws the patterns that result when two sine waves are superimposed at right angles, i.e., the program draws the patterns resulting from horizontal \((x)\) and vertical \((y)\) motions given by

\[
x(t) = \sin(f_x \omega t) \quad y(t) = \sin(f_y \omega t + \phi)
\]  \(G.3\)

where the frequency ratio \(f_x, f_y,\) and the phase \(\phi\) are specified by the user.

G.1.4.2 Instructions

1. Copy \texttt{lisjus.pro} from $\text{HEADEM/idl}$ to a directory in your account, make that directory your default directory, launch IDL, and then execute the statement

   \texttt{IDL> \textasciitilde \text{lisjus}}

2. The program requests the following data:

   \begin{itemize}
   \item \texttt{# columns} The number of columns of graphs to be displayed in the plot window
   \item \texttt{# rows} The number of rows of graphs to be displayed in the plot window
   \end{itemize}

   These items are entered in a single line and separated by commas

   \texttt{# columns, # rows: 3, 2}

   The illustrated values will result in the plot window being divided into six segments in an array that is three columns wide and to rows high.

3. The program requests the following data:

   \begin{itemize}
   \item \texttt{f_x} frequency of the signal on the \textit{x} axis
   \item \texttt{f_y} frequency of the signal on the \textit{y} axis
   \item Phase (in degrees) phase \(\phi\) in Eq. (G.3)
   \item \texttt{# cycles} the number of cycles in the \textit{x} oscillation for which the pattern is to be traced
   \end{itemize}

   These items are entered in a single line and separated by commas:

   \texttt{f_x, f_y, phase, # cycles: 2, 3, 0, 2}

   Only the ratio of the two frequencies is important. The illustrated values 2, 3 result in a pattern in which the signal on the \textit{x} axis completes two cycles in the time in which the signal on the \textit{y} axis completes three cycles. The phase is entered in degrees. To draw a complete pattern, \texttt{# cycles} should be set to the number of cycles required by the \textit{x} signal to complete the entire pattern.
4. The program draws the requested pattern in the upper left region of the plot window and returns to step 3 for another set of controlling data, which will result in the new pattern being drawn in the next available region of the plot window. Once a pattern has been drawn in each of the available regions (here six), drawing an additional pattern will result in erasing the plot window and starting again in the upper left region. Any number of figures can be drawn. To terminate execution, enter the values 9999,0,0,0 in response to the prompt at step 3.
G.1.5 Instructions for antena for IDL

G.1.5.1 Function

ANTENA plots polar graphs of intensity as a function of angle for an array of up to ten point sources in a plane. The position, amplitude, phase, and (common) wavelength for each source are specified by the user.

G.1.5.2 Instructions

1. Copy antena.pro from $HEADEM/idl to a directory in your account, make that directory your default directory, launch IDL, and then execute the statement

   IDL> @antena

2. The program requests the number of sources and the wavelength of their output. Two numbers are entered in a single line and separated by a comma:

   Number of sources, wavelength: 2, 0.5

   The number of sources cannot exceed 10.

3. For each source, the program requests the following data:

   \begin{itemize}
   \item $X$ = X coordinate of source location in X-Y plane
   \item $Y$ = Y coordinate of source location in X-Y plane
   \item AMP = amplitude of source
   \item Phase for source = phase of source (in degrees)
   \end{itemize}

   These items are entered in a single line and separated by commas:

   \begin{itemize}
   \item X, Y, AMP, Phase for source 1: 0, 1, 1, 0
   \item X, Y, AMP, Phase for source 2: 0, -1, 1, 0
   \end{itemize}

   All $X$ and $Y$ coordinates must be scaled so that $-10 \leq X \leq 10$ and $-10 \leq Y \leq 10$, and the amplitudes must be scaled so no single amplitude exceeds 1. The program will mark the location of each source as soon as the last source has been entered. \textit{Note: The wavelength and the X and Y coordinates locating the sources must be supplied in the same units. For the sample input above, the essential feature is that the separation of the sources (two units along the Y axis) is four times the wavelength (.5 units).}

4. The program requests the following data:

   \begin{itemize}
   \item Start (degrees) = the desired starting angle
   \item Stop (degrees) = the desired stopping angle
   \item # steps = the number of steps to be made in incrementing from Start to Stop
   \end{itemize}

   These items are entered in a single line and separated by commas:

   \begin{itemize}
   \item Start (degrees), Stop (degrees), # steps: 0, 360, 360
   \end{itemize}

5. The program plots the specified graph and returns control to the IDL prompt.
Appendix H

Answers to Selected Problems

H.0 Chapter 0

P0.2: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is the volume of the parallelepiped bounded by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

P0.6: See Tables E.3 and E.4 on page 454.

P0.8: $\hat{\mathbf{A}} = \frac{1}{7}(3 \hat{\mathbf{e}}_1 + 2 \hat{\mathbf{e}}_2 - 6 \hat{\mathbf{e}}_3)$; $\mathbf{A} \cdot \mathbf{B} = -3$; $\mathbf{A} \times \mathbf{B} = 2 \hat{\mathbf{e}}_1 - 9 \hat{\mathbf{e}}_2 - 2 \hat{\mathbf{e}}_3$; 107.6°.

P0.11: $r_2 - r_1$

P0.13: $S(r) = x^2 + y^2 + x^2 = r^2$

P0.14: $S(r, r') = \frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}$

$Q(r, r') = \frac{(x-x') \hat{i} + (y-y') \hat{j} + (z-z') \hat{k}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$

P0.16: Spheres centered at $r_0$.

P0.20: (a) 1; (b) $\frac{1}{2}$; (c) $\frac{13}{12}$; (d) 1

P0.23: $U(r) = \frac{a}{[x^2 + y^2 + z^2]^{3/2}} = \frac{a}{r}$

P0.30: $U(r) = U(r_0) - \int_{r_0}^{r} f(r') dr'$

H.1 Chapter 1

P1.6: (c) $1.60210 \times 10^{-19}$ C = $4.80298 \times 10^{-10}$ statC = $1.60210 \times 10^{-20}$ abC

P1.7: $k_3$ (modified Gaussian) = $1/c$

P1.9: $1.24 \times 10^{-36}$

P1.10: (c) 55.69°; (d) $2.95 \times 10^{-7}$ C
P1.11: Maximum fraction of electrons transferred \( \approx 7 \times 10^{-14} \)

P1.12: \( s = \frac{\mu_0 I^2 \ell}{2\pi mg \sin \theta} \)

H.2 Chapter 2

P2.1: (a) \( \approx 10^{14} \) years; (b) 10 light years

P2.3: \( \rho = \frac{Q}{4\pi R^3/3}; \sigma = \frac{Q}{4\pi R^2}; \lambda = \frac{Q}{2\pi R} \)

P2.5: \( 6.25 \times 10^{17} \) electrons/s

P2.6: (a) \( \omega Q/2\pi \)

P2.8: \( J(r) = \frac{qN}{4\pi r^2} r \)

P2.9: \( J(r) = \frac{3\omega Q r}{4\pi a^2} \Phi \)

P2.10: \( J \cdot S \)

P2.11: (a) \( 4a^2 b; \) (b) 0

P2.20: \( Q(t) = \frac{1}{2\pi} \alpha a^4 t (1 - e^{-\beta b}) \)

P2.24: Cookian units: \( \nabla \cdot J + \frac{1}{c} \sqrt{\frac{K_1}{K_2}} \frac{\partial \rho}{\partial t}; \) modified Gaussian units: \( \nabla \cdot J + \frac{1}{c} \frac{\partial \rho}{\partial t} \)

P2.25: \( J = \frac{h}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*) \)

P2.26: (a) \( A/m; \) (c) \( \frac{\partial}{\partial t} \int \sigma dS = -\oint j \cdot dl \times \hat{n} \)

H.3 Chapter 3

P3.3: \( y = \frac{qEa}{mv^2} \left( b - \frac{1}{2} a \right) \)

P3.6: \( v_0 = E/B \)

P3.10: (c) static equilibrium at \( \theta = 0 \) (stable) and \( \theta = \pi \) (unstable).

P3.12: \( \mathbf{p} = 3qa_0 \hat{k} \)

P3.13: static equilibrium at \( \theta = 0 \) (stable) and \( \theta = \pi \) (unstable).

P3.14: \( \mathbf{m} = \frac{1}{2} QR^2 \omega \)
H.4 Chapter 4

P4.1: 1720 N/C directed away from the vacant point along the diagonal through that point.

P4.3: \( \mathbf{F} = \frac{Qp}{4\pi\varepsilon_0} (-2\cos\alpha \mathbf{\hat{r}} + \sin\alpha \mathbf{\hat{\theta}}) \)

P4.7: \( \mathbf{E}(0,0,b) = \frac{Q}{4\pi\varepsilon} \left( \frac{b}{r^2 + a^2} \right)^{3/2} \)

P4.8: (a) \( \mathbf{E}(0,y,z) = \frac{Q/\pi a^2}{4\pi\varepsilon} \int_0^{2\pi} \int_0^a \mathbf{r}' d\mathbf{r}' \mathbf{d}\phi' \left[ \mathbf{\hat{r}} - \mathbf{\hat{\phi}}' \right] \left[ (y - y') \sin \phi' \right] + \mathbf{\hat{z}} \left[ z - z' \right] \)

P4.12: \( \mathbf{E} = \frac{\rho b}{2\varepsilon_0} \frac{x}{b} \mathbf{\hat{r}}, \quad \tau < b; \quad \cdots = -\frac{\rho b}{2\varepsilon_0} \frac{x}{b} \mathbf{\hat{r}}, \quad \tau > b \)

P4.21: \( Zq^2/\pi\varepsilon_0 m v^2 \)

P4.22: \( 2^{2/3}V = 1.587V \)

P4.23: \( 6 \times 10^5 \) volts

P4.25: \( V(r,\theta) = \frac{qa^2}{4\pi\varepsilon_0 r^3} (3\cos^2 \theta - 1); \quad \mathbf{E}(r,\theta) = \frac{3qa^2}{4\pi\varepsilon_0 r^4} \left\{ (3\cos^2 \theta - 1) \mathbf{\hat{r}} + 2\cos \theta \sin \theta \mathbf{\hat{\theta}} \right\} \)

P4.26: (a) \( V(\mathbf{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^a \frac{\mathbf{r}' d\mathbf{r}' d\phi'}{((x - x') \cos \phi' + (y - y') \sin \phi')^2 + z^2} \)

where \( \sigma \) is the charge density on the disc.

(b) \( V(0,0,z) = \frac{\sigma}{2\varepsilon_0} \left( \sqrt{a^2 + z^2} - |z| \right) \)

P4.27: (a) \( V(\tau,z) = \frac{\lambda}{4\pi\varepsilon_0} \left( \frac{d\tau'}{\sqrt{(\tau'^2 + (z - z')^2)^{1/2}} \right) \)

(b) \( V(\tau,z) = \frac{\lambda}{4\pi\varepsilon_0} \ln \left( \frac{z + a}{z - a} \right); \quad z > a \)

P4.28: \( \mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2} (1 + \alpha r)e^{-\alpha r} \mathbf{\hat{r}} \)

\( \rho = -\frac{Qa^2}{4\pi r} e^{-\alpha r} + \left( \text{point charge of strength } Q \text{ at origin} \right) \)

P4.29: \( V(z) = V_0 z/d; \quad \mathbf{E} = -(V_0/d) \mathbf{\hat{k}}; \quad \sigma(z = 0) = -\sigma(z = d) = -V_0\varepsilon_0/d \)
P4.30: Spheres centered on the charge.

P4.32: $6.7 \text{ V/m}$

P4.37: $W = -\sqrt{2}q^2/4\pi\varepsilon_0 a$

P4.40: (a) $W = Q^2/8\pi\varepsilon_0 a$; (b) $a = Q^2/4\pi\varepsilon_0 mc^2 = 1.41 \times 10^{-13} \text{ m}$

P4.42: $(Q_{ij}) = \begin{pmatrix} 2qa^2 & 0 & 0 \\ 0 & 2qa^2 & 0 \\ 0 & 0 & -4qa^2 \end{pmatrix}$

To find $V(r, \theta)$ and $E(r, \theta)$, replace $q$ by $-q$ in the expressions given as the answers to P4.25.

P4.44: (a) $Q = 2\pi a \lambda_0; \quad p = \frac{1}{2} Q a \hat{\jmath}$; $(Q_{ij}) = \begin{pmatrix} \frac{1}{2} Q a^2 & 0 & 0 \\ 0 & \frac{1}{2} Q a^2 & 0 \\ 0 & 0 & -Q a^2 \end{pmatrix}$

(b) $V(r, \theta) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{Qa}{8\pi\varepsilon_0 r^2} \sin \theta \cos \theta + \frac{Qa^2}{16\pi\varepsilon_0 r^3} (1 - 3 \cos^2 \theta) + \cdots$

P4.46: (a) $Q = 2\pi \int_0^\pi \int_0^\infty \rho(r, \theta) r^2 \sin \theta \, dr \, d\theta$

(b) $p_z = 2\pi \int_0^\pi \int_0^\infty \rho(r, \theta) r^3 \cos \theta \sin \theta \, dr \, d\theta$

(c) $Q_{33} = 2\pi \int_0^\pi \int_0^\infty \rho(r, \theta) r^4 (3 \cos^2 \theta - 1) \sin \theta \, dr \, d\theta$

(d) $V(r, \theta) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{p_z \cos \theta}{4\pi\varepsilon_0 r^2} + \frac{Q_{33}}{8\pi\varepsilon_0 r^3} \left\{ \frac{1}{2} (3 \cos^2 \theta - 1) \right\} + \cdots$

(e) $Q = -q; \quad p = 0; \quad Q_{33} = 12qa_0^2$

P4.49: (a) $\mathbf{E}(r, r') = \frac{1}{4\pi\varepsilon_0} \frac{r-r'}{|r-r'|}$

(c) $\mathbf{E}_{\text{dipole}}(r) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{3}{|r-r_0|^5} \mathbf{p} \cdot (r-r_0) (r-r_0) - \frac{\mathbf{p}}{|r-r_0|^4} \right]$

P4.50: $\frac{q}{4\pi\varepsilon_0 r^2} \left[ 1 + \frac{2}{a_0} \frac{r}{a_0} + \frac{2}{a_0^2} \frac{r^2}{a_0^2} \right] e^{-2r/a_0} \hat{\mathbf{r}}$

P4.51: (a) $\mathbf{E}(r) = \frac{Q}{4\pi\varepsilon_0 a^2} \frac{r}{a} \hat{\mathbf{r}}, \quad r < a; \quad \mathbf{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}, \quad r > a$

(c) $V(r) = \frac{Q}{4\pi\varepsilon_0 a} \left[ \frac{3}{2} - \frac{1}{2} \frac{r^2}{a^2} \right], \quad r < a; \quad V(r) = \frac{Q}{4\pi\varepsilon_0 r}, \quad r > a$

P4.56: Force $= \frac{1}{2} \epsilon_0 A \left( \frac{\Delta V}{d} \right)^2$
H.5 Chapter 5

P5.3: attractive force of magnitude \( \frac{B + z}{B + 0} = \frac{\mu_0 I I' h}{2\pi} \left( \frac{1}{a} - \frac{1}{b} \right) \) when \( I \) and \( I' \) are both positive.

P5.4: (b) \( \frac{53/2}{16} \begin{bmatrix} \frac{1}{1 + \left( \frac{z}{a} - \frac{1}{2} \right)^2}^{3/2} + \frac{1}{1 + \left( \frac{z}{a} + \frac{1}{2} \right)^2}^{3/2} \end{bmatrix} \)

P5.7: current = \( \frac{q_0 v}{2\pi a} \); magnetic dipole moment = \( \frac{1}{2} q_0 v a \); \( B \)-field at center = \( \mu_0 q_0 v / 4\pi a^2 \approx 1.2 \times 10^5 \) G at nucleus of hydrogen atom.

P5.10: \( \Phi_m = \frac{\mu_0 I H}{2\pi} \ln \frac{b}{a} \)

P5.13: \( B(r) = \frac{\mu_0 I}{2\pi a} \hat{\phi}, \quad r < a; \quad B(r) = \frac{\mu_0 I}{2\pi r} \hat{\phi}, \quad a < r < b; \quad B(r) = 0, \quad b < r \)

P5.16: \( A(r) = \frac{\mu_0 I}{4\pi} \ln \left| \frac{b - z + \sqrt{r^2 + (b - z)^2}}{a - z + \sqrt{r^2 + (a - z)^2}} \right| \)

P5.19: (b) \( \Lambda = \frac{1}{2} B_0 xy \)

P5.20: \( A = \frac{\mu_0 I}{2\pi} \ln \tau \)

P5.21: \( A = \frac{\mu_0 m I}{2\pi} \hat{\phi}, \quad r < a; \quad A = \frac{\mu_0 m I a^2}{2\pi} \hat{\phi}, \quad r > a \)

P5.25: (b) \( Q^{(m)}_{ij} = \begin{pmatrix} 0 & 0 & Q \hat{j} \\ 0 & 0 & -Q \hat{i} \\ Q \hat{j} & -Q \hat{i} & 0 \end{pmatrix} \) where \( Q = 3\pi a^2 Ib \)

(c) \( A = \frac{\mu_0 Q}{4\pi r^3} \cos \theta \sin \theta \hat{\phi}; \quad B = \frac{\mu_0 Q}{4\pi r^4} [(3 \cos^2 \theta - 1) \hat{r} + 2 \cos \theta \sin \theta \hat{\theta}] \)

P5.28: (b) \( V^{(m)} = m \bullet r / 4\pi r^3; \quad (c) V^{(m)} = -I \phi / 2\pi; \quad (d) V^{(m)} = -\frac{m}{2\pi a^2 (a^2 + z^2)^{1/2}} \frac{z}{\sqrt{a^2 + z^2}} \hat{k} \)

H.6 Chapter 6

P6.3: \( v = mgR/w^2B^2 \), where \( g \) = acceleration of gravity.

P6.4: \( \varepsilon_{\text{mot}} = \frac{1}{2} \omega BR^2 \)

P6.5: (a) \( \varepsilon = \omega BA \sin \omega t \), where \( t = 0 \) when the axis of the loop coincides with the field.
P6.6: (a) clockwise viewed from \( z = \infty \).
(b) counterclockwise viewed from \( z = \infty \).
(c) no induced emf.

P6.7: (a) \( \mathcal{E} = -\mu_0 h \frac{dI}{dt} \ln \frac{b}{a} \), where the positive direction for flux is into the page.

P6.11: \( L = \frac{\mu_0 h}{2\pi} \ln \frac{b}{a} \int \int \frac{1 - \xi \cos \phi'}{[1 - 2\xi \cos \phi' + \xi^2]^{3/2}} \)

P6.12: \( M = \frac{\mu_0 h}{2\pi} \ln \frac{b}{a} \)

P6.26: \( S = -\frac{EI}{2\pi b} \hat{k} \)

P6.29: (a) \( V(r) = \begin{cases} \frac{Q}{4\pi \varepsilon_0 a} \left( \frac{3}{2} \frac{1}{2} \frac{r^2}{a^2} \right) & r < a \\ \frac{Q}{4\pi \varepsilon_0 r} & r > a \end{cases} \)

(b) \( E(r) = \begin{cases} \frac{Q}{4\pi \varepsilon_0 a^3} & r < a \\ \frac{Q}{4\pi \varepsilon_0 r^3} & r > a \end{cases} \)

P6.30: (c) \( E = -(V_0/d) \hat{k} \); (d) \( \sigma(x = 0) = -\varepsilon_0 V_0/d; \sigma(z = d) = \varepsilon_0 V_0/d \)

P6.31: Let \( \alpha = -(V_b - V_a)/\ln(b/a) \). Then

\[
V = V_a + \alpha \ln \frac{r}{a}; \quad E = -\frac{\alpha}{r} \hat{k}
\]

\[
\sigma(r = a) = -\varepsilon_0 \alpha/a; \quad \sigma(r = b) = \varepsilon_0 \alpha/b
\]

P6.32: Let \( \alpha = \frac{ab(V_b - V_a)}{a - b} \) and \( \beta = \frac{aV_a - bV_b}{a - b} \). Then

\[
V = \beta + \frac{\alpha}{r}; \quad E = \frac{\alpha}{r^3} \hat{r}; \quad \sigma(r = a) = \varepsilon_0 \alpha/a^2; \quad \sigma(r = b) = \varepsilon_0 \alpha/b^2
\]

P6.37: (b) \( T_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \left( \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) \delta_{ij} \)

P6.39: (b) \( \frac{d^2V}{dz^2} = -\frac{J}{\varepsilon_0 \sqrt{m/2e}} V^{-1/2} \)

H.7 Chapter 7

P7.4: \( \nu_{\text{visible}} \approx 6 \times 10^{14} \text{ Hz}; \quad \nu_{\text{microwave}} \approx 3 \times 10^9 \text{ Hz.} \)

P7.5: \( \mathbf{B} = \frac{\kappa E_0}{\omega} \left[ \hat{i} \sin(\kappa y - \omega t) - \hat{k} \cos(\kappa y - \omega t) \right] = \frac{\kappa}{\omega} \hat{j} \times \mathbf{E} \)

P7.6: \( \mathbf{B} = \frac{1}{c} f(z - ct) \hat{j} \)

P7.8: \( \langle p_r \rangle = \langle u_{\text{EM}} \rangle \)

P7.9: (a) amplitude \( \approx 1000 \text{ V/m}; \text{ pressure } \approx 4 \times 10^{-6} \text{ N/m}^2 (4 \times 10^{-11} \text{ atmospheres}) \)
(b) \( \approx 3.6 \times 10^{21} \text{ photons/sec m}^2 \)
P7.10: $\langle p_r \rangle = \epsilon_0 E_{x0}^2$

P7.11: (a) $\langle p_r \rangle = 2\langle u_{EM} \rangle \cos^2 \theta$; (b) $\langle p_r \rangle = \frac{1}{3} \left( \text{total energy density in the fields present} \right)$

P7.12: $\tan \alpha = E_{y0}/E_{x0}$, where $\alpha$ is measured clockwise from the positive $x$ axis.

P7.14: With $\theta$ measured counterclockwise from the positive $x$ axis, one axis of the ellipse is at an angle satisfying

$$\tan 2\theta = \frac{2E_{x0}E_{y0} \cos \Phi}{(E_{x0})^2 - (E_{y0})^2}$$

and the other axis is at $90^\circ$ to the first axis.

P7.17: (a) $\Delta = \frac{2\pi}{N}$; (b) secondary maximum $\approx 0.45$ as high as primary maximum.

P7.22: (a) $A(\kappa) = c \int_{-\infty}^{\infty} E_x(0, t) e^{i\kappa t} dt$; (b) $A(\kappa) \frac{2E_0 \sin \kappa ct}{\kappa}$; $u(\kappa) = \frac{4E_0^2 \sin^2 \kappa ct}{\pi \mu_0 c^2 \kappa^2}$

P7.23: (a) $\frac{|A(\omega/c)|^2}{\pi \mu_0 c^3}$; (b) $\frac{2|A(2\pi/\lambda)|^2}{\mu_0 c^2 \lambda^2}$

P7.24: (a) $\mathcal{E} = E_{y0} \hat{j} e^{i(kz - \omega t)}$; $\mathcal{B} = \frac{\kappa E_{y0}}{\omega} \hat{k} e^{i(kx - \omega t)}$

(b) $\mathcal{E} = E_{z0} \hat{k} e^{i(kx \cos \theta + y \sin \theta - \omega t)}$

$$\mathcal{B} = \frac{\kappa E_{z0}}{\omega} (i \sin \theta - j \cos \theta) e^{i(kx \cos \theta + y \sin \theta - \omega t)}$$

(c) $\mathcal{E} = E_0 (\hat{j} - i \hat{k}) e^{i(kx - \omega t)}$; $\mathcal{B} = \frac{\kappa E_0}{\omega} i(\hat{j} - i \hat{k}) e^{i(kx - \omega t)}$

P7.27: $u(r, t) = \frac{q(r - at)}{r} + \frac{Q(r + at)}{r}$, where $q$ and $Q$ are arbitrary functions of the indicated arguments.

P7.29: $A = \frac{E_{x0}}{\omega} i \sin(kz - \omega t + \phi)$; $V = 0$

P7.31: $\pi a^2 \langle u_{EM} \rangle$ is the energy density in the incident wave.

P7.32: (b) $\begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$; (c) rotates angle of polarization toward the $x$ axis by amount $\phi$.

P7.36: $x(t) = \frac{qE_{x0}/m}{(\omega^2 - \omega_0^2) + i(\omega/m)} e^{-i\omega t}$, where $\omega_0^2 = \frac{k}{m}$

H.8 Chapter 8

P8.5: $A_n = \frac{2V_0[1 - (-1)^n]}{n \pi \sinh(n \pi a/b)}$

P8.9: $V(\tau, \phi) = -E_0 \tau \left(1 - \frac{a^2}{\tau^2}\right) \cos \phi$; $\sigma(\phi) = 2\epsilon_0 E_0 \cos \phi$
P8.10: \( V(\tau, \phi) = \frac{V_0}{\pi} \sum_{m=0}^{\infty} \left( \frac{\tau}{a} \right)^{4m+2} \frac{\sin(4m+2)\phi}{2m+1} \) where \( 0 \leq \phi \leq \frac{1}{2} \pi \).

P8.12: \( J_0(x) = \frac{\sin x}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{x}{2} \right)^{2n} \)

P8.13: \( \sigma(z) = \frac{2e_0V_0}{\pi \sqrt{a^2 - z^2}} \); \( Q = 8e_0V_0a \)

P8.14: \( p = 4\pi e_0E_0a^4 \hat{k} \); \( Q = 3\pi e_0E_0a^2 \); \( s = \frac{4}{3}a \)

\[ E = E_0 \left( 1 + \frac{2a^4}{r^3} \right) \cos\theta \hat{r} - E_0 \left( 1 - \frac{a^3}{r^3} \right) \sin\theta \hat{\theta} \]

P8.17: Set \( b_0 = Q/4\pi e_0 \) in Eqs. 8.46 and 8.47.

P8.18: \( V(r, \theta) = \frac{Q}{4\pi e_0 r} \left[ 1 - \frac{a^2}{2r^2} P_2(\cos\theta) + \frac{3a^4}{8r^4} P_4(\cos\theta) - \frac{5a^6}{16r^6} P_6(\cos\theta) + \cdots \right] \)

P8.20: \( \Re\{z^n\} = r^n \cos n\phi \); \( \Im\{z^n\} = r^n \sin n\phi \)

P8.21: \( 4e_0 \) on \( x \) axis; \(-4e_0 \) on \( y \) axis.

P8.24: \( \sigma(y, z) = -\frac{q}{{4\pi e_0}} \frac{d^2 - a^2}{(d^2 + y^2 + z^2)^{3/2}} \); \( Q = -q \); \( F = -\frac{q^2}{4\pi e_0(2d)^2} \hat{i} \)

P8.26: Charges are required at \((a, b, 0), (-a, b, 0), (a, -b, 0)\) and \((-a, -b, 0)\). NEEDS SIGN OF CHARGES.

P8.28: \( \sigma(\theta) = -\frac{q}{{4\pi e_0}} \frac{d^2 - a^2}{(d^2 + a^2 - 2ad \cos \theta)^{3/2}} \)

P8.31: The charge having the smaller magnitude lies inside the sphere. Let \(|q'| < |q|\). Then the radius of the sphere is \( a = s/(|q'/q| - |q'/q|) \) and its center is located on the line joining the two charges and a distance \( b = (|q'/q|) a \) from the charge \( q' \).

P8.38: \( V(x, y) = \frac{1}{4} \left[ V(x + d, y) + V(x - d, y) + V(x, y + d) + V(x, y - d) + \frac{d^2 \rho(x, y)}{\epsilon_0} \right] + O(d^4) \)

P8.39: \( V(x, y, z) = \frac{16V_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sinh \alpha_{mn} \pi z}{mn \sinh \alpha_{mn} \pi c} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \)

where, \( m \) and \( n \) assume only odd values and \( \alpha_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \)

P8.40: \( \left( \begin{array}{c} \text{product} \\ \text{solution} \end{array} \right) = J_n(\alpha r) \left[ A \cos n\phi + B \sin n\phi \right] \left[ C e^{-\alpha x} + D e^{\alpha x} \right] \)

P8.41: \( \left( \begin{array}{c} \text{product} \\ \text{solution} \end{array} \right) = \left\{ A \rho^n + \frac{B}{r^{n+2}} \right\} P^n_m(\cos\theta) \left[ C \cos m\phi + D \sin m\phi \right] \)

P8.43: \( V(x, y) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \kappa}{\kappa} e^{-\kappa y} \cos \kappa x \, d\kappa \)
**H.9  Chapter 9**

**P9.1:** (time)$^{-1}$

**P9.2:** 4.8 m

**P9.5:** $\tau = \epsilon_0/g \approx 2.5 \times 10^{-19}$ s for Al

**P9.6:**
(a) $1.74 \times 10^{-7}$ m/s; (b) $1.17 \times 10^5$ m/s; (c) $\approx 7 \times 10^{-15}$ s; (d) $8.2 \text{ Å} = 0.82 \text{ nm}$; (e) $\mu \approx -1.23 \times 10^{-3}$ mks units, $E$ for drift velocity $\approx 1.4 \times 10^{-4}$ V/m, $E$ for thermal velocity $\approx 9.5 \times 10^7$ V/m

**P9.7:** $\Delta V = V_b - V_a = -wvB$

**P9.8:** $b = q/\mu$; approach to terminal velocity is characterized by a time constant $m/b \approx 6.8 \times 10^{-15}$ s

**H.10  Chapter 10**

**P10.3:**
(a) $\frac{-\hat{n}qd/4\pi\epsilon_0r^3}{4\pi\epsilon_0}$; (c) with $r \approx 10^{10}$ m, $\alpha_0 \approx 10^{-40}$ mks units; (d) $\approx 10^{-34}$ C m; (e) $\approx 6 \times 10^{-16}$ m

**P10.4:** $\approx 6 \times 10^{11}$ V/m

**P10.6:** $1.2 \times 10^{-4}$ C/m$^2$

**P10.9:** $\sigma_b = ba$ on all surfaces; $\rho_b = -3b$

**P10.10:**
(a) $\sigma_b(z = \pm \frac{1}{2}L) = \pm P$, $\sigma_b$(cylindrical side) = 0; $\rho + b = 0$

(b) $V(0, 0, z) = \frac{PR}{2\epsilon_0} \left[ \sqrt{1 + \left( \frac{z}{R} - \frac{L}{2R} \right)^2} - \left| \frac{z}{r} - \frac{L}{2R} \right| \right.$

$\left. - \sqrt{1 + \left( \frac{z}{R} + \frac{L}{2R} \right)^2} - \left| \frac{z}{r} + \frac{L}{2R} \right| \right]$

(c) $V(0, 0, z) \rightarrow \frac{PLR^2}{4\pi\epsilon_0}$

(d) dipole moment = $\pi P L R^2$

**P10.11:**
(a) $\sigma_b(\theta) = P \cos \theta$, $\rho_b = 0$; (b) dipole moment = $\frac{4}{3}\pi a^3 P \hat{k}$

**P10.13:** $F_{qq'} = qq'/4\pi\epsilon r^2$

**P10.16:** $C = \frac{A\epsilon_0}{d} \left| \frac{K_2 - K_1}{\ln(K_2/K_1)} \right|

**P10.17:**
(a) $E = \frac{\Delta V}{d + \left( \frac{1}{K} - 1 \right)t}$; $E' = \frac{1}{K} E$; $D = D' = \epsilon_0 E$
(b) \( \sigma_{\text{plate}} = D = \varepsilon_0 E; \quad \sigma_{\text{dielectric}} = \chi_\varepsilon E' = \left(1 - \frac{1}{K}\right) \sigma_{\text{plate}} \)

(c) \( C = \frac{\varepsilon_0 A/d}{1 + \left( \frac{1}{K} - 1 \right) \frac{r}{d}} \)

P10.18: (a) \( D = \frac{q}{4\pi r^2} \hat{r}; \quad E = \frac{1}{\varepsilon} D; \quad P = \frac{\chi_\varepsilon}{\varepsilon} D \)

(b) \( \rho_\varepsilon = 0; \quad \sigma_\varepsilon(a) = -\frac{\chi_\varepsilon q}{4\pi \varepsilon a^2}; \quad \sigma_\varepsilon(b) = \frac{\chi_\varepsilon q}{4\pi \varepsilon b^2} \)

(c) \( D = \frac{q}{4\pi r^2} \hat{r}; \quad E = \frac{1}{\varepsilon_0} D \)

(d) \( E(b^+) - E(b^-) = \frac{\sigma_\varepsilon(b)}{\varepsilon_0} \hat{r} \)

(e) \( C = 4\pi \varepsilon \frac{ab}{a-b} \)

P10.20: \( \approx 1.3 \times 10^{-10} \) m

P10.21: (b) \( \alpha_\varepsilon = 2.10 \times 10^{-40} \) mks units; (c) \( K_\varepsilon = 1.48 \)

P10.24: \( D = E + 4\pi P, \quad \frac{1}{c^2}E + 4\pi P, \quad E + 4\pi P \) in cgs-esu, cgs-emu, Gaussian units, and Heaviside-Lorentz units, respectively.

H.11 Chapter 11

P11.1: \( B_m \ll 4 \times 10^6 \) G

P11.2: \( B_m \approx 5 \times 10^7 \) G (CHECK THE 5)

P11.3: (c) \( M_B = 0.922 \times 10^{-23} \) J/T; \( \omega_0 \approx 4.1 \times 10^{16} \) s\(^{-1} \), assuming a radius of \( 0.528 \times 10^{-10} \) m; \( B_m \ll 5 \times 10^9 \) G

P11.5: current = \( M/n \)

P11.6: (a) \( J_m = 0; \quad j_m = M \sin \theta \hat{\phi}; \quad (b) \quad m = \frac{4}{3\pi} R^3 M \)

P11.13: \( \sigma_m(z = \pm \frac{1}{\varepsilon} L) = \pm M; \) no other equivalent poles. \( V^{(m)}(0,0,z) \) is obtained by setting \( \varepsilon_0 = 1 \) and replacing \( P \) by \( M \) in the answer to P10.10(b).

P11.14: \( \rho_m = 0; \quad \sigma_m = M \cos \theta; \quad m = \frac{4}{3\pi} R^3 M \)

P11.23: (a) \( \mathbf{H} = NI \hat{k}, \quad \mathbf{B} = \mu NI \hat{k}, \quad \mathbf{M} = \chi_m NI \hat{k} \); (c) \( j_m = \chi_m NI \dot{\phi} \)

P11.25: (b) \( T_c = 0.361 \) °K; (c) \( \gamma = 963 \)

P11.26: (b) 0.17, 1.10, 1.30 T, respectively.

P11.33: \( \mathbf{H} = \varepsilon^2 \mathbf{B} - 4\pi \mathbf{M}, \quad \mathbf{B} = 4\pi \mathbf{M}, \quad \mathbf{B} = 4\pi \mathbf{M}, \) and \( \mathbf{B} - \mathbf{M} \) in cgs-esu, cgs-emu, Gaussian units, and Heaviside-Lorentz units, respectively.
**H.12  Chapter 12**

P12.4:  \[ F_{qq'} = \frac{qq'}{4\pi \varepsilon} \frac{r - r'}{|r - r'|^3} \]

P12.6:  (b) speed = \(1/\sqrt{\varepsilon \mu}\);  \( n = \sqrt{K_m K_e} \)

P12.10:  force = \((\varepsilon - \varepsilon_0)(\Delta V)^2 b / 2d\); pulls slab in.

P12.18:  \( \mu_1 \cot \theta_1 = \mu_2 \cot \theta_2 \)

P12.19:  \( H_{\text{material}} = \frac{NI}{2\pi r + \lambda_m d} \);  \( H_{\text{gap}} = K_m H_{\text{material}} \);  \( B_{\text{gap}} = B_{\text{material}} = \mu H_{\text{material}} \)

P12.20:  \( E_{\text{in}} = \frac{3\varepsilon_0 E_0}{\epsilon + 2\epsilon_0} (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \)

\[ E_{\text{out}} = \left( 1 + \frac{2(\varepsilon - \varepsilon_0) a^3}{\epsilon + 2\epsilon_0} \right) E_0 \cos \theta \hat{r} - \left( 1 - \frac{\varepsilon - \varepsilon_0}{\epsilon + 2\epsilon_0} a^3 \right) E_0 \sin \theta \hat{\theta} \]

\( \rho_b = 0; \quad \sigma_b = \frac{3\varepsilon_0 \chi_e}{\epsilon + 2\epsilon_0} \cos \theta; \quad \) dipole moment = \( 4\pi \varepsilon_0 a^3 \frac{\varepsilon - \varepsilon_0}{\epsilon + 2\epsilon_0} E_0 \)

P12.21:  \( V_{\text{in}}(\tau, \phi) = -\frac{2}{K_e + 1} E_0 \tau \cos \phi \);  \( V_{\text{out}}(\tau, \phi) = -\left( 1 - \frac{K_e - 1 \hat{a}^2}{K_e + 1 \hat{a}^2} \right) E_0 \tau \cos \phi \)

\[ E_{\text{in}} = \frac{2E_0}{K_e + 1} (\cos \phi \hat{\tau} - \sin \phi \hat{\phi}) = \frac{2E_0}{K_e + 1} \hat{i} \]

\[ E_{\text{out}} = \left( 1 + \frac{K_e - 1 \hat{a}^2}{K_e + 1 \hat{a}^2} \right) E_0 \cos \phi \hat{\tau} - \left( 1 - \frac{K_e - 1 \hat{a}^2}{K_e + 1 \hat{a}^2} \right) E_0 \sin \phi \hat{\phi} \]

P12.24:  \( B_{\text{in}} = \frac{3K_m}{K_m + 2} B \)

\[ B_{\text{out}} = B + \frac{\mu_0}{4\pi r^2} [3(\mathbf{m} \cdot \hat{r}) \hat{r} - \mathbf{m}], \quad \text{where} \quad \mathbf{m} = \frac{4\pi a^3 (K_m - 1)}{\mu_0 (K_m + 2)} \mathbf{B} \]

P12.25:  \( H_{\text{in}} = -\frac{1}{3} M; \quad B_{\text{out}} = \mu_0 H_{\text{out}}; \quad B_{\text{in}} = \frac{2}{3} \mu_0 M; \quad H_{\text{out}} = \frac{a^3}{3r^3} [3(\mathbf{M} \cdot \hat{r}) \hat{r} - \mathbf{M}] \]

P12.26:  \( V^{(m)} = \text{constant} + \frac{I}{2} \left[ \frac{1}{2} \left( \frac{a}{r} \right)^2 P_1(\cos \theta) - \frac{3}{8} \left( \frac{a}{r} \right)^4 P_3(\cos \theta) \right. \]

\[ + \left. \frac{5}{16} \left( \frac{a}{r} \right)^6 P_5(\cos \theta) + \ldots \right] \]

P12.29:  \( L = \frac{1}{3} \)

P12.30:  \( d_1 = d_2 = d; \quad q_1 = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} q; \quad q_2 = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} q \)
H.13 Chapter 13

P13.3: (a) $2.07 \times 10^4 \text{ s}^{-1}$; (b) $4.14 \times 10^{16} \text{ s}^{-1}$; (c) $K_e = 1.0000727$ from (a); $K_e = 1.000291$ from (b)

P13.11: (a) $\omega \ll 6.5 \times 10^9 \text{ s}^{-1}$; (b) 0.77 m

P13.20: $T \rightarrow \sqrt{\frac{4}{\mu_0 g \omega}}$

P13.21: $\sigma = 0; \quad j = 2\sqrt{\frac{T}{\mu_t}} \Re\{\varepsilon_{20} e^{-i\omega t}\} \hat{i}$

P13.27: In the coordinate system of Fig. 13.8, $\sigma = 0$ and

$$j = 2 \frac{\kappa \cos \theta_i}{\omega \mu_i} \Re\{\varepsilon_{20} e^{-i(\kappa_i \sin \theta_i + \omega t)}\} \hat{i}$$

P13.28: Let $\kappa' = 2\pi n/\lambda$. Then

$$[T; R] = \frac{4n^2; (n^2 - 1)^2 \sin^2 \kappa'd}{4n^2 + (n^2 - 1)^2 \sin^2 \kappa'd}$$

P13.30: (a) $R_\perp = \left(\frac{n_2^2 - n_1^2}{n_2^2 - n_1^2}\right)^2$; (b) $R_\perp = 0.148, \quad \theta_B = 56.31^0$

P13.32: $\sigma = -\sigma_2 = \Lambda \varepsilon_0 \sin \left(\frac{n\pi y}{b}\right) \cos(\kappa_z z - \omega t); \quad j_{(4)} = -\frac{n\pi A}{\omega \mu_0 b} \sin(\kappa_z z - \omega t) \hat{i}$

$$j_{(4)} = \frac{A}{\omega \mu_0} \left[\kappa_z \sin \left(\frac{n\pi y}{b}\right) \cos(\kappa_z z - \omega t) \hat{k} + \frac{n\pi}{b} \cos \left(\frac{n\pi y}{b}\right) \sin(\kappa_z z - \omega t) \hat{j}\right]$$

P13.34: $\mathbf{E}_{m0}^{TE}(r, t) = A \sin \left(\frac{m\pi x}{a}\right) \hat{j} e^{i(k_z z - \omega t)}$

$$\mathbf{H}_{m0}^{TE}(r, t) = \frac{A}{i\omega \mu_0} \left[-i\kappa_z \sin \left(\frac{m\pi x}{a}\right) \hat{i} + \frac{m\pi}{a} \cos \left(\frac{m\pi x}{a}\right) \hat{k}\right] e^{i(k_z z - \omega t)}$$

P13.35: Let $\kappa_z = m\pi/a$ and $\kappa_y = n\pi/b$. Then

$$\mathbf{E}_{mn}^{TE}(r, t) = C \left[\kappa_y \cos \kappa_x x \sin \kappa_y y \hat{i} - \kappa_x \sin \kappa_x x \cos \kappa_y y \hat{j}\right] e^{i(k_z z - \omega t)}$$

$$\mathbf{H}_{mn}^{TE}(r, t) = \frac{C}{i\omega \mu_0} \left[i\kappa_x \kappa_z \cos \kappa_y y \cos \kappa_x x \hat{i} + i\kappa_z \kappa_y \cos \kappa_x x \sin \kappa_y y \hat{j}\right.$$

$$\left.\left.-\left(\kappa_x^2 + \kappa_y^2\right) \cos \kappa_x x \cos \kappa_y y \hat{k}\right] e^{i(k_z z - \omega t)}$$

where $\kappa$ and $\omega$ satisfy $(\omega/c)^2 = \kappa_x^2 + \kappa_y^2 + \kappa_z^2$.

P13.36: $\mathbf{E}(r, t) = A J_1(\kappa r) \hat{\phi} e^{i(k_z z - \omega t)}$

$$\mathbf{H}(r, t) = \frac{A}{i\omega \mu_0} \left[-i\kappa_z J_1(\kappa r) \hat{r} + \kappa_x J_0(\kappa r) \hat{k}\right] e^{i(k_z z - \omega t)}$$

where $\kappa^2 + \kappa_x^2 = (\omega/c)^2$ and $J_1(\kappa r) = 0$. 
H.14. CHAPTER 14

P13.38: \( \mathcal{H}(r, t) = \mathcal{H}_0(r) e^{-i\omega t} \) where
\[
\mathcal{H}_{0x}(x, y, z) = \frac{\kappa_x E_x - \kappa_z E_y}{i\omega \mu_0} \sin \kappa_x x \cos \kappa_y y \cos \kappa_z z
\]
\[
\mathcal{H}_{0y}(x, y, z) = \frac{\kappa_z E_x - \kappa_x E_y}{i\omega \mu_0} \cos \kappa_x x \sin \kappa_y y \cos \kappa_z z
\]
\[
\mathcal{H}_{0z}(x, y, z) = \frac{\kappa_x E_y - \kappa_y E_x}{i\omega \mu_0} \cos \kappa_x x \cos \kappa_y y \sin \kappa_z z
\]

P13.43: \( \frac{v_p}{c} = \frac{1}{A + B/\lambda^2}; \quad \frac{v_g}{c} = \frac{A - B/\lambda^2}{A + B\lambda^2} \)

P13.47: \( \kappa = \sqrt{\frac{\mu g^2}{2\epsilon} Q \left[ \sqrt{1 + \frac{1}{Q^2}} + 1 \right]} \); \( \alpha = \sqrt{\frac{\mu g^2}{2\epsilon} Q \left[ \frac{1 + \frac{1}{Q^2} - 1}{\sqrt{1 + \frac{1}{Q^2}} + 1} \right]} \);
\[
|K| = \sqrt{\frac{\mu g^2}{\epsilon} Q \left[ \frac{1 + \frac{1}{Q^2}}{\sqrt{1 + \frac{1}{Q^2}} + 1} \right]} \); \( \tan \phi = \frac{\sqrt{1 + \frac{1}{Q^2}} - 1}{\sqrt{1 + \frac{1}{Q^2}} + 1} \); \( \frac{u_E}{u_B} = \left[ 1 + \frac{1}{Q^2} \right]^{-1/2} \)

P13.48: \( \zeta = \frac{\gamma}{\sqrt{2}} \left[ \left( \sqrt{1 + \frac{1}{Q^2} + 1} \right)^{1/2} + i \left( \sqrt{1 + \frac{1}{Q^2} - 1} \right)^{1/2} \right] \)

P13.51: \( \epsilon_{10} = \tan(\theta_i - \theta_t) \hat{\epsilon}_{10}; \quad \epsilon_{0} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_t + \theta_i) \cos(\theta_t - \theta_i)} \hat{\epsilon}_{0} \)

P13.56: \( \omega_{np} = c \sqrt{\frac{\alpha_n^2}{a^2} + \frac{p^2\pi^2}{b^2}}, \) where \( \alpha_n \) is the \( n \)-th root of \( J_1(x) = 0 \).

P13.57: \( \epsilon_{0y} = \frac{i}{\omega \epsilon_0} \left( 1 - \frac{\kappa_x^2 \epsilon_0^2}{\omega^2} \right)^{-1} \left( -\frac{\partial \mathcal{H}_{0x}}{\partial x} + \frac{\kappa_x \partial \epsilon_{0z}}{\omega \mu_0} \right) \)
\[
\mathcal{H}_{0x} = \frac{i}{\omega \mu_0} \left( 1 - \frac{\kappa_x^2 \epsilon_0^2}{\omega^2} \right)^{-1} \left( \frac{\kappa_x \partial \mathcal{H}_{0z}}{\omega \epsilon_0} - \frac{\partial \epsilon_{0z}}{\partial y} \right)
\]
\[
\mathcal{H}_{0y} = \frac{i}{\omega \epsilon_0} \left( 1 - \frac{\kappa_x^2 \epsilon_0^2}{\omega^2} \right)^{-1} \left( \frac{\kappa_x \partial \mathcal{H}_{0z}}{\omega \mu_0} + \frac{\partial \epsilon_{0z}}{\partial x} \right)
\]

P13.58: (a) \( \kappa^2 = \mu_0 \epsilon_0 (\omega^2 - \omega_p^2); \) (e) \( n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \)

H.14 Chapter 14

P14.3: \[
\begin{align*}
\{ \mathbf{E}(r, t) \} &= -\frac{\mu_0 p_0 \omega \kappa}{4\pi} \cos(\kappa r - \omega t) \sin \theta \left\{ \begin{array}{c}
\hat{\phi} \\
\hat{i}
\end{array} \right\} \\
\{ \mathbf{B}(r, t) \} &= \frac{\mu_0 p_0 \omega \kappa}{4\pi} \cos(\kappa r - \omega t) \sin \theta \left\{ \begin{array}{c}
\hat{\phi} \\
\hat{i}
\end{array} \right\}
\end{align*}
\]
P14.7: Let \( \mathbf{m}_0 = \pi a^2 \delta \mathbf{0} \hat{\mathbf{k}} \). Then

\[
\mathcal{A}_0(r) = \frac{i \kappa \mu_0 e^{i \kappa r}}{4 \pi r} \left( 1 - \frac{1}{i \kappa r} \right) (\hat{\mathbf{r}} \times \mathbf{m}_0)
\]

\[
\mathcal{B}_0(r) = \frac{\kappa^2 \mu_0 e^{i \kappa r}}{4 \pi r} \left( [\hat{\mathbf{r}} \times \mathbf{m}_0] \times \hat{\mathbf{r}} + \frac{\mu_0 e^{i \kappa r}}{4 \pi r^3} \right) (1 - i \kappa r) [3(\hat{\mathbf{r}} \cdot \mathbf{m}_0) \hat{\mathbf{r}} - \mathbf{m}_0]
\]

\[
\mathcal{E}_0(r) = \frac{\kappa^2 \mu_0 e^{i \kappa r}}{4 \pi r} \left( 1 - \frac{1}{i \kappa r} \right) (\mathbf{m}_0 \times \hat{\mathbf{r}})
\]

P14.10: (b) \( V = \frac{q}{4 \pi \epsilon_0 a} \left( 1 - \frac{v}{c} \cos \theta \right) \)

P14.15: (b) \(|\mathbf{E}(r, \phi, z, t = 0)| = \frac{q}{4 \pi \epsilon_0 r^2} \left[ \frac{1}{1 - \frac{v^2}{c^2}} - \frac{v^2}{c^2} \sin^2 \theta \right]^{3/2} \)

P14.17: (b) \( \frac{dE}{dt} = -\frac{1}{\tau} E \) where \( \tau = \frac{3 \pi m^3 \epsilon_0 c^3}{q^4 B^2} \)

P14.18: \( \frac{dP(t')}{d\Omega} = \frac{q^2 \epsilon_0^2}{16 \pi^2 \epsilon_0 c^3} \frac{\sin^2 \theta}{(1 - \frac{v}{c} \cos \theta)^5} \)

P14.19: \( \frac{d}{dE/dt} \approx 4 \times 10^{-14} \)

P14.20: (b) \( \approx 5000 \text{ MeV} \)

P14.21: \( \frac{dW}{d\Omega d\omega} = \frac{q^4 \omega^2 \sin^2 \theta}{16 \pi^3 \epsilon_0 c^5} \cos^2 \omega T \frac{\pi^2/T^2}{2} \left[ \frac{\pi^2/T^2}{\omega^2 - (\pi^2/T^2)} \right]^2 \)

P14.22: \( \frac{dW}{d\Omega} = \frac{2q^4 \omega^4 r_0^2}{3 \pi^2 \epsilon_0 c^3} \left[ \frac{\sin^2[(\omega - \omega_0)T]}{((\omega - \omega_0)T)^2} + \frac{\sin^2[(\omega + \omega_0)T]}{((\omega + \omega_0)T)^2} \right] \)

P14.23: (c) \( \approx 1.3 \times 10^{-11} \text{ s} \)

P14.24: (a) \( 6.3 \times 10^{-24} \text{ s} \)

P14.26: (c) \( V(r, t) = \frac{1}{(2\pi)^4 \epsilon_0} \int \tilde{\rho}(\kappa, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d\kappa_x d\kappa_y d\kappa_z d\omega \)

P14.27: Let \( r_0 = q^2 / 4 \pi \epsilon_0 mc^2 \). Then

(a) \( \langle dP(t')/d\Omega \rangle = \frac{1}{2} \epsilon_0 E_0^2 c r_0^2 [1 - \sin^2 \theta \cos^2(\phi - \psi)] \)

(b) \( d\sigma/d\Omega = r_0^2 [1 - \sin^2 \theta \cos^2(\phi - \psi)] \)

(c) \( (d\sigma/d\Omega)_{\text{unpolarized}} = \frac{1}{2} r_0^2 (1 + \cos^2 \theta) \)

(d) \( \sigma_{\text{Thomson}} = 8 \pi r_0^2 / 3 \)
H.15  Chapter 15

P15.3: \[
\left(1 - \frac{v^2}{c^2}\right) \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} + \frac{2v}{c^2} \frac{\partial}{\partial t} \frac{\partial}{\partial x'} - \frac{1}{c^2} \frac{\partial}{\partial t^2} \right] u = 0
\]

P15.7: speed' = c; \quad \tan \theta' = \frac{\sqrt{1 - \beta^2} \sin \theta}{\cos \theta - \beta}

P15.22: \(v = c^2E \times B/E^2\)

P15.24: \[
\begin{bmatrix} V \\ A \end{bmatrix} = \frac{q}{4\pi \epsilon_0} \left[ (x-ct)^2 + (1-\beta^2)(y^2 + z^2) \right]^{1/2} \begin{bmatrix} 1 \\ v \hat{i}/c^2 \end{bmatrix}
\]

P15.30: \[
F'_x = F_x - \frac{v(V_y F_y + V_z F_z)}{c^2 - V_z v}; \quad F'_y = \frac{F_y}{\gamma(1 - V_x v/c^2)}; \quad F'_z = \frac{F_z}{\gamma(1 - V_x v/c^2)}
\]

P15.34: \(L = \begin{pmatrix}
\gamma \gamma_1 (1 + \beta \beta_1) & 0 & 0 & i \gamma \gamma_1 (\beta + \beta_1) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-i \gamma \gamma_1 (\beta + \beta_1) & 0 & 0 & \gamma \gamma_1 (1 + \beta \beta_1)
\end{pmatrix}\)

P15.35: \(L = \begin{pmatrix}
\gamma \beta_x^2 + \beta_y^2 & (\gamma - 1) \beta_x \beta_y & 0 & i \gamma \beta_x^2 \beta_x \\
(\gamma - 1) \beta_x \beta_y & \beta_x^2 + \gamma \beta_y^2 & 0 & i \gamma \beta_x^2 \beta_y \\
0 & 0 & 1 & 0 \\
-i \gamma \beta_x^2 \beta_x & -i \gamma \beta_x^2 \beta_y & 0 & \gamma \beta_x^2
\end{pmatrix}\)

P15.37: \(F = \begin{pmatrix}
0 & -i \frac{E_x}{c} & i \frac{E_y}{c} & B_x \\
i \frac{E_x}{c} & 0 & -i \frac{E_y}{c} & B_y \\
i \frac{E_y}{c} & -i \frac{E_x}{c} & 0 & B_z \\
-B_x & -B_y & -B_z & 0
\end{pmatrix}\)

P15.38: \(E' = \frac{\lambda y}{2\pi \epsilon_0} \begin{pmatrix} \hat{j} + z' \hat{k} \\ y' \end{pmatrix}; \quad B' = \frac{\mu_0 \lambda y v}{2\pi} \begin{pmatrix} z' \hat{j} - y' \hat{k} \\ y'^2 + z'^2 \end{pmatrix}\)

P15.39: \(F_{\mu \nu}^0 = 9(\kappa_\mu \alpha_\nu - \kappa_\nu \alpha_\mu)\)

P15.41: \(x(t) = \frac{mc^2}{qE} \left\{ \sqrt{1 + \left( \frac{qE}{mc} t \right)^2} - 1 \right\}\)

H.16  Appendix A

PA.1: (a) equations inconsistent, e.g., \(2x + 3y = 5\) and \(4x + 6y = 8\)

(b) equations redundant, e.g., \(2x + 3y = 5\) and \(4x + 6y = 10\)

PA.3: \(x_1 = -1, \quad x_2 = 2, \quad x_3 = 5\)

PA.4: \(x_1 = \frac{2}{3} \alpha, \quad x_2 = \alpha, \quad x_3 = \frac{5}{3} \alpha\) for \(\alpha\) arbitrary
PA.9: $x^T y = 7$; $(Bx^T) = (-6, 10, -2)$

\[
yx^T = \begin{pmatrix} -1 & 2 & 5 \\ 1 & -2 & -5 \\ -2 & 4 & 10 \end{pmatrix}; \quad A(A + B) = \begin{pmatrix} 0 & 2 & 2 \\ 8 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix}
\]

PA.13: $A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

PA.15: \(\lambda_1 = -\sqrt{2}, \quad x_1^T = \alpha(1, -\sqrt{2}, 1)\)
\(\lambda_2 = 0, \quad x_2^T = \beta(1, 0, -1)\)
\(\lambda_3 = \sqrt{2}, \quad x_3^T = \gamma(1, \sqrt{2}, 1)\)

where $\alpha$, $\beta$, and $\gamma$ are arbitrary.

H.17 Appendix B

PB.3: \(\frac{1}{2}mv^2 \left[ 1 + \frac{3}{4} v^2 + \frac{5}{8} \frac{v^4}{c^4} + O(\frac{v^6}{c^6}) \right]\)

PB.4: $|f| < 0.1$

PB.5: See Table 8.1.

PB.6: $\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots$

\(e^x = 1 + x + \frac{1}{2!} x^2 + \cdots\)

\((1 + x)^q = 1 + qx + \frac{q(q - 1)}{2!} x^2 + \cdots\)

H.18 Appendix D

PD.2: \(z_1z_2 = 7 + i, \quad \frac{z_2}{z_1} = -\frac{1}{25} (1 + 7i), \quad z_2 = \sqrt{2}e^{-i\pi/4}\)

PD.11: \(\cos 3\theta = 4\cos^3\theta - 3 \cos \theta, \quad \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta\)

PD.12: $\sinh z = \sinh x \cos y + i \cosh x \sin y$

PD.15: $x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

PD.18: $f(\kappa) = 2 \frac{\sin \kappa a}{\kappa}$
Index

The user of this index should be aware that not only textual discussions but also some of the problems are indexed. Problems are identified by the problem number and page, e.g., the notation P10.12(278) refers to problem P10.12 on page 278. The user seeking problems on a given topic should also scan the problems at the end of any section to which a textual reference is given; not all of those problems are indexed. A brief index to frequently used information is located on page 530.

A

abampere (unit of current), 35, 38, 41, 
P1.4(43), P1.7(43)
abbreviation of units, 454
abcoulomb (unit of charge), 38, 41, P1.4(43), 
P1.7(43)
aberration of starlight, 405, P15.7(412), 
P15.39(428)
Abraham, Max, 399
Abraham-Lorentz equation, 399
absolute value
  of complex number, 447
absorption, see attenuation in conductors
action at a distance versus field, 69
adaptive methods for ODEs, 75–76
addition
  of complex numbers, 445
  of matrices, 435
  of velocity, P15.1(406), P15.7(412), 
P15.34(427)
adjoint of a matrix, 436
algebra
  of complex numbers, 445–448
  of matrices, 435–436
  of scalars, 1
  of vectors, 1–3
ampere (unit of current), 38, 41, 43, 
P1.3(42), P1.4(43), P1.5(43)
Ampere’s Circuital Law, see circuital law
Ampere, André Marie, 35
ampere/(volt-meter) (unit of conductivity), 
244
ampere/meter (unit of magnetic field), 298
Amperian current, see current, bound
amplitude of plane wave, 188, 197, 340
analog computer, 77
analog solution
  of Newton’s Second Law, 77
analytic function, 224–226
  importance to Laplace’s equation, 
  225–226
angle
  of incidence, 350, 351
  of reflection, 350, 351
  of refraction, 350
  of transmission, 350, 351
angular distribution of energy, see radiation
  fields
angular frequency, 188
angular momentum and magnetic moment, 
P3.22(83), P11.3(285)
anisotropy, 249, 257, 299, 304, P9.8(251), 
P10.25(277), P12.8(317)
antenna, 199
  instructions for use of
    in IDL, 492
  listing of, 481–482
anticommutative property, 3
antihermitian matrix, 436
antisymmetric matrix, 436
arcfield
  listing of
    in IDL, 96
arcpot
  listing of, 111
INDEX

argument
  of complex number, 447

arrowhead
  listing of
    in IDL, 466–480

associated Legendre equation, P8.41(240)

associative law, 2, 5, 435, 445, 446

asymptotic boundary condition, 212

attenuation, see electromagnetic wave, attenuation

attenuation in conductors, 341–342

Avogadro’s number, 43, 453

Avogadro, Amadeo Carlo, 453

B

bar magnet, see also magnetized rod, 127,
  P11.31(308)

Bessel equation, 218

Bessel functions, 218, P8.12(223),
  P8.40(240), P13.36(367)

Bessel, Friedrich, 218

betatron, P6.10(166)

binomial theorem, 439–440

Biot, Jean-Baptiste, 130

Biot-Savart Law, 128–130, 423–426

Bohr magneton, P11.3(285)

Bohr radius, 453

Bohr, Niels Henrik David, 453

Boltzmann’s constant, 239, 256, 280, 453

Boltzmann, Ludwig, 453

bound charge, 261–262

bound current, see current, bound

bound electric charge, see charge, bound
  electric; charge density, bound

bound magnetic charge, see magnetic
  monopoles

boundary conditions
  asymptotic, 212
  at material discontinuities, 320–325,
    336–339
  at perfect conductor, 338

Dirichlet, 211, 234

homogeneous, 212

internal, 212

mixed, 212

Neumann, 212

on complex fields, 336–339

on displacement field, 321–322

on electric field, 322–323

on electrostatic potential, 211–212,
  324–325

on Laplace’s equation, 211–212

on magnetic induction, 322

  on magnetic intensity, 323

  on Maxwell’s equations, 320–325

  on potential, 323–324

  on surface of conductor, 212

  periodic, 217

  summary, 324

boundary value problem, 212

bremsstrahlung, P14.18(397), P14.19(397),
  P14.21(397)

Brewster’s angle, 357, P13.29(359)

Brewster, David, 357

Busch method for measuring e/m, P3.15(82)

C

calcbfield
  listing of
    in IDL, 471–475

calcbloop
  listing of, 475–480

calcefield
  listing of, 466–471

candela (unit of luminous efficacy), 43

capacitance
  definition and units of, 114
  in presence of dielectric, 271–273,
    P10.16(274), P10.17(274),
    P10.18(274)
  of cylindrical capacitor, P4.35(116)
  of parallel plates, P4.34(116)

capacitor, 114

  and displacement current, P6.16(172)

  cylindrical, P4.35(116), P6.31(181)

  energy stored in, P4.38(119)

  force on dielectric slab in, P12.10(317)

  force on plates of, P4.56(124)

  parallel plate, 271–273, P4.17(106),
    P4.29(115), P4.34(116), P6.30(181),
    P10.16(274), P10.17(274)

  spherical, P4.36(117), P6.32(181),
    P10.18(274)

Cartesian coordinates, 6–8

Cauchy equation, P13.43(373)

Cauchy, Augustin-Louis, 225

Cauchy-Riemann conditions, 225

causality, 379

Cavendish, Henry, 31

cavity resonator(s), 363–366

  cuboid
    fields in, P13.38(367), P13.55(376)
    resonant frequencies of, P13.39(367)

  cylindrical
    fields in, P13.56(376)
    resonant frequencies of, P13.56(376)
INDEX

rectangular parallelepiped
fields in, 363–366
resonant frequencies of, 366
cgs units, see units
charge
bound electric, 255, 261–262, 269, 312
bound magnetic, see magnetic monopoles
definition and units of, 29–30, 32–33, 37–39
from charge density, 49–50
interaction of point, 31–33
Lorentz invariance of, 414
on electron, 453
on proton, 453
charge and current distribution(s), see also specific distribution of interest
description of, 47–57
electric field of, 86–98
electrostatic potential of, 107–110
energy stored in, 117–119, 166–169, P4.41(119), P4.48(122)
force and torque on, 69–81, P3.18(83)
magnetic induction of, 129–138
magnetic vector potential of, 147–150
power input to, 81, 167, 176–177, 315–316
charge carriers, 243
charge density, 47, 78, 86
as component of four-vector, 414–415
bound, 261–262
definition and units of, 47, 49–50
for a point, 50
free, 269, 270, 311, 321–325
from boundary conditions, 322, 324, 338
from electrostatic field, 110
from electrostatic potential, 110
from particle density field, 50
linear, 49, 86
on conductors, 104, 212, 222, 338,
P4.20(115), P6.30(181), P6.31(181), P6.32(181), P6.13(223), P8.9(223), P8.21(227), P8.24(230), P8.28(231), P8.34(234), P12.5(314), P13.21(357), P13.27(358), P13.32(366)
on polarized dielectrics, 261–262
surface, 49, 86, 91–92, 261
from boundary conditions, 338–339
total, 269
volume, 49, 50, 86, 261
charged disc
electric field of, P4.8(99), P4.26(115)
electrostatic potential of, P4.26(115), P8.13(223)
charged line(s)
equipotentials of, P8.32(231)
electric field of, 102–104, P4.6(99), P4.19(114), P8.22(227)
electrostatic potential of, P4.19(114), P4.27(115), P4.47(122), P8.22(227)
Lorentz transformation of field of, P15.38(428)
charged ring
charge density on, P2.3(51)
electric field of, P4.7(99)
electrostatic potential of, P4.44(122), P4.57(124), P8.18(224)
rotating, P2.6(52)
charged rod
electric field of, P4.12(105)
charged sheet, 91–92, P4.13(105), P4.53(124)
electrostatic potential of, P4.52(123), P4.53(124)
charged sphere
charge density on, P2.3(51)
electric field of, P4.9(100), P4.51(123), P6.29(180)
electrostatic potential of, P4.22(115), P4.23(115), P4.51(123), P6.29(180)
energy stored in, P4.40(119)
rotating, 54, P3.14(82)
Child’s Law, 239, P6.39(183)
Child, Clement D., 183
circuital law, 142–145, 158
at material discontinuities, 323
generalized, 172–174, 312–313
in presence of matter, 297, 298, 312–313
circular frequency, 190
circular polarization, see polarization
Clausius, Rudolf Julius Emanuel, 257
Clausius-Mossotti relation
analog for relative permeability, 301–303
for dielectric constant, 257, 274–276
goal coaxial cable, P5.13(146)
 coercive force, 306
coherence, P7.34(208)
collision time, 248, 249
column matrix, 436
column vector, 436
commutative law, 2, 3, 5, 435, 436, 445, 446
complex conjugate, 436, 446, 448
complex field
for monochromatic sources, 331, 381–385
for plane waves, 197–198, 205–206, 339–341, 345
complex numbers, 445

511
absolute value of, 447
algebra of, 445–448
Cartesian form of, 447
graphical representation of, 447
imaginary part of, 224, 445
magnitude of, 447
polar form of, 447
real part of, 224, 445
complex variable, 447–448
and Laplace’s equation, 224–226
function of, 224–226, 447–448
components of vectors, 4–8, 409
Compton scattering, P7.36(209), P14.27(400)
Compton, Arthur Holly, 210
computer algorithm
for mapping fields, 93–95, 136
electfield, 94
magnetfield, 136
magnetloop, 136
conductance, 247
conducting cylinder(s), see also capacitor, cylindrical
image of line charge in, 230
in constant E-field, P8.9(223)
conducting disc
electrostatic potential of, P8.13(223)
conducting plate(s)
electric field of, P4.17(106), P4.29(115), P6.30(181)
electrostatic potential of, P4.29(115), P6.30(181), P8.27(231)
force on point charge near, P8.24(230)
images in, 228–229, P8.24(230)
semi-infinite, P8.23(227)
conducting sphere(s), see also capacitor, spherical
force on point charge near, P8.28(231)
images in, 229–230, P8.28(231), P8.29(231)
in constant E-field, 220, P8.16(224), P8.17(224), P8.30(231)
conducting wedge, P8.10(223), P8.26(231)
conduction, 243–250
anisotropic, 249, P9.8(251)
Drude theory of, 332–333, P9.8(251)
macroscopic description of, 244–247
microscopic description of, 247–250
steady-state, 243–244
conductivity, 249
anisotropic, 249, 251
complex, 333–334, P13.5(336), P13.58(376)
definition and units of, 244
frequency dependence of, 332–333
of plasma, P13.58(376)
tables of values, 245
conductivity tensor, 249
conductor, 104–105, 243, 245, 338, 349
approach to equilibrium in, P9.5(250)
as equipotential region, 114
boundary conditions at, 212, 338–339
cavity in, P4.16(106), P8.29(231)
charge density on, see charge density
current density on, see surface currents
energy loss in, 247, 341–342, P13.5(336), P13.54(375)
properties of E-field near, 104–105, 114
wave propagation in, 341–343
conformal mapping, 226, P8.42(240)
conservation
of charge, 63–65
of energy, see also energy theorem, 16, 107–108, 422
of momentum, 422
conservative force field, 15–18
constants
values of physical, 453
constitutive relation, 244, 311, 332
for conductors, see Ohm’s Law
for dielectrics, 270–271, 314
for magnetic matter, 298, 314
continuity
equation of, see equation of continuity
conversion of units, 39–41
coordinate system(s)
common, 6–8
general orthogonal, P0.37(27), P2.23(67)
operators in various, see operator desired rotation of, P0.34(26), PA.14(437)
coulomb (unit of charge), 38, 41, P1.4(43)
Coulomb gauge, 148
Coulomb’s Law, 425
for magnetic poles, P11.31(308)
in dielectrics, P10.13(273), P12.4(314)
in vacuum, 31–33, 37, 85–86, 100–102, 127, 129, 423–426, P6.24(175)
Coulomb, Charles-Augustin de Coulomb, 31
Cramer’s rules, 432, 433
Cramer, Gabriel, 432
cross product, see also triple vector product,
 triple scalar product, 3, 5, 454,
P0.6(6), P0.8(6), P0.33(26)
in MATHEMATICA, 5
cross section, P14.27(400)
Curie temperature, 304, P11.25(303)
Curie, Pierre, 303
curl, 18, P0.24(21)
4D-analog, 416–417
coordinate-free definition of, 22–24, P0.37(27)
in MATHEMATICA, 18
of curl, 66
of gradient, 66
vector field with zero, 66
current
across an arbitrary surface, 51–52, 54–55
as charge in motion, 36
bound, 255, 286–292, 296, 297, 312
definition and units of, 33–39
energy stored in, 166–169
force on ... in wire, 79, 128–129
from current density, 55, P2.7(53)
intersection of two, 34–36, 128–129
steady, 64
surface, see surface currents
current density, 47, 51–57, 78
and polarization, 312
as component of four-vector, 414–415
boundary conditions on, 323–324
complex, 332
currents from, 55, P2.7(53)
definition and units of, 52–53
free, 298, 311
from magnetic induction, 149
from magnetic vector potential, 149
from particle density and velocity fields,
56–57
surface, see surface currents
total, 297, 312
volume, 52–53
current distributions, see charge and current
distributions
current loop, see magnetic dipole
self-inductance of, P6.11(169)
currentscope, 36
curve fitting
to parabola, PA.6(434)
cutoff frequency, 361, 366
cutoff wavelength, 361
cyclotron radiation, P14.20(397), P14.22(398)
cyclotron, P3.4(73)
cylinder, see conducting cylinder(s);
dielectric cylinder
cylindrical coordinates, 6–8

D
d’Alembert, Jean le Rond, 411
d’Alembertian (∇²), 411
Debye equation, 276
Debye shielding length, 239
Debye, Peter Joseph William, 239
degeneracy, 366
delta function, 50
depolarization factor, P12.29(329)
determinant, 431–433
Laplace development of, 433
properties of, 432–433
second-order, 431
third-order, 432
diagonal matrix, 436
diamagnetic material, see magnetic material,
diamagnetic
diamagnetism, 280–281, 299
dielectric, see also insulator, 245, 253
and capacitance, 271–273
anisotropic, 257
linear, 271
wave propagation in, 339–345
dielectric constant
and index of refraction, 343–344
and polarizability, 275
complex, P13.2(335)
definition, 271
frequency dependence of, 334
frequency-dependent, P13.2(335)
static, 271, 276, 334
table of values, 272
dielectric cube
bound charges on polarized, P10.9(262)
dielectric cylinder, P10.10(262)
electrostatic potential of polarized, P10.10(262)
in uniform E-field, P12.21(328)
dielectric permittivity, 271
frequency dependence of, 332–334
dielectric polarizability, 255–257
and dielectric constant, 275
deformational electric, 256
orientational electric, 256
dielectric polarization, 253, 260, 264–271, 312
definition and units of, 259
general strategy, 253–254
macroscopic description, 259
microscopic description, 255–257
spontaneous, P10.22(276)
dielectric slab (film)
force on ... in capacitor, P12.10(317)
reflection and transmission at, P13.28(358), P13.53(375)
dielectric sphere
bound charges on polarized, P10.11(263)
in uniform E-field, 327–328, P12.22(328)
dielectric susceptibility, 271
dielectric(s)
INDEX

and capacitance, P10.16(274), P10.17(274), P10.18(274)
and method of images, P12.30(329)
anisotropic, P10.25(277), P12.8(317)
differential operators, see specific operator of interest
dipole, see electric dipole; magnetic dipole
dipole approximation, P14.21(397)
dipole moment
  average electric, 256
  average magnetic, 280
electric, 79, 90, 255–257, 260
  induced electric, 256
  magnetic, 80–81, 149, 279–286
  permanent electric, 256
  permanent magnetic, 280
dipole-dipole interaction, P4.54(124), P5.29(153)
Dirac delta function, see delta function
Dirac, Paul Adrien Maurice, 50
directional derivative, 22
Dirichlet boundary conditions, 211, 234
Dirichlet, Peter Gustav Lejeune, 211
disc dynamo, P6.4(164)
dispersion, 367–373
  of triangular pulse, 370–373
dispersion relation, 340
displacement current, 170–172, 312, P6.16(172)
displacement current density, 172
displacement elements, differential, 14
displacement field, 269–273
  boundary conditions on, 321–322, 324, P10.18(274)
  definition and units of, 313, P10.24(277)
distributive law, 2, 3, 5, 435, 436, 446
divergence, 59, P2.23(67)
in 4D, 411
  of curl, 66
  of gradient, see Laplacian
vector field with zero, 66
divergence theorem, 58–61, 119, 141, 143, 151, 261, 444
domain
  magnetic, 303–304
Doppler effect, relativistic, P15.39(428)
Doppler, Christian, 428
dot product, see also triple scalar product, 5, 410, 454, P0.6(6), P0.8(6), P0.33(26)
in MATHEMATICA, 5
drift velocity, 48, 244, 247–249
in crossed E and B, 73

Drude theory of conduction, 332–333, P9.8(251)
Drude, Paul Karl Ludwig, 332
dyne (unit of force), 14

E

eigenvalue and eigenvector, PA.15(437)
Einstein, Albert, 407

electfield, 94, 113
  instructions for use of in IDL, 483–485
  listing of, 466–471
electric charge, see charge
electric current, see current
electric dipole
  electric field of, 89–90
  electrostatic potential of, 108–109, 220
  equipotentials of, 89, P4.31(116)
  far field of, 150
  fields of oscillating, 382–385, P14.3(385)
  force on, P3.10(81), P3.18(83), P4.3(99)
  orientation of . . . in E-field, 255–256,
    P3.10(81), P10.2(257)
  potential energy of, P4.18(114),
    P4.41(119), P4.54(124)
  radiation from oscillating, 381–385,
    P14.3(385)
  torque on, 79, P3.10(81)
  vector potential of oscillating, 382–383
electric dipole moment
  and E-field of dipole, 89–90
  and potential of a dipole, 108, 121
  definition and units of, 79, 121
  induced, 256, P3.12(82), P10.3(258)
  invariance of . . . to coordinate translation, P3.19(83)
  of polarized dielectric, 259
  permanent, 256, P10.2(257)
electric energy density, see energy density, in electric field
electric field
  force on point charge in, 70
  refraction at dielectric interface, 325
  units of, 70
electric field(s), see also capacitor; cavity resonator(s), charge and current distribution(s); charged disc;
  charged line(s); charged ring;
  charged rod; charged sheet; charged sphere; conducting plate(s);
  conducting sphere(s); dielectric cylinder; electric dipole; electric quadrupole; electrostatic field;
INDEX

hydrogen atom; magnetic dipole; monochromatic source(s); plane wave(s); point charge(s); polarized dielectric; quadrupole lens; wave guide

boundary conditions on, 320, 322–323, 336–339, P10.18(274)
definition and units of, 69–70
electrostatic potential of uniform, 220, P4.18(114)
energy stored in, 117–119, 176–177, 315–316
figures showing, 88, 89, 92, 227
from potential(s), 108, 179, 377, 382, 414
from sources, 85–91
in cavity in dielectric, P12.23(329)
near conductor, 104–105, 114, P4.20(115)
relativistic transformation of, 419, P15.31(427)
tracing, P4.58(125)
electric flux, see flux of electric field of distributed charge, 101–102
of point charge, 101
electric force, see also force, 69, 78–81, 85–86
electric force density, 78
electric monopole, see point charge
electric multipole, see also electric dipole; electric monopole; electric quadrupole, electric octupole, 90–91
formed by differentiation, 90–91
electric octupole, 90
electric polarizability, see dielectric polarizability, P10.2(257), P10.3(258)
electric quadrupole
electric field of, P4.25(115), P4.42(122)
electrostatic potential of, 220, P4.25(115), P4.42(122)
formation of, 90
formed by differentiation, 90–91
in uniform E-field, P10.5(259)
electric quadrupole moment tensor, 121
electric susceptibility, see dielectric susceptibility
electromagnetic field, see also cavity resonator(s); charge and current distribution(s); electric dipole; magnetic dipole; monochromatic source(s); plane wave(s); point charge; wave guide, 158
as intermediary between source and detector, 69
definition and units of, 69–72
energy stored in, 176–177, 315–316
energy transported by, 176–177, 315–316
force on
point charge, 70, 71
gauge invariance of, 179, 319
macroscopic, 255, 267
microscopic, 254, 256, 267, 274–276, 301–303
momentum of moving charge, 389–392
power input from, 81
relativistic considerations, 72
electromagnetic field tensor, 416–417, 419
electromagnetic force, 69
electric and magnetic parts of, 70
in Newton’s Second Law, 72–73
on current in wire, 79
on general distributions, 78–81
on line charge, 78
on surface charge, 78
electromagnetic force density, 78
electromagnetic induction, see Faraday Law
electromagnetic spectrum, 192
electromagnetic theory includes optics, 357
electromagnetic torque
on general distributions, 78–81
electromagnetic units, see units, electromagnetic
electromagnetic wave, see also plane wave; spherical wave, 188–192
= light, 191
attenuation, 341–342
complex representation of, 197–198, 331–334, 339–341
group velocity of, 341
in linear matter, 331–373
in non-conductor, 344–345
period of, 190
phase velocity of, 341
plane, 339–341
polarization of, 195–202
propagation vector for, 189, 341–344
speed of propagation, 190–191
speed of propagation of, 189, 342
wavelength of, 190, 342

 electromotance, see electromotive force
electromotive force (emf), 158–160, 305, 312
electron charge, 453
electron rest mass, 453
electroscope, 33
electrostatic field
corresponding to f(z) = 2, 226
exterior to a polarized dielectric, 262
from charge density, 110
from electrostatic potential, 108, 110
from sources, 85–91
in a conductor, 104–105
in presence of dielectrics, 269–273
inside dielectric sphere in constant field, 328
interior to a polarized dielectric, 263–268
just outside a conductor, 104–105
of charged semicircle, 95–98
of electric dipole, 89–90
of electric monopole, 87–88
of electric quadrupole, 90
of infinite line charge, 102–104
of infinite sheet, 91–92
on surface of conductor, 114
perpendicular to equipotentials, 113
electrostatic potential, see also capacitor;
charge and current distribution(s);
charged disc; charged line; charged sphere;
conducting cylinder; conducting disc;
conducting sphere; conducting wedge;
dielectric cylinder; dielectric sphere;
electric dipole; electric quadrupole; Laplace’s equation;
point charge; polarized dielectric;
quadrupole lens; scalar potential;
semi-infinite plate
and conductors, 114
boundary conditions on, 211–212,
324–325
compared with magnetic vector
potential, 150
continuity across discontinuity, 324–325
corresponding to f(z) = z^2, 226
definition and units of, 107–108
exterior to a polarized dielectric,
260–262
from charge density, 110
from electrostatic field, 107–112
from source distribution, 108–110
from sources, see Poisson’s equation
in circle, 217–218
in cylinder, 218
in rectangle, 215–217, P8.5(222)
in sphere, 218–222
in square, 232–234
multipole expansion of, 120–121
of conducting sphere in uniform E,
220–222
of dielectric sphere in constant field,
327–328
of linear quadrupole, 220
of point charge, 220
of point charge and conducting sphere,
229–230
of point charge and plane conductor,
228–229
of point dipole, 220
of semicircular arc, 110–112
of uniform field, 220, P4.18(114)
superposition of, 212
electrostatic potential energy, 108, 117–119
≠ electrostatic potential, 108
electrostatic units, see units, electrostatic
elliptical polarization, see polarization
emf, see electromotive force
electric potential, see also potential energy
conservation of, see conservation of
energy; energy theorem
distribution of . . . in, see plane wave;
radiation fields
radiated from, see radiation from
relativistic, 423, PB.3(440), P15.12(413)
stored in, see capacitor; charge and
current distribution(s); charged
sphere; current; electric field;
electromagnetic field; inductor;
magnetic induction; plane wave;
toroid
transported by, see electromagnetic field;
plane wave(s); wave guide
energy density, P7.8(194), P7.26(206)
in electric field, 119
in electromagnetic field, 176, 316, 341,
422
in magnetic induction, 151
in monochromatic field, 334, 345
in wave guide, 362
energy loss, see conductor; ferromagnetism;
radiation from; wave guide
energy storage
in electric field, 118–119
in electromagnetic field, 176–177,
315–316, 420–422
in magnetic induction, 151, 166–169
energy theorem, 176–177, 315–316
relativistic form of, 420–422
energy transport
by plane waves, 192–193
in electromagnetic field, 176–177,
315–316
in general plane wave, 204–205
in wave guide, 362–363
equality
of complex numbers, 445
INDEX

of matrices, 435

equation of continuity, 63–65, 157, 315,
  414–415, 418, P2.24(67), P2.26(67),
  P6.23(175)
at material discontinuities, 323–324

equation of motion, 73, 399

equipotentials, see also charged line; electric
dipole; level contours; point charge;
quadrupole lens, 92, 112–114
  ⊥ to field lines, 113, 225, P0.26(21)
of various sources (figures), 89, 92, 236

erg (unit of work), 14

erth, 403–405

Euler formula, 448, PD.13(449)

Euler method, see also improved Euler
  method, 74–75

Euler, Leonard, 74

event in space-time, 403

exact differential, 17–18

F

far field, 150

farad (unit of capacitance), 114

Faraday disc dynamo, 164

Faraday Law, 158–164, 173, 174, 282, 305
  at material discontinuities, 322–323
  in presence of matter, 311–312
  of electromagnetic induction, 163
  restricted, 106–107, 157, 269, 270

Faraday, Michael, 158, 163

ferromagnetism, 303–307, P11.25(303),
  P12.9(317), P12.27(329)

field lines
  ⊥ to equipotentials, 113, 225, P0.26(21)
definition of, 11
electric (figures), 88, 89, 92
magnetic induction (figures), 131, 137
tracing, P4.58(125), P5.33(154)

field point, 87

field tensor, see electromagnetic field tensor

field tray, 238, P9.4(250)

field(s), see also specific field of interest
complex, 197–202, 331–334, 339–341, 345
differentiation of, 10
integration of, 10
representation of, 8–11
versus action at a distance, 69

Fitzgerald, George Francis, 412

Fizeau experiment, P15.8(412)

Fizeau, Armand Hippolyte Louis, 412

flux, see also electric flux; magnetic flux
definition of, 57
  of electric field, 100–102
  of magnetic induction, 141, 160–164

flux law, see magnetic flux law

force, see also capacitor; charge and current
distribution(s); conducting plate;
conducting sphere; current;
dielectric slab; electric dipole;
electric force; Lorentz force;
magnetic dipole; magnetic force;
point charge; radiation pressure
from potential energy, 107

Lorentz transformation of, 423–424

transformation of . . . in relativity,
  P15.30(426)
work done by, 13–14

force density, 78, 178, 317, 420

force field, 13–20, P0.30(25)

four-current, 414–415, 417, P15.24(420)

four-momentum, 423, P15.12(413),
  P15.40(429)

four-potential, 416, 417, P15.24(420),
  P15.39(428)

four-vectors, see also four-current;
  four-momentum; four-potential;
  wave number-frequency four-vector;
  vectors, under Lorentz
  transformation, 409–410
  spatial components of, 409
  temporal component of, 409

Fourier integral theorem, 451

Fourier inversion formula, 452

Fourier series, 217, 449–450
  periodicity of, 450

Fourier transform, 451–452

Fourier, Jean-Baptiste Joseph, 217

frame of reference, 72, 403–408
  inertial, 407

free charge, 104, 255

free charge density, see charge density, free

free current, 255

free current density, see current density, free

free current density, see current density, free

frequency, 188–190

frequency distribution, see plane wave;
radiation fields

Fresnel equations, 353–357

Fresnel, Augustin-Jean, 355

G

Galilean transformation
  applied to wave equation, P15.3(406)
  equations for, 403
  inadequacy of, 403–406
  of velocity and acceleration, P15.1(406)
Galilei, Galileo, 403

gauge, see Coulomb gauge; Lorentz gauge; radiation gauge
gauge invariance, 148, 179
gauge transformation, 148, 179, 319, P5.19(151), P15.36(427)
gauss (unit of magnetic induction), 71, 298
Gauss’s Law, see also Maxwell’s equations, 157, 173, 269
applied to line charge, 102–104
at material discontinuities, 321–322
for displacement, 270
for magnetic induction, see magnetic flux law
integral form of, 100–105
Gauss, Johann Carl Friedrich, 102
Gaussian pillbox, 103, 104, 321
Gaussian units, see units, Gaussian
generalized circuital law, see circuital law, generalized
generator, P6.5(164)
geometric sum, 200
Gerlach, Walter, 84
getpointB
listing of in IDL, 471–480
getpointE
listing of, 466–471
gradient, 17, 65
as four-vector, 409–410
coordinate-free definition of, 22, P0.37(27)
in MATHEMATICA, 18
of divergence, 65
gravitational constant, 453
Green’s theorem, 444, P8.45(241)
Green, George, 241
Gregory, James, 440
group velocity, 341, 363, 368, P13.42(373)

H

Hall effect, 249
Hall probe, 304, P9.7(251)
Hall, Edwin, 249
Heaviside, Oliver, 39
Heaviside-Lorentz units, see units, Heaviside-Lorentz
helicity, 196
Helmholtz coil, 134–135, 137, P5.4(139)
henry (unit of inductance), 168
Henry, Joseph, 158, 163
Hermitian matrix, 436
hertz (unit of frequency), 190
Hertz potential, P12.12(319)
Hertz, Heinrich, 319
homogeneous boundary condition, 212
homogeneous equation, 433
hydrogen atom
charge density in, P2.4(51)
electric field in, P4.50(123)
electric multipole moments of excited state of, P4.46(122)
induced electric dipole moment of, P3.12(82)
magnetic induction at nucleus of, P5.6(140)
plum pudding model of, P4.7(99)
radiative collapse of, P14.23(398)
hyperbolic functions, 216, PD.12(449), PD.13(449)
hysteresis, 299, 305–307, P12.27(329)
hysteresis loop, 306–307
energy dissipated in, 307

I

$i \sqrt{-1}$, 446
identities, vector, 443–444
identity matrix, see unit matrix
IDL
for evaluating fields, 96–98
for evaluating potentials, 111–112
for plotting fields, 96–98
for solving ODEs, 76
IDL functions/procedures
antena.pro, see antena
arcfield.pro, see arcfield
arcpot.pro, see arcpot
arrowhead.pro, see arrowhead
calcbfield.pro, see calcbfield
calcbloop.pro, see calcbloop
calcefield.pro, see calcefield
electfield.pro, see electfield
getpointB.pro, see getpointB
getpointE.pro, see getpointE
lisjus.pro, see lisjus
loopax.pro, see loopax
looprad.pro, see looprad
ludiffeq_23.pro, see ludiffeq_23
ludiffeq_45.pro, see ludiffeq_45
lugen_grid.pro, see lugen_grid
magnetfield.pro, see magnetfield
magnetloop.pro, see magnetloop
qsimp.pro, 97–98, 112, 138
image(s)
in dielectrics, P12.30(329)
method of, 227–230
of line charge in conducting cylinder, 230
of point charge
  in conducting plate, 228–229,
  P8.24(230)
  in conducting sphere, 229–230
improved Euler method for ODEs, 75–76
index of refraction
  complex, 342, 351, 354, P13.13(345)
  dielectric constant and, 343–344
  ordinary, 342–344, 352, P12.6(314)
inductance, see also current loop; toroid, 169,
  P6.12(169), P6.13(169)
  definition and units of, 167–168
  in presence of magnetic matter, 300
  of plasma, P13.58(376)
induction, see electromagnetic induction
induction zone, 383
inductor
  energy stored in, 167–169, P5.23(152),
  P6.15(169)
inhomogeneous equation, 433
inhomogeneous wave equation, see wave
  equation, inhomogeneous
insulator, see also dielectric, 245, 253
interference pattern, see also superposition
  of linear array of \(N\) sources, 199–202
intermediate zone, 383
internal boundary conditions, 212
invariance, see electric dipole moment;
  electromagnetic field(s); gauge
  transformation; Lorentz invariance;
  symmetry arguments
inverse matrix, 436, PA.13(437)
irrotational vector field, 66

**J**

Jacobi, Carl Gustav Jacob, 411
Jacobi matrix, 411
Jones calculus, P7.32(207)
Jones, Robert Clark, 208
joule (unit of work), 14
Joule heat, 247
Joule, James Prescott, 108
Joule/Coulomb (unit of electric potential),
  108
jumping ring, P6.8(165)

**K**

Kelvin (unit of temperature), 43
Kepler, Johannes, 371
kilogram (unit of mass), 43
Kronecker delta, 120, 421, P0.33(26)
Kronecker, Leopold, 120

**L**

Landé \(g\) factor, P11.29(307)
Landé, Alfred, 307
Langevin function, P10.2(257), P11.1(285)
Langevin, Paul, 258
Laplace development
  of determinant, 433
Laplace’s equation, 109, 180, 211–238, 326,
  P5.28(153)
  and complex variables, 224–226
  boundary conditions for, 211–212
  extrema in solutions to, P8.4(214)
  in 1-D, 215
  in 2-D, 215–226
  in 2-D Cartesian coordinates, 215–217,
    P8.46(241)
  in 2-D cylindrical coordinates, 217–218
  in 2-D spherical coordinates, 218–222
  in 3-D, P8.39(240), P8.40(240),
    P8.41(240)
  in a rectangle, 215–217
  numerical methods for, 231–234,
    P8.35(237), P8.36(238)
  solution of \ldots by analogy, 238
  uniqueness of solution to, 212–213
Laplace, Pierre-Simon, 110
Laplacian \((\nabla^2)\), 65, 109, 110, 180, 211,
  P2.23(67)
Larmor frequency, 282
Larmor’s formula, 394, P15.40(429)
Larmor, Joseph, 394
Law of Biot-Savart, 128–130
law of reflection, 351
Legendre polynomials, 219, 221, 222,
  P8.14(223), P8.41(240), PB.5(441)
Legendre’s equation, 219
Legendre, Adrien-Marie, 219
Lenz’s Law, 163
Lenz, Heinrich Friedrich, 163
level contours, see also equipotentials, 11
Liénard’s formula, 395, P15.40(429)
Liénard, Alfred-Marie, 388
Liénard-Wiechert potentials, 385–388
light = electromagnetic wave, 191
line integral, 14, P0.35(27)
linear charge density, see charge density,
  linear
linear dielectric, 271, 314
linear equations (algebraic), 431–432
linear materials, 244, 271, 299, 314
linear polarization, see polarization
lisjus
  instructions for use of
  in IDL, 490–491
  listing of, 480
loopax
  listing of, 136, 475–480
looprad
  listing of, 136, 475–480
Lorentz condition, 180, 319, 378, 414, 416, 418, P14.1(381)
Lorentz force, 72, 157, 313, 424–425, P15.31(427)
Lorentz gauge, 180, 377, P6.28(180), P7.29(207)
Lorentz invariance
  of 4D divergence, 411
  of 4D dot product, 410
  of 4D volume element, 411
  of charge, 415
  of d’Alembertian, 411
  of equation of continuity, 415
  of Lorentz condition, 416
  of Maxwell’s equations, 405–406, 416–417
  of trace, 411
  of wave equation, 207, P15.5(412)
Lorentz transformation, 406–411
  applied to wave equation, P7.28(207), P15.5(412)
  as rigid rotation, 409, P15.35(427)
  in four-space, 408–409
  inverse of, 408–409
  matrix form of, 408–409
  of electromagnetic field, 419, P15.32(427)
  of force, 423–424
  of four-current, 414–415, P15.25(420)
  of four-gradient, 409–410
  of four-momentum, P15.12(413)
  of four-potential, 415–416, P15.24(420)
  of four-vector, 409, P15.35(427)
  of scalar, 409
  of second rank tensor, 410
  of space-time intervals, P15.6(412), P15.11(413)
  of velocity, P15.7(412), P15.34(427)
Lorentz, Hendrik Antoon, 39
Lorentz-Fitzgerald contraction, 412
loss, see energy loss
LSODE
  for solving ODEs, 76
ludiffeq_23
  listing of, 456–461
ludiffeq_45, 461–466
lugen_grid, 455–456
  use of
  in IDL, 134
M
macroscopic description
  of charge and current distributions, 47–57
  of conduction, 243–247
  of dielectric material, 259
  of diia/paramagnetic material, 285–286
  of electromagnetic field, see macroscopic
magnetic charge, see magnetic monopoles
magnetic dipole, 137
  far field of, 150
  fields of oscillating, P14.7(385)
  force on, 80, P3.13(82), P3.23(83), P5.29(153)
  magnetic induction of, 137, 149, P5.4(139), P5.7(140), P5.33(154)
  magnetic scalar potential of, P5.28(153), P12.26(329)
  magnetic vector potential of, P5.17(150)
  orientation of ... in B-field, 280, P3.13(82), P11.1(285), P11.29(307)
  potential energy of, P3.21(83)
  precession of ... in constant B-field, P3.22(83)
  radiation from oscillating, P14.7(385)
  torque on, 80–81, P3.13(82)
  vector potential of, 149, 152–153
  vector potential of oscillating, P14.7(385)
magnetic dipole moment
  and B-field of dipole, 149, P5.7(140)
  and angular momentum, P3.22(83), P11.3(285)
  and vector potential of dipole, 149, 152
  definition and units of, 80–81, 152
  induced, 281–285
  of magnetized matter, 286
  of orbiting particle, P5.6(140)
  of rotating sphere, P3.14(82)
  permanent, 280
magnetic domain, 303–304
magnetic energy density, see energy density
magnetic field, see magnetic intensity;
magnetic induction
magnetic flux, 160–164, 167–169
   definition and units of, 141
magnetic flux law, 141–142, 157, 173, 174,
   297, 313
   at material discontinuities, 322
in presence of matter, 298
magnetic force, see also force, 70–72
magnetic force density, 78
magnetic induction, see also charge and
current distribution(s); coaxial
cable; electric dipole; Helmholz
coil; magnetic dipole; magnetic
quadrupole; magnetized matter;
magnetized sphere; monochromatic
source(s); plane wave(s), point
charge; solenoid; straight rod;
straight wire; toroid
boundary conditions on, 322, 324
definition and units of, 70–72
energy stored in, 151, 166–169, 176–177,
   315–316
exterior to magnetized material, 292–294
figures of, 131, 137
force on
   point charge, 71
from current density, 149
from magnetic vector potential, 147,
   149, 179, 377, 382, 414
from sources, 129–138
gauge invariance of, 148
in Rowland ring, 299–301
interior to magnetized material, 294–297
magnetic vector potential of uniform,
   P5.22(151)
of arbitrary current loop, 149–150
of circular current loop, 132–138
of long cylinder, 144–145
of long straight wire, 131–132
of magnetic dipole, 149–150
relativistic transformation of, 419,
   P15.31(427)
source of, 127–128
tracing, P5.33(154)
units of, 71
magnetic induction field, see magnetic
induction
magnetic intensity, see also cavity
   resonator(s); plane wave(s); toroid;
   wave guide
boundary conditions on, 323, 336–339
definition and units of, 298, 313,
   P11.33(309)
magnetic material
diamagnetic, 279, 281–284
paramagnetic, 279, 280
magnetic monopoles, 127, 170, P6.35(182),
   P11.31(308)
equivalent ... for magnetic materials,
   294
non-existence of, 127, 142, 171, 313
magnetic multipoles, see magnetic dipole;
magnetic monopoles; magnetic
quadrupole
magnetic permeability, see permeability,
magnetic
magnetic polarizability, 279–285
deformational, 281–284
orientational, 280
magnetic polarization
   macroscopic description, 285–286
   microscopic description, 279–285
magnetic poles, see magnetic monopoles;
magnetic dipole; magnetic
quadrupole
magnetic pressure, P5.31(153)
magnetic quadrupole, 137, P5.25(153)
magnetic quadrupole moment, 153
magnetic scalar potential, see also magnetic
dipole; magnetized matter; straight
wire, 293–294, 319, 326
multipole expansion of, P11.30(308)
magnetic scalar potential, P5.28(153),
   P11.22(301)
magnetic sphere in uniform B-field,
   P12.24(329)
magnetic susceptibility, 299
table of values of, 300
magnetic vector potential, see also charge
   and current distribution(s); electric
dipole; magnetic dipole; magnetic
quadrupole; magnetized matter;
plane wave(s); point charge;
solenoid; straight wire, 147–150,
   179, 377–378
as component of four-vector, 415–416
compared with electrostatic potential,
   150
exterior to magnetized material, 286–292
for time-dependent fields, 318–319
from current density, 149
from magnetic induction, 149
gauge invariance of, 148
multipole expansion of, 152–153
of uniform field, P5.19(151)
retarded, see retardation; retarded fields;
potential, retarded
magnetization, 285–292, 297
definition and units of, 285–286
general strategy, 279
spontaneous, P11.25(303)
magnetization current, see current, bound
magnetization curve, 304–307
magnetized matter
bound charges on, 293
bound currents on, 287
cavity in, P11.11(292), P11.17(294)
dipole moment of, 286
magnetic induction of, 292–297
magnetic scalar potential of, 292–294
magnetic vector potential of, 286–292
magnetized rod
bound currents on, P11.5(292)
bound poles on, P11.31(308)
compared to solenoid, P11.5(292)
magnetized sphere
bound currents on, 292
dipole moment of, P11.6(292)
magnetic induction of, P12.25(329)
magnetloop, 136
instructions for use of
in IDL, 488–489
listing of, 475–480
magnetoelectric induction, 170
magnetostatic field
in presence of magnetic material, 297–299
magnitude
of complex number, 447
of vector, 1, 3, 5
mapping
magnetic induction, 136
MATHEMATICA functions
Cross, 5
Curl, 18
Dot, 5
Grad, 18
SetCoordinates, 18
MATLAB
for solving ODEs, 76
matrix algebra, 435–436
Maxwell’s equations
boundary conditions on, 320–325
decoupled, 181
for complex amplitudes, 332, 334
for monochromatic waves, 331–334
for static fields in matter, 311
for the potentials, 178–180, 377
in covariant form, 414–417
in differential form, 173–174
in integral form, 173
in matter (mks units), 311–314
in presence of matter, 313–314
in relativistic notation, 417
in vacuum (Gaussian units), 185–186
invariance of . . . to Lorentz transformation, 405–406, 416–417
plane wave solutions, 188–192
summary of, 175, 418
symmetries of, 170–172, P6.20(175), P12.2(314)
wave equation from, 188–189
Maxwell, James Clerk, 158, 170, 191
mechanical view of universe, 403–406
meter (unit of length), 41, 43
method of images, 227–230
mho (unit of conductance), 247
mho/meter (unit of conductivity), 244
Michelson, Albert Abraham, 404
Michelson-Morley experiment, 404
microscopic description
of charge and current distributions, 47–57
of conduction, 243–244, 247–250
of dia/paramagnetic material, 279–285
dielectric material, 255–257
microscopic electromagnetic field, see electromagnetic field, microscopic
mixed boundary condition, 212
mks units, see Maxwell’s equations; units, rationalized mks
mobility of charge carriers, 248, 249, 333
modes, see also transverse electric mode; transverse magnetic mode, 359
momentum
in electromagnetic field, see also plane wave(s); radiation pressure, 177–178, 317–318
relativistic, 423, P15.12(413)
momentum density, 178, 193–194, 317–318, 422
momentum theorem, 177–178, 317–318, P15.29(422)
momentum transport
by electromagnetic field, 420–422
by plane waves, 193–194
monochromatic fields, see also plane waves, 331
boundary conditions on, 336–339
INDEX

in cavity resonators, 363–366
in matter, 331–332, 339–345
in vacuum, 188–192, 205–206, 381–385
in wave guides, 359–363
monochromatic source(s)
fields of, 381–382
radiation from, 381–385
retarded potentials of, 381
monochromatic wave in vacuum, 188–192
monopole, see electric monopole; magnetic monopole; point charge
Morley, Edward Williams, 404
Mossotti, Ottaviano-Fabrizio, 257
motion in various fields, see trajectory in...
motional emf, 160–163, 312
multiple reflection, interference from,
P7.20(203)
multiple slit interference, 201–202
multiplication
of a matrix by a scalar, 435
of complex numbers, 446
of two matrices, 435
multipole, see electric dipole; electric monopole; electric multipole;
magnetic dipole; magnetic monopole; magnetic multipole;
magnetic quadrupole
multipole expansion, P11.30(308)
of electrostatic potential, 120–121
of magnetic vector potential, 152–153
mutual inductance, see inductance

N

n-pole as derivative of (n – 1)-pole, 90
nabla (\(\nabla\)), 65
natural units, see units, natural
near field, 150
near zone, 383
Neumann boundary conditions, 212,
P8.35(237)
Neumann functions, 218
Neumann, Carl Gottfried, 212
neutron rest mass, 453
newton (unit of force), 13, 38
newton-meter (N-m, unit of torque), 19
Newton’s Second Law, 13, 15, 73
numerical solution of, 74–76
with electromagnetic force, 72–73
Newton, Isaac, 13
Newton/Coulomb (unit of electric field), 108
non-trivial solution of homogeneous linear equation, 433
normal derivative, 212
numerical integration
to evaluate fields, 95–98, 136–138
to evaluate potentials, 110–112
numerical method(s) for
determining charge densities, P8.34(234)
field tracing, P4.58(125), P5.33(154)
Laplace’s equation, 231–234, P8.35(237)
line integrals, P0.35(27)
ordinary differential equations, 74–76
Poisson’s equation, P8.38(240)
wave equation, P7.35(208)
wave propagation in dispersive medium,
370–373, P13.44(373)
Numerical Recipes
for solving ODEs, 76
numerical solution
of Laplace’s equation, 231–234
of Newton’s Second Law, 74–76
of ODEs
via adaptive methods, 75–76
via Euler method, 74–75
via improved Euler method, 75–76
O
observation time, 386
OCTAVE
for solving ODEs, 76
oersted (unit of magnetic field), 298
Oersted, Hans Christian, 127
ohm (unit of resistance), 247
Ohm’s Law, 245–249, 314, 338
limitations of, 249
Ohm, Georg Simon, 247
ohmic conductors, 244
operator, see specific operator of interest
optics is included in electromagnetic theory,
357
orthogonal matrix, 436, PA.12(437)
oscilloscope, P3.3(73)
P
parallel plate capacitor, see capacitor,
parallel plate
paramagnetic material, see magnetic material, paramagnetic
paramagnetism, 280, 299
Parseval’s theorem, 396, 452
Parseval, Marc-Antoine, 396
particle density, 78
particle density field, 48, 50, 56–57
particle in various fields, see trajectory in
particle velocity field, see trajectory in
path difference, 201
path independence, 15
penetration length, 342–345, P13.11(345)
period of electromagnetic wave, 190
periodic boundary condition, 217
permeability, see also relative permeability
differential, 305
frequency dependence of, 332
measuring ... with Rowland ring, 299–301
of ferromagnets, 305
of free space, 38, 453
relative, 299, 302
static magnetic, 299–301
permittivity, see also dielectric permittivity
complex, P13.2(335)
frequency dependence of, 333–334
frequency-dependent, P13.2(335)
of free space, 39, 453
relative, 271, 275
static, 271
permittivity tensor, 316, P10.25(277), P12.8(317)
phase, 188, 195–196
of complex number, 447
phase difference, 195
phase shift on, see reflection; transmission
phase velocity, 341, 343–345, 367, 370
physical constants, 453
pillbox, see Gaussian pillbox
Planck’s constant, 453
Planck, Max Karl Ernst Ludwig, 453
plane of incidence, 350–355
plane wave(s), see also electromagnetic waves, plane
complex representation of, 197–198, 339–345
definition of, 191–192
energy stored in, 341, 345, P7.8(194)
energy transported by, 192–193, 334, 345, P7.8(194)
fields in, 188–192, 205–206, 341, 344–345
four-potential for, P15.39(428)
frequency distribution of energy in, 204–205, P7.23(205)
in 3-D, 205–206
in dielectric, 343–344
in imperfect conductor, 339–345
in relativistic notation, P15.39(428)
in vacuum, 188–192, 205–206
interference of, 198–202
magnetic vector potential for, P7.29(207)
momentum transported by, see also radiation pressure, 193–194
polarization of, 195–198, 352–353
INDEX

polarization charge, see charge, bound electric
polarization current density, 312
polarized cube, see dielectric cube
polarized cylinder, see dielectric cylinder
polarized dielectric
bound charges on, 261–262
cavity in, P12.23(329)
dipole moment of, 259
electric field of, 260–268
electrostatic potential of, 260–262
polarized sphere, see dielectric sphere
position vector, 8
potential, see also electrostatic potential;
polarization potential; scalar potential; vector potential
as component of four-vector, 415–416
boundary conditions on, 323–324
for time-dependent fields, 318–319
retarded, see retarded potential
potential energy, see also electric dipole;
energy, stored in; magnetic dipole;
point charge, 15–17, 107–108, 151
potential theory, 211–240
power, 81, 167, 176–177, 315–316
definition of, 20
dissipated in a resistor, 247
power loss, see energy loss
Poynting vector, see also plane waves;
radiation fields, 177, 178, 192–193,
206, 316, 317, 341, 356, 362, 384,
393, 394, 422, P7.25(206),
P7.26(206)
for monochromatic field, 334
from complex field, 198–199, 334
Poynting, John Henry, 177
precession of magnetic dipole, P3.22(83)
predictor-corrector scheme, 75, 136
pressure, see magnetic pressure; radiation pressure
principal maxima, 201, P7.17(202)
principle of relativity, 405–406
principle of superposition, see also
superposition, 86, P6.19(174),
P12.1(314)
product of matrices, see matrix algebra
product of vectors, see vectors
propagation vector, see electromagnetic wave, propagation vector
proper time interval, 423, P15.11(413)
proton charge, 453
proton rest mass, 453
PYTHON
for solving ODEs, 76

Q

Q (quality factor), P13.47(374), P13.48(374)
qsimp, see IDL functions/procedures
quadrupole, see electric quadrupole;
magnetic quadrupole; multipole expansion
quadrupole lens, 226, 227, P8.21(227)

R

radiation
frequency distribution of, 395–396
from accelerated charge, 393–396,
P14.18(397), P14.21(397)
from circular motion, P14.17(397),
P14.22(398)
from linear accelerator, P14.18(397),
P14.19(397)
from monochromatic sources, 381–385
from oscillating electric dipole, 382–385,
P14.3(385)
from oscillating magnetic dipole,
P14.7(385)
from point charge in circular accelerator,
P14.20(397)
radiation fields, see electric dipole; magnetic dipole; monochromatic source; point charge
angular distribution of energy in, 384,
394, P14.4(385), P14.18(397)
frequency distribution of energy in,
394–396, P14.21(397), P14.22(398)
polarization of, P14.3(385), P14.7(385)
Poynting vector in, 384, 393–394
radiation pressure, 194, P7.8(194),
P7.10(195), P7.11(195), P7.31(207)
radiation reaction, 398–399
radiation zone, 383
radius of electron, P4.40(119)
rationalized mks units, see units, rationalized mks; Maxwell’s equations
real numbers, 446
reality of the fields, 185
reciprocal of complex number, 446
red shift, P15.39(428)
reflection
at normal incidence, 348–350
at oblique incidence, 350–357
at plane interface, 346–357
interference from multiple, P7.20(203)
law of, 351
phase shift on, 350, 356
polarization by, 356
total internal, 352, P13.29(359), P13.52(375)

reflection coefficient
at normal incidence, 349–350
at oblique incidence, 350–353, 355–356
definition of, 347
for dielectric film, P13.28(358), P13.53(375)

refraction, index of, see index of refraction
relative permeability, 299
relative permittivity, see dielectric constant
relativistic energy and momentum, 423, P15.12(413)

relativity
principle of, 405–406
special, 406–411

relaxation time, P9.5(250)
remanence, 306
resistance, 247, P14.6(385)
resistivity, 244
tables of values, 245
resistivity, P13.54(375)
resistor, 247
power dissipated in, 247

resonance, see cavity resonators
rest mass
of electron, 453
of neutron, 453
of proton, 453

restricted Faraday Law, see Faraday Law, restricted
retardation, 378–381

retarded fields of, see electric dipole; magnetic dipole; monochromatic source; point charge(s)
retarded location, 386
retarded potential, see also electric dipole; magnetic dipole; monochromatic source; point charge(s), 378–381
and Lorentz condition, P14.1(381)
retarded time, 380, 381, 386, 387, 394, 396

Riemann, Bernhard, 225

right-hand rule
for field of straight wire, 132
for relating surface elements to path elements, 24, 58, 143
for solenoid, P5.14(146)
in cross product, 3
in Stokes’ theorem, 58

rotation of coordinates, see coordinate system(s)
row matrix, 436
row vector, 436
Rowland ring, 299, 300, 304, 305

Rowland, Henry Augustus, 299

S
saturation, 304–305, P11.29(307)
Savart, Félix, 130
scalar field, see field
scalar potential(s), see also electrostatic potential; magnetic scalar potential, 179, 318–319, 377
compared with vector potential(s), 150
retarded, see retarded potential
wave equation for, 180, 319, 377, 414
scalar product, see dot product; triple scalar product
scalars, 1–8, 409
algebra of, 1
under Lorentz transformation, 409
scattering by electron, P7.36(209), P14.27(400)
search coil, P6.5(164)
second (unit of time), 41, 43
secondary maxima, 201, P7.17(202)
self-inductance, see inductance
semi-infinite plate, P8.23(227)
semiconductor, 245
separation constant, 216, 219
separation of variables
for Laplace’s equation in 2-D, 215–222
for Laplace’s equation in 3-D, 222
shielding, 239–240, P8.3(214)
Simpson, Thomas, 371
skew Hermitian matrix, 436
skin depth, 342–344, P13.11(345)
Snell’s law, 351, 356
Snellius, Willebrord, 351
solenoid
compared to magnetized rod, P11.5(292)
magnetic induction of, P5.5(140), P5.14(146), P11.23(301)
magnetic vector potential of, P5.21(151)
solenoidal vector field, 66
solid angle, 61–62
solution of Laplace’s equation
by analogy, 238
by experiment, 238
using complex variables, 224–226
using method of images, 227–230
using numerical methods, 231–234
using separation of variables, 215–222
solving linear equations, 431–433
source distribution, 69, 86
source point, 87
space contraction, P15.6(412)
INDEX

spatial components of four-vectors, 409
special relativity, 406–411
  postulates of, 407
spectral analyzer, 205
spectral decomposition, 203–205
spectral function, 203–205, 370
speed of light, 37, 41, 71, 189–191, 453,
P15.8(412)
invariance of, 405
speed of propagation of electromagnetic
  wave, see speed of light
sphere, see charged sphere; conducting
  sphere; dielectric sphere
spherical coordinates, 6–8
spherical wave, P7.27(207)
spin, 280
square matrix, 436
standing waves, 363–366
statampere (unit of current), 37, 41, P1.4(43)
statcoulomb (unit of charge), 33, 37, 41,
P1.4(43)
static versus uniform field, 49
static zone, 383
steradian, 62
Stern, Otto, 84
Stern-Gerlach experiment, P3.23(83)
Stokes’ theorem, 57–58, 142, 298, 444,
P5.30(153)
Stokes, George, 58
straight rod, current-carrying
  magnetic induction of, 144–145
straight wire, current-carrying
  magnetic induction of, 131–132
  magnetic scalar potential of, P5.28(153)
  magnetic vector potential of, P5.16(150),
P5.20(151)
stress tensor, 183, 420, 422, P6.37(182),
P12.28(329)
stress-energy-momentum tensor, 420–422
subtraction of complex numbers, 445
summation convention, 408
superconductor, 245
superposition, see also electric force;
  electromagnetic field(s);
  electrostatic potential; plane
  wave(s), 189
  for Laplace’s equation, 212, 218, 220
  of N plane waves, 199–202
  of waves of different frequencies, 203–205
  of waves of same frequency, 195–202
  principle of, 86, P6.19(174), P12.1(314)
surface charge density, see charge density,
surface
surface charge(s), see charge density, surface
  from boundary conditions, P13.27(358),
P13.32(366)
surface current density, see current density,
surface
surface currents, P2.26(67), P5.34(155)
  and continuity, P2.26(67)
  bound, 286–290, 292
  from boundary conditions, 323–324, 337,
  338
  impossibility of ... on imperfect
  conductors, 338
  on conductors, P13.21(357),
P13.27(358), P13.32(366)
surface integral, 55, 57–62
susceptibility, see dielectric susceptibility;
  magnetic susceptibility
symmetric matrix, 436
symmetry
  applied to Laplace’s equation, 215
  essential idea of, 102
  in Maxwell’s equations, 170–172,
P6.20(175)
  of cylindrical current, 144
  of infinite charged sheet, 91
  of line charge, 102–104
  of sphere in uniform E, 221, 327
  spherical, 378
  with Gauss’s Law, 102–104, P4.12(105),
P4.13(105)
symmetry arguments, 102–104, 144–145,
  170–172
systems of units, see units
T
Taylor expansion, 440
Taylor, Brook, 440
TE mode, see wave guide, TE mode
temporal component of four-vectors, 409
tensors, 1
  Lorentz transformation of, 410
tesla (unit of magnetic induction), 71
tesla, Nikola, 71
test charge, see also point charge, 69–72, 159,
  254
Thomson cross-section, P14.27(400)
Thomson, Joseph John, 401
time dilation, P15.6(412)
TM mode, see wave guide, TM mode
toroid
  magnetic induction in, P5.15(146),
P12.19(326)
magnetic intensity in, P12.19(326)
self-inductance of, P6.15(169)
torque, see also charge and current
distribution(s): electric dipole;
magnetic dipole; point charge(s)
definition of, 19
work done by, 19–20
total charge density, see charge density, total
total internal reflection, 352, P13.29(359), P13.52(375)
trace, 411, PA.10(437)
trajectory in
constant B-field
non-relativistic, 73, P3.4(73), P3.5(73), P3.7(77)
relativistic, P15.42(429)
constant E-field
non-relativistic, 73, P3.2(73), P3.3(73)
relativistic, P15.41(429)
constant E and B-fields
non-relativistic, 73
Coulomb field, 73
crossed E- and B-fields, P3.6(74), P3.16(82)
plane wave, P7.36(209)
prescribed fields, 72–73, P3.8(77), P3.9(77)
transformation, see force; Galilean transformation;
Lorentz transformation
transformer, P6.15(169)
transmission
at normal incidence, 348–350
at oblique incidence, 350–357
at plane interface, 346–357
phase shift on, 350, 356
transmission coefficient
at normal incidence, 349–350
at oblique incidence, 350–353, 355–356
definition of, 347
for dielectric film, P13.28(358), P13.53(375)
transpose of a matrix, 436
transpose electric modes, 360, 363, 366
transpose magnetic modes, 363, 366
transpose wave, 195, 340
triangle inequality, PD.6(448)
triple scalar product, 3, P0.7(6)
triple vector product, 3, 4, P0.9(6), P0.33(26)

U
uniform versus static field, 49
uniqueness theorem, 212–213, 228

unit matrix, 436
unit vector(s), 2, 4–5, 454
unitary matrix, 436
units, see also concept or quantity whose units are desired
abbreviation of names of, 454
conversion of, 39–41
electromagnetic (all systems), 70
electromagnetic (cgs-emu), 38, 71, 80, 269, 288, 298
electrostatic (cgs-esu), 37–38, 71, 80, 269, 288, 298
Gaussian, 39, 71, 80, 269, 288, 298
Heaviside-Lorentz, 39, 71, 80, 269, 288, 298
natural, 39
rationalized mks, 38–39, 71, 80, 85, 100, 114, 269, 287, 298
systems of, 39–41, P1.7(43)
updated definitions of, 41–43

V
vector algebra, 1–5
vector field, see also field
irrotational (zero curl), 66
solenoidal (zero divergence), 66
vector identities
summary, 443–444
vector potential(s), see also magnetic vector potential, 179, 318–319, 377
compared with scalar potential(s), 150
wave equation for, 180, 319, 377, 414
vector product, see also cross product; triple vector product, 454
vectors, 1–8
algebra of, 1–3
behavior of ... under coordinate rotations, P0.34(26), PA.14(437)
column and row, 436
components of, 4–5, 409
cross product of, 3, 5, 454, P0.6(6)
definition of, 1, 409
dot product of, 2, 5, 410–411, 454, P0.6(6)
longitudinal part of, P0.5(5)
magnitude of, 1, 3, 5
transverse part of, P0.5(5)
triple products of, see triple scalar product; triple vector product
under Lorentz transformation, 409–410
unit, see unit vector(s)
velocity, see addition of velocity; group velocity; phase velocity; world velocity
velocity field, 48, 56–57, 78
velocity selector, P3.6(74)
volt (mks unit of electric potential), 108, 159
volume charge density, *see* charge density, volume
volume current density, *see* current density, volume
volume currents
bound, 290–292
volume elements
differential, 10, 411

**W**

Watt, James, 20, 247

wave equation
- for fields, 181, 188–189
- for potentials, 179–180, 319, 377, 415–416, 418
- from Maxwell’s equations, 188–189
- Galilean transformation applied to, P15.3(406)
- general solution for potentials in vacuum, 378–381
- general solution of, P7.1(192)
- in 3D, P7.27(207)
- inhomogeneous ... in matter, 377
- inhomogeneous ... in vacuum, 180, 181
- invariance of ... to Lorentz transformation, P7.28(207), P15.5(412)
- numerical method for, P7.35(208)

wave guide, 359–363
- cylindrical fields in, P13.36(367)
- energy loss in, 363, P13.54(375)
- rectangular charge and current on walls of, P13.32(366)
- TE mode, 359–363, 366
- TM mode, 363, 366
- wave number, 188
- wave number-frequency four-vector, P15.39(428)
- wave propagation
  - in dielectric, 343–344
  - in dispersive medium, 368–373
  - in imperfect conductor, 339–345
  - in plasma, P13.58(376)
  - in vacuum, 187–206
- wavelength
  - definition of, 190
  - in dielectrics, 343–345
  - in imperfect conductors, 341–343
  - in vacuum, 190, 192
  - in wave guides, 362
  - of electromagnetic wave, 190
- weber (unit of magnetic flux), 141
- Weber, Wilhelm Eduard, 71
- weber/m^2 (unit of magnetic induction), 71, 141
- wedge, *see* conducting wedge
- Weiss theory, P11.25(303)
- Weiss, Pierre-Ernest, 303
- Wiechert, Emil Johann, 388
- work, *see also* potential energy
  - definition of, 13–14
  - done by force, 13–14
  - done by torque, 19–20
  - to assemble charge distribution, 117–119
  - to establish a current, 151, 167–169
- work-kinetic energy theorem, 15
- world velocity, P15.11(413)

**Y**

Yukawa potential, 239, P4.28(115)
- Yukawa, Hideki, 239

**Z**

zones for fields of monochromatic sources, 383
Brief Index to Frequently Used Information

Abbreviation of unit names p. 454
Component form of dot and cross products p. 5
Displacement elements p. 14
Dot and cross products between unit vectors p. 454
Maxwell’s equations
  covariant form of p. 418
  in matter p. 313
  in vacuum p. 175, 185
Physical constants p. 453
Relationships among coordinates and unit vectors p. 9
Relationships among \( \mathbf{J} \), \( \mathbf{B} \), and \( \mathbf{A} \) p. 149
Relationships among \( \rho \), \( \mathbf{E} \), and \( \mathbf{V} \) p. 110
Vector derivatives See table below
Vector identities pp. 443–444
Volume elements p. 10

\[ \begin{array}{|c|cccc|}
\hline
\text{Coordinate System} & \text{Cartesian} & \text{Cylindrical} & \text{Spherical} & \text{General Orthogonal} \\
\hline
\text{Operator} \downarrow & \text{Eq. (0.48), p. 17} & \text{Eq. (0.66), p. 22} & \text{Eq. (0.67), p. 22} & \text{P0.37, p. 27} \\
\text{Gradient} & \text{Eq. (0.53), p. 18} & \text{Eq. (0.73), p. 24} & \text{Eq. (0.74), p. 24} & \text{P0.37, p. 27} \\
\text{Curl} & \text{P0.24, p. 21} & \text{Eq. (0.67), p. 22} & \text{Eq. (0.67), p. 22} & \text{P0.37, p. 27} \\
\text{Divergence} & \text{Eq. (2.24), p. 59} & \text{Eq. (2.29), p. 60} & \text{Eq. (2.30), p. 60} & \text{P2.23, p. 66} \\
\text{Laplacian} & \text{Eq. (2.51), p. 66} & \text{Eq. (2.52), p. 66} & \text{Eq. (2.53), p. 66} & \text{P2.23, p. 66} \\
\hline
\end{array} \]